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Abstract

Some results on the properties and solutions of the steady state form of the neoclassical transport equations are presented for regimes of interest to feasibility experiments and reactors. An analytical solution is obtained for the special case $dT/dr = 0$. For $\frac{dT}{dr} \neq 0$, an analytic solution is found for the particle density, $n(r)$, in terms of the current, $j(r)$, and the temperature, $T(r)$. For a prescribed $j(r)$, a single differential equation remains to find the temperature profile. The numerical results suggest that neoclassical scaling does not admit self-sustained fusion plasma operation ($n \sim 10^{14}$/cc, $T \sim 10$ keV) except in plasmas of uninterestingly small sizes (I_p <500 kA).

I. Introduction

The transport equations derived by Rosenbluth, Hazeltine and Hinton[1] follows much work, initiated by Galeev and Sagdeev[2], on the theory of collisional diffusion in axisymmetric toroidal devices in the low collision frequency regime. These equations have been used in computer codes[3-5] to simulate the time and space (1-D) evolution of current Tokamak discharges. In some codes[3,4], a variety of atomic effects (ionization, charge-exchange, impurity radiation, etc.) have been added to simulate experimental conditions more properly. The steady state behavior of the neoclassical equations without atomic effects has been recently studied by Wiley and Hinton[6] and by MacMahon and Ware[7]. In the latter work, the equations were solved numerically assuming the net particle flux is zero (the trapped particle pinch effect balances the usual outward diffusion) and the plasma is heated only by joule heating. Sigmar and Rutherford[8], following up previous work by Bickerton, Connor, and Taylor[9], have recently considered steady-state solutions of the neoclassical equations in which a seed current, possibly from neutral beams, allows the equilibrium current to be the so called bootstrap current.[9] There is an externally generated heat input but no particle input.

In this paper, we consider the steady state profiles of density and temperature in regimes suitable for feasibility experiments and reactors. An external source of particles is assumed to maintain the density while internal heating, as in a reacting plasma, maintains the energy balance. For such large systems, neoclassical theory will lead to long energy containment times compared to the electron-ion rethermalization time so that one expects approximately equal electron and ion temperatures. This has been found by MacMahon and Ware,[7] and also by McAlees and Conn[10] in a study of neutral beam heating of Tokamaks using a mixture of neoclassical and pseudoclassical[11] transport equations. In section II, the basic equations are presented and an analytic solution is developed for the case of a flat temperature profile. In section III, the case where $\frac{dT}{dr} \neq 0$ is analysed by developing an analytic relation between n(r) and T(r). This leads, for an assumed current profile, to an equation involving a linear second order differential operator for the temperature profile. Numerical solutions of this equation are presented for cases of interest. Finally, in section IV, the interpretation of the results is presented along with the connection to earlier work.

II. The Steady State Equations and an Analytic Solution

For $T_i = T_e$, the steady state neoclassical transport equations[1] are

$$\frac{1}{r}\frac{d}{dr}(r\Gamma) = S_p(r) \qquad (1)$$

$$\frac{1}{r}\frac{d}{dr}(rQ_T) = S_E(r) - L_E(r) \qquad (2)$$

where the particle flux is

$$\Gamma = c(r)\left(\frac{-2.24n}{T^{1/2}}\frac{dn}{dr} + \frac{.62n^2}{T^{3/2}}\frac{dT}{dr}\right), \qquad (3)$$

and the total heat flux is

$$Q_T = \tilde{Q}_i + \tilde{Q}_e + 5\Gamma T = c(r)\left(-\frac{32.3n^2}{T^{1/2}}\frac{dT}{dr} - 5.52nT^{1/2}\frac{dn}{dr}\right). \qquad (4)$$

