# Twisted Coil Geometry

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FUSION TECHNOLOGY INSTITUTE

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## INTRODUCTION

Twisted coils of complicated shape are employed in various magnetic plasma confinement devices, especially those in the general class of stellarator-type devices. In design of these devices, ad hoc winding pack envelopes are frequently assumed. A more exact treatment shows that the above ad hoc winding pack definitions do not represent the true shape taken by windings as successive turns and layers of turns are wound. As a result, errors in calculation of magnetic fields and in shapes of coil cases, etc., occur. In the present work, the mathematical techniques of differential geometry are applied to the problem. Except where otherwise noted, solutions are given for the limiting case of conductors with small cross section compared to bend radii and winding pack transverse dimensions.

## WINDING PACK COORDINATES

A convenient set of parameters for defining a twisted winding pack is based on the actual sequence in which it is wound. First, a starting layer surface is specified by a vector function $\vec{R}(p,q)$ that gives the coordinates of a surface point relative to a fixed rectangular set of axes, as a function of two general non-orthogonal parameters p and q. Successive layers form a family of parallel surfaces with the parametric form

$$\vec{R}^{*}(p,q,h) = \vec{R}(p,q) + h\hat{N}(p,q) \qquad (1)$$

where h is the winding depth and $\hat{N}$ is the unit surface normal. Starting turns in each layer are specified by defining a starting turn surface that intersects the family surfaces parallel to the starting layer surface; starting turns for each layer are the curves of intersection of the above surfaces.

In a particular layer, successive turns form a family of geodesic parallels. Analytic expressions can usually not be found for the geodesic parallels; instead, a particular member of a family of geodesic parallels is generated by laying off equal arc lengths along the family of geodesic curves that intersect the starting turn orthogonally. The geodesic parallel is the locus of the endpoints of these arcs. After starting layer and starting turn surfaces have been given, specification of the winding pack envelope is completed by giving a total winding depth H and a width for each layer that is the constant arc length along the orthogonal geodesics from the starting turn to the last turn. The layer widths are given by a function W(h) of the winding depth parameter h. The above specification is the most general one possible for winding with conductors of constant height and width in a particular turn and constant height throughout a particular layer.

The foregoing procedure may be generalized to allow the starting layer and starting turn surfaces to intersect at the center of the pack, in which case turns and layers are "unwound" to get the actual starting layer and starting turn surfaces.

In order to make the relation between the starting layer and starting turn surfaces explicit, it is convenient to take the arc length $s_0$ along the curve of intersection of the starting layer and starting turn surfaces to be the winding length parameter. The starting turn surface can then be defined ab initio in terms of $s_0$ and one other parameter, or if needed, a coordinate transformation can be made from some other pair of parameters to a pair that includes $s_0$. Assuming the former, the starting turn surface has the parametric form $\vec{R}_s(s_0,u)$ and its curve of intersection with the starting layer surface is given by known parametric expressions, say $p = f_0(s_0)$, $q = g_0(s_0)$.

Denoting starting turns on subsequent layers by $p = f(s_0,h)$, $q = g(s_0,h)$, and $u = e(s_0,h)$, the functions f, g, and e must obey the vector equation

$$\vec{P}(s_0,h) = \vec{R}\,[f(s_0,h),g(s_0,h)] + h\hat{N}\,[f(s_0,h),g(s_0,h)]$$
$$= \vec{R}_s\,[s_0,e(s_0,h]\ . \tag{2}$$

Equation (2) is, in general, a set of three coupled transcendental equations in the unknowns f, g, and e; $\vec{R}$, $\hat{N}$, and $\vec{R}_s$ are known functions. Solution of Eq. (2) for all possible values of $s_0$ and h defines functions f, g, and e of the parameters $s_0$ and h (see Fig. 1). Numerical solution of Eq. (2) (which will be required in almost all cases of practical interest) is most easily performed by first differentiating it with respect to h, holding the parameter $s_0$ fixed, and using the following well-known expressions for the derivatives of the normal vector with respect to the surface parameters:

$$\hat{N}_1 = \frac{FM - GL}{V^2}\vec{R}_1 + \frac{FL - EM}{V^2}\vec{R}_2 \tag{3}$$

$$\hat{N}_2 = \frac{FN - GM}{V^2}\vec{R}_1 + \frac{FM - EN}{V^2}\vec{R}_2\ . \tag{4}$$

In Eqs. (3) and (4), the first order quantities $E = \vec{R}_1^2$, $F = \vec{R}_1\cdot\vec{R}_2$, $G = \vec{R}_2^2$, and $V^2 = EG - F^2$, and the second order quantities $L = \vec{R}_{11}\cdot\hat{N}$, $M = \vec{R}_{12}\cdot\hat{N}$, and $\hat{N} = \vec{R}_{22}\cdot\hat{N}$ have been introduced. The resultant system of first order equations to be integrated with respect to h is

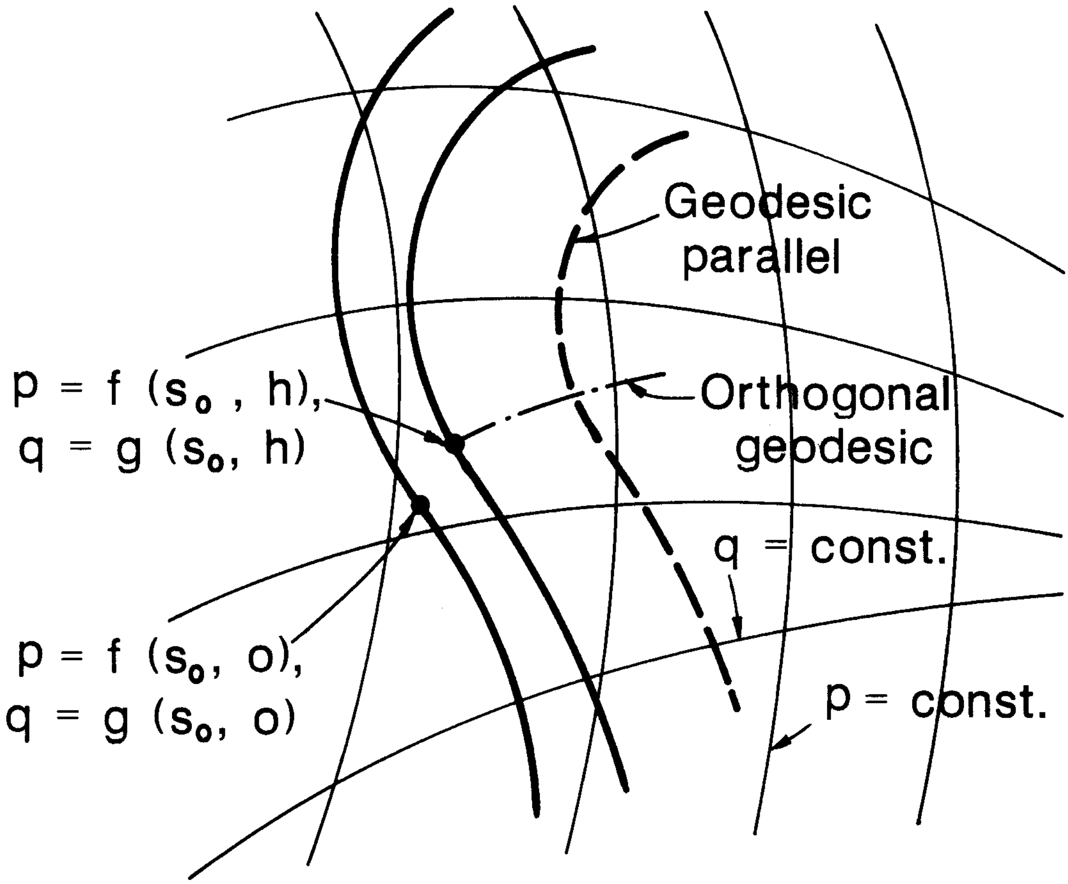


Fig. 1. Construction of geodesic parallels on a parallel surface that is a distance h from the starting surface. The curves shown for the parallel surface ($p = f(s_0,h)$, $q = q(s_0,h)$ and its geodesic parallel) are represented by their respective normal projections on the starting surface.

$$\frac{\partial f}{\partial h}\left[\left(1 + h\,\frac{FM - GL}{V^2}\right)\vec{R}_1 + h\,\frac{FL - EM}{V^2}\,\vec{R}_2\right] + \frac{\partial g}{\partial h}\left[\left(1 + h\,\frac{FM - EN}{V^2}\right)\vec{R}_2 + h\,\frac{FN - GM}{V^2}\,\vec{R}_1\right] + \hat{N} = \frac{\partial \vec{R}_s}{\partial e}\,\frac{\partial e}{\partial h} \; . \tag{5}$$

Equation (5) is reduced to a set of three scalar equations, each containing only one unknown derivative, by taking scalar products with vectors orthogonal to the vectors multiplying the remaining two unknowns. The initial conditions for integration are $f(s_0,0) = f_0(s_0)$, $g(s_0,0) = g_0(s_0)$, and $e(s_0,0) = 0$.