We have taken the ion mass number as 2.5 to model a 50-50 D-T plasma and have defined c(r) as

$$c(r) = \frac{8\sqrt{2\pi}}{3} \left(\frac{e^2\sqrt{m_e}\ \ln\Lambda}{B_\theta^2(r)} \right) \left(\frac{r}{R}\right)^{1/2} \quad .$$

using the same notation as in reference (1). In equation (1), the trapped particle pinch effect is neglected assuming large poloidal beta[12] ($\beta_p > 1$) and the steady state is therefore maintained by a particle source, $S_p(r)$. Equation (2) is obtained by adding the separate energy equations of Reference 1 and using $T_i = T_e$ in the resulting equation. We have also included terms for an energy source, $S_E(r)$, and an energy loss, $L_E(r)$, to account for alpha heating, external heating, and radiation cooling. Joule heating is negligible compared to alpha heating in a D-T plasma at $T \sim 10$ keV. Equations (1)-(5) are not strictly valid over the entire cross section of the plasma. For small r, one should use "plateau" diffusion coefficients[2]. For large devices, however, the transition between "plateau" and "banana" diffusion occurs at such small values of r (where the gradients are small) that little error is made using "banana" diffusion coefficients for $r \to 0$. At the plasma edge, a transition to the plateau or Pfirsch-Schluter[13] regimes would take place if the temperature becomes very low. However, in the analysis to be presented, the boundary condition at the plasma edge, $r = a$, is taken to be zero particle density, $n(a) = 0$. It will be shown that in such a case, $T(a)$ cannot be arbitrarily specified and in fact remains relatively large. Thus, the plasma edge in this model using only equations (1)-(4) remains in the banana regime.

For Tokamak plasmas bounded by a divertor with excellent particle collection characteristics, one might expect that the appropriate boundary condition at the plasma edge, r = a, is $dT/dr \cong 0$. As a first case, we consider under what conditions equations (1) and (2) possess solutions with $dT/dr = 0$ throughout the plasma. Assuming $\frac{dT}{dr} = 0$ in (3) and substituting into (1) gives

$$\frac{-2.24}{T^{1/2}} \frac{1}{r} \frac{d}{dr} \left(r\, c(r) n \frac{dn}{dr}\right) = S_p(r). \tag{6}$$

Setting $\frac{dT}{dr} = 0$ in (4) and substituting into (2) gives

$$-5.54\, T^{1/2} \frac{1}{r} \frac{d}{dr} \left(r\, c(r)\, n \frac{dn}{dr}\right) = S_E(r) - L_E(r). \tag{7}$$

Divide (7) by (6) to obtain

$$\frac{5.54}{2.24} T = \frac{S_E(r) - L_E(r)}{S_p(r)} \quad . \tag{8}$$

Since T is independent of r, the right hand side of equation (7) must also be independent of r. Thus, for uniform temperature, the particle source function $S_p(r)$ must have the same spatial dependence as the net energy source $S_E(r)-L_E(r)$. In a self-sustaining D-T reactor, the energy source is alpha heating, which is a function of r through $n^2(r)$. (We neglect here for simplicity alpha particles born near the plasma edge which can be lost before slowing down.) Bremsstrahlung radiation also varies as $n^2(r)$ so that $S_E(r) - L_E(r)$ is proportional to $n^2(r)$. Thus, $S_p(r)$ must also be proportional to $n^2(r)$ if the temperature is to be uniform. Under this condition, the density profile is determined by the equation

$$\frac{1}{r}\frac{d}{dr}\left[r\ c(r)\ \frac{dn^2}{dr}\right] = -\ \alpha^2 n^2(r) \tag{9}$$

where $\alpha^2 > 0$ is an arbitrary constant. To obtain a solution to equation (9), assume the toroidal current profile is uniform. Then $c(r) \propto r^{-3/2}$ and equation (9) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r^{-1/2}\ \frac{dn^2}{dr}\right) = -\alpha^2 n^2(r). \tag{10}$$

This can be transformed using $y = n^2(r)$, $u = r^{3/2}$, to

$$\frac{d^2 y}{du^2} + \frac{4\alpha^2}{9}\ u^{1/3} y = 0 \tag{11}$$

which is a special form of Bessel's equation[14]. Imposing the boundary conditions, $\Gamma(0) = 0$ and $y(a) = n^2(a) = 0$, the solution is

$$n^2(r) = n_o^2\ \left(\frac{r}{a}\right)^{3/4}\ J_{-3/7}\ \left(\gamma\left(\frac{r}{a}\right)^{7/4}\right) \tag{12}$$

where $n_o = n(o)$, $J_{-3/7}$ is the Bessel function of index -3/7, a is the plasma radius and $\gamma = 1.7$ is the first zero of $J_{-3/7}$. This solution is shown in Figure 1 together with $(1 - (r/a)^3)^{1/2}$, a simple and good fit.

III. General Density and Temperature Profiles

For the case $\frac{dT}{dr} \neq 0$, it is possible to proceed part of the way analytically. In particular, an analytic relationship between the density and temperature profiles can be derived such that the remaining equation for the temperature profile is much simplified, involving a linear second order differential operator. To proceed, write equation (3) as

$$\Gamma = c(r)\left(\frac{-2.24n}{T^{1/2}}\frac{dn}{dr} + \frac{.56n^2}{T^{3/2}}\ \frac{dT}{dr} + \frac{.06n^2}{T^{3/2}}\frac{dT}{dr}\right). \tag{13}$$

By neglecting the last term in the brackets, a 10% correction to the cross term contribution to the particle flux and an even smaller correction to Γ if $\frac{dT}{dr} \simeq 0$, as we shall find, the particle flux Γ can be written as a perfect derivative

$$\Gamma = -1.12\ c(r)\ \frac{d}{dr}\left(\frac{n^2}{T^{1/2}}\right) \quad . \tag{14}$$

Therefore, eqn. (1) can now be integrated twice to obtain

$$n^2(r) = [D + b(r)]\ T(r)^{1/2}, \tag{15}$$

where

$$D = \frac{n^2(a)}{T(a)^{1/2}} \tag{16}$$

and

$$b(r) = \frac{1}{1.12} \int_r^a \frac{dr'}{r'c(r')} \int_o^{r'} dr''\ r''\ S_p(r''). \tag{17}$$

Substituting this relationship between n(r) and T(r) into equation (4), one finds that Q_T depends linearly on T, namely,

$$Q_T = -33.7(D + b(r))\ \frac{dT}{dr} - 2.76c(r)T\ \frac{db}{dr} \ . \tag{18}$$

Thus, the left hand side of the energy conservation equation, (2), will involve a linear second order differential operator on T and the number of coupled differential equations to be solved has been reduced by one. However, c(r) and b(r) depend on the plasma current profile, j(r), and, in the general case, must be solved self consistently with the neoclassical Ohm's law. In the case of a specified, though arbitrary current profile, the linear, second order differential equation for T(r), eqn. (2), can be solved numerically in a straight forward way. Thus, for the remainder of this paper, we shall assume a constant current profile and proceed to examine the implications of this model problem.

As an example of interest for a D-T burning plasma, consider alpha heating for the energy source, $S_E(r)$, and bremsstrahlung radiation for $L_E(r)$. Then

$$S_E(r) - L_E(r) = \frac{1}{4}\ (D + b(r))T^{1/2}\ \langle\sigma_f v\rangle_{D-T} Q_\alpha$$

$$-f\ T(r)\ [D + b(r)] \tag{19}$$

where $\langle\sigma_f v\rangle_{D-T}$ is the fusion reaction rate, Q_α is 3520 KeV and $f = 4.8 \times 10^{-37}$ when n is in cm^{-3} and T is in kev. The equation for T(r) becomes

$$\frac{-33.7}{r}\frac{d}{dr}\{r\ c(r)\ [D + b(r)]\ \frac{dT}{dr} + .0815\ r\ c(r)\ \frac{db}{dr}\ T\}$$

$$= [D + b(r)]\ [\frac{\alpha}{4}\ \sqrt{T}\ \langle\sigma_f v\rangle - fT] \quad . \qquad (20)$$

Equation (20) has been solved assuming a uniform particle source, S_p, a uniform toroidal current density, and zero density at the plasma edge, n(a) = 0. Using x = r/a, equation (17) for b(x) gives

$$b(x) = \frac{S_p\ a^2}{7.84d}\ (1 - x^{7/2}) \qquad (21)$$

where

$$d = \frac{8\sqrt{2\pi}}{3}\ \frac{e^2 m_e^{1/2} \ln\Lambda}{A^{1/2} B_\theta^2(a)} \qquad (22)$$

and A is the torus aspect ratio. Equation (20) thus becomes

$$\frac{d^2T}{dx^2} - \frac{3.79\ x^{5/2} + \frac{1}{2x}\ (1 - x^{7/2})}{1 - x^{7/2}}\ \frac{dT}{dx}$$