The starting coordinates $p_0$, $q_0$ for orthogonal geodesics are given by the solutions to Eq. (5). For complete specification of the geodesics, the starting direction must also be given; this requires knowledge of the tangent vector to the starting turn curve. Differentiating Eq. (2) with respect to $s_0$, one obtains

$$(\vec{R}_1 + h\hat{N}_1)\,\frac{\partial f}{\partial s_0} + (\vec{R}_2 + h\hat{N}_2)\,\frac{\partial q}{\partial s_0} = \frac{\partial \vec{R}_s}{\partial s_0} + \frac{\partial \vec{R}_s}{\partial e}\,\frac{\partial e}{\partial s_0} \; . \tag{6}$$

Equation (6) is solved for the unknowns $\partial f/\partial s_0$, $\partial g/\partial s_0$, and $\partial e/\partial s_0$ by taking scalar products with orthogonal vectors as before. The three expressions are evaluated after each iteration during the integration of Eq. (5) and substituted in Eq. (6); the left hand side, apart from a normalization factor, is the tangent to the starting turn at a point with parameter values $p = f$, $q = g$ on the parallel surface and $s_0$, h on the starting turn surface. Denoting the first order quantities for the parallel surface by $E^* = \vec{R}_1^{*2}$, $F^* = \vec{R}_1^* \cdot \vec{R}_2^*$, $G^* = \vec{R}_2^{*2}$, the starting derivatives for geodesics on the parallel surface that are orthogonal to the starting curve are given by

$$p_0' = - \frac{\frac{\partial f}{\partial s_0} F^* + \frac{\partial g}{\partial s_0} G^*}{[(\frac{\partial f}{\partial s_0})^2 E^* + 2 (\frac{\partial f}{\partial s_0})(\frac{\partial g}{\partial s_0}) F^* + (\frac{\partial g}{\partial s_0})^2 G^*]^{1/2} V^*} \tag{7}$$

$$q_0' = \frac{\frac{\partial f}{\partial s_0} E^* + \frac{\partial g}{\partial s_0} F^*}{[(\frac{\partial f}{\partial s_0})^2 E^* + 2 (\frac{\partial f}{\partial s_0})(\frac{\partial g}{\partial s_0}) F^* + (\frac{\partial g}{\partial s_0})^2 G^*]^{1/2} V^*} \quad . \tag{8}$$

The above first order quantities are readily obtained from Eqs. (1), (3) and (4).

Using the notation $p = \alpha(s_0,h,w)$, $q = \beta(s_0,h,w)$ for orthogonal geodesics (which depend explicitly on w, the arc length along them, and on $s_0$ and h through initial values $p_0$, $q_0$, $p_0'$ and $q_0'$), the parameterization of the winding pack is

$$\vec{P}(s_0,h,w) = \vec{R}\,[\alpha(s_0,h,w),\beta(s_0,h,w)] + h\hat{N}\,[\alpha(s_0,h,w),\beta(s_0,h,w)] \quad . \tag{9}$$

The explicit method of finding $\alpha$ and $\beta$ is given in the next section. The magnetic field generated by a current density J(h,w) is given by the Biot-Savart law which takes the following form (MKS units)

$$\vec{B}(\vec{X}) = \frac{\mu_0}{4\pi} \int_0^H \int_0^L \int_0^{W(L)} \frac{J(h,w)(\hat{N} \cdot \frac{\partial \vec{P}}{\partial h}) \left|\frac{\partial \vec{P}}{\partial s_0}\right| \hat{T} \times (\vec{X} - \vec{P})}{|\vec{X} - \vec{P}|^3} \, dw\, ds_0\, dh \tag{10}$$

where $\hat{T}$ is the unit tangent vector to a turn and L is the total length of the starting turn. The vector $\hat{T}$ is given by vector product of $\hat{N}$ with the unit vector $\partial\vec{P}/\partial w$:

$$\hat{T} = \hat{N} \times \frac{\partial \vec{P}}{\partial w} . \tag{11}$$

As $\vec{P}$ is, in general, not available in analytic form, the three partial derivatives of $\vec{P}$ must be obtained by numerical means; the procedure is outlined in the next section.