$$- \frac{.573\ x^{3/2}}{1 - x^{7/2}}\ T = -1785\ \delta\ x^{3/2}\ [\ \frac{Q_\alpha}{4}\ \sqrt{T}\ \langle\sigma_f v\rangle - .048\ T] \quad (23)$$

which depends on a single parameter, δ, given by

$$\delta = \frac{.04\ I^2\sqrt{A}}{\ln\Lambda} \quad . \qquad (24)$$

Here, I is the plasma current in MA, and Q_α and T and measured in keV. Therefore, for a given aspect ratio machine, the plasma temperature essentially will depend only on the plasma current, remembering that a uniform current profile has been assumed.

Equation (23) possesses a regular singular point at $x = 1$, which arises from $n(1) = 0$, and can be made to have a regular singular point at $x = 0$ by a change of variable. At $x = 1$, one solution is well-behaved and the other is divergent. At $x = 0$, one solution possesses $Q_T = 0$, the other does not. (Rather than $Q_T(0) = 0$, the boundary condition at $r = 0$ is sometimes incorrectly assumed to be $\frac{dT}{dr} = 0$ at $r = 0$. We have in fact found numerically many solutions with zero derivative at $r = 0$ but, of course, only one with $Q_T(0) = 0$.) A numerical solution to equation (23) which has the property $Q_T(0) = 0$ and is well-behaved at $x = 1$ is shown in Figure 2. It is interesting to note that, because of the singular behavior at $x = 1$, only one solution is admissible. Since $n(a) = 0$, it is not possible to arbitrarily choose the edge temperature. Rather, $T(a)$ is determined by connecting the well-behaved solution at the edge to the solution satisfying $Q_T = 0$ at the origin. Once $T(r)$ is known, the density profile is determined from equation (15). This is also shown in Figure 2 together with the previously derived analytic solution, equation (12). Since the temperature profile is effectively flat, equation (12) agrees with the numerical solution except near the edge.

IV. Discussion

The solution in Figure 2 for $\delta = .931 \times 10^{-3}$ has a centerline temperature of 11 keV, a temperature of reactor interest. Assuming other parameters of reactor interest, such as a safety factor, $q(a)$, of 2.5, poloidal $\beta_p = 2$, and toroidal field strength, B_ϕ, equal to 50 kG, the center line density is 10^{14} per cm^3. However, for $A = 3$ and $\ln\Lambda \sim 18$, this value of δ corresponds to a plasma current of only 500 kA and a plasma minor radius of just 15 cm. Thus, neoclassical theory admits interesting reactor grade plasmas only in uninterestingly small systems. The reasons are clear and have been discussed previously by several authors using space-independent models.[15,16] Essentially, the only way to achieve a plasma energy equilibrium at temperatures in the 10 to 15 keV range, assuming hydrogenic bremsstrahlung losses, is to have high particle leakage rates and high conduction losses. This has been achieved here essentially by reducing the plasma radius, since the confinement time varies as a^2 or, equivalently, as I^2. Parenthetically, even at 500 kA, this equilibrium could not be maintained since such a current is insufficient to contain the alpha orbits.[17]

It is of interest to determine how anomolous the transport coefficients must be, relative to the neoclassical values, to achieve plasmas of interesting size at 10^{14} per cm^3 density and 11 keV. Enhancing the collision frequency by a factor, S_c, will scale all the transport coefficients. The effect is to alter δ such that

$$\delta = \frac{.04\ I^2(MA)\ \sqrt{A}}{S_c\ \ln\Lambda} .$$

For a plasma characterized by $\beta_p = 2$, $q(a) = 2.5$, $B_\phi = 50$ kG, $A = 3$, $n = 10^{14}$/cc, and $T = 11$ keV, a 5 MA discharge requires $S_c = 100$ and a 10 MA discharge means $S_c = 400$. One would conclude therefore, that neoclassical scaling is two to three orders of magnitude better than is desirable in a fusion reactor. While this conclusion is based on the simplified model considered herin, it is consistent with numerical results obtained using the full set of neoclassical equations.[4]

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Figure Captions

Figure 1 - Analytic solution for the density profile when $dT/dr = 0$ and current profile is constant.

Figure 2 - Temperature and density profiles found by solving eqn. (23) and using the analytic relationship, eqn. (15) between $n(r)$ and $T(r)$. Assumptions are that $j(r)$ and $S_p(r)$ are uniform in space.

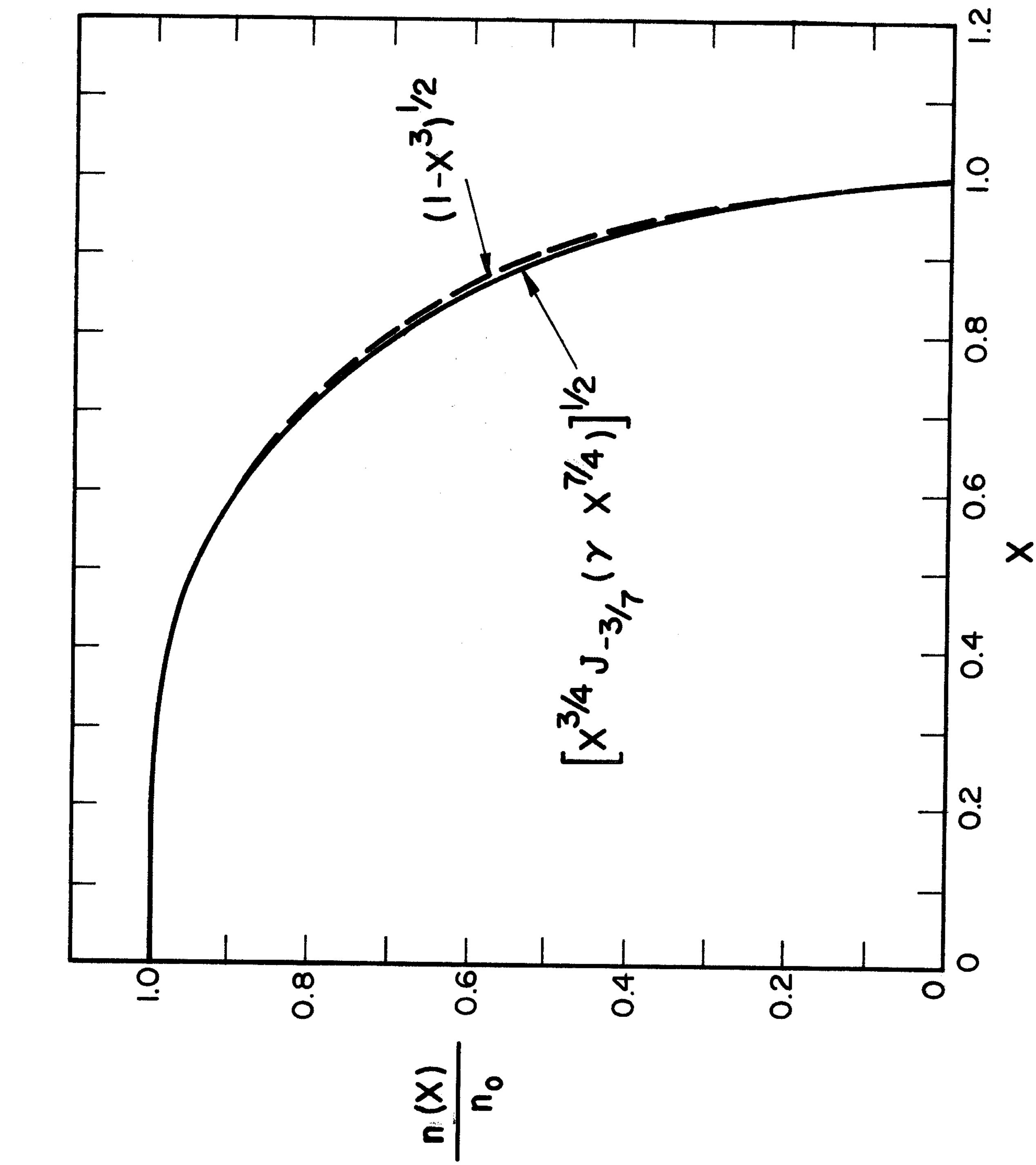
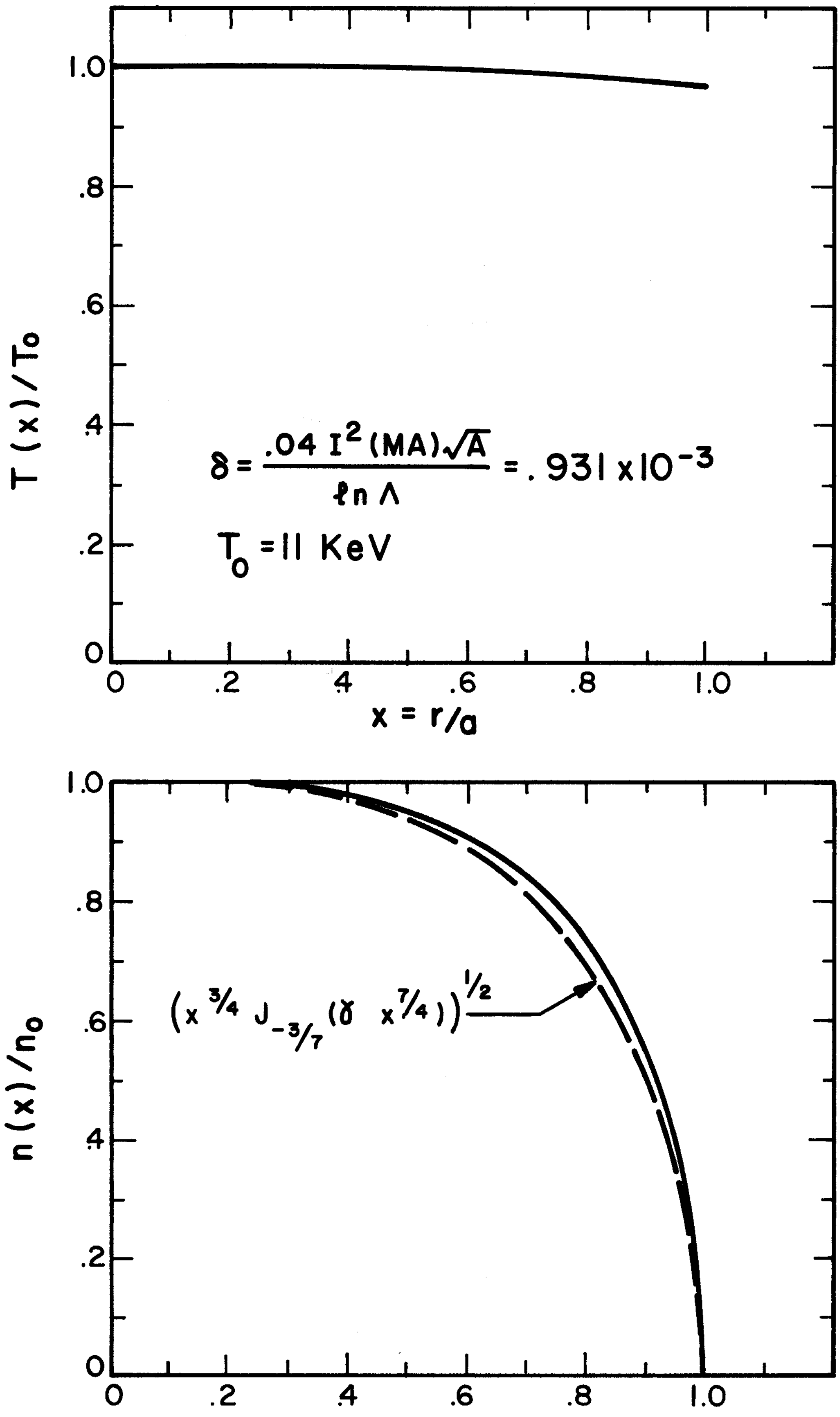


Figure 1



1.0
.8
.6
.4
.2
0
T (x) / To
δ = .04 I² (MA)√A / ℓn Λ = .931 x10⁻³
T₀ = 11 KeV
0
.2
.4
.6
.8
1.0
x = r/a
1.0
.8
.6
.4
.2
0
n (x) / n₀
(x ³⁄₄ J₋₃⁄₇ (γ x⁷⁄₄))^½
0
.2
.4
.6
.8
1.0
x = r /a

Figure 2