GEODESICS ON PARALLEL SURFACES

When the two parametric coordinates of a geodesic curve are given in terms of its arc length w, the equations for geodesics take the well-known form[1]

$$\begin{aligned} p'' + {}^{1}\Gamma_{11}p'^{2} + 2\,{}^{1}\Gamma_{12}p'q' + {}^{1}\Gamma_{22}q'^{2} &= 0 \\ q'' + {}^{2}\Gamma_{11}p'^{2} + 2\,{}^{2}\Gamma_{12}p'q' + {}^{2}\Gamma_{22}q'^{2} &= 0 \end{aligned} \tag{12}$$

where the primes indicate differentiation with respect to w. The coefficients are functions of the first and second partial derivatives of the Cartesian coordinates $\vec{R}(p,q)$ of the surface with respect to the parameters p and q and are a type of Christoffel symbol. Equations (12) are coupled second-order equations. From the theory of differential equations it is known that if initial values $p_0$, $q_0$ and initial derivatives $p_0'$, $q_0'$ are given, the solution is unique. That is, given a surface, geodesics are uniquely specified by giving their direction at any one point along them. In some cases, it is desirable to eliminate the variable w from Eqs. (12) and express one coordinate in terms of another, e.g., $q = \phi(p)$. In this case, the equations for geodesics become the single equation

$$\frac{d^2\phi}{dp^2} + {}^1\Gamma_{22}\left(\frac{d\phi}{dp}\right)^3 + \left(2\,{}^1\Gamma_{12} - {}^2\Gamma_{22}\right)\left(\frac{d\phi}{dp}\right)^2 + \left({}^1\Gamma_{11} - 2\,{}^2\Gamma_{12}\right)\frac{d\phi}{dp} - {}^2\Gamma_{11} = 0 \ . \quad (13)$$

The normal vector to the surface is given by the expressions

$$\hat{N} = (\vec{R}_1 \times \vec{R}_2)/V \quad (14)$$

$$V = |\vec{R}_1 \times \vec{R}_2| \ . \quad (15)$$

The Christoffel symbols are given by the expressions

$${}^i\Gamma_{jk} = (\vec{R}_{jk} \cdot \vec{F}_i)/V \quad (16)$$

where $\vec{F}_1 = \vec{R}_2 \times \hat{N}$ and $\vec{F}_2 = -\vec{R}_1 \times \hat{N}$.

On the parallel surfaces, the Christoffel symbols are defined in the same way, with $\vec{R}^*$ (Eq. 1) replacing $\vec{R}$. The Christoffel symbols for the parallel surface, when it is parameterized in terms of the fixed distance h between the surfaces and the parameters p and q of the starting surface, now involve third partial derivatives in addition to first and second derivatives. The normal vector $\hat{N}$ is the same for the same values of p and q on the two surfaces. One can show that for the parallel surface,

$$\vec{F}_1^* = \vec{F}_1 + \frac{h}{V}(M\vec{R}_2 - N\vec{R}_1) \quad (17)$$

$$\vec{F}_2^* = \vec{F}_2 + \frac{h}{V}(M\vec{R}_1 - L\vec{R}_2) \tag{18}$$

$$V^* = V - \frac{h}{V}(GL + EN - 2\,FM) + \frac{h^2}{V}(LN - M^2)\ . \tag{19}$$

The three second partial derivatives for the parallel surface are given by the expressions below containing Christoffel symbols for the starting surface:

$$\vec{R}_{11}^* = \vec{R}_{11} + \frac{h}{V}[(2\,L\,{}^1\Gamma_{11} + 2\,M\,{}^2\Gamma_{11} - \hat{N}\cdot\vec{R}_{111})\vec{F}_1 + (2\,L\,{}^1\Gamma_{12} + 2\,M\,{}^2\Gamma_{12} - \hat{N}\cdot\vec{R}_{112})\vec{F}_2 + \frac{EM^2 - 2\,FLM + GL^2}{V}\hat{N}] \tag{20}$$

$$\vec{R}_{12}^* = \vec{R}_{12} + \frac{h}{V}[(M\,{}^1\Gamma_{11} + N\,{}^2\Gamma_{11} + M\,{}^2\Gamma_{12} + L\,{}^1\Gamma_{12} - \hat{N}\cdot\vec{R}_{112})\vec{F}_1 + (L\,{}^1\Gamma_{22} + M\,{}^2\Gamma_{22} + N\,{}^2\Gamma_{12} + M\,{}^1\Gamma_{12} - \hat{N}\cdot\vec{R}_{221})\vec{F}_2 + \frac{EMN - FM^2 - FLN + GLM}{V}\hat{N}] \tag{21}$$

$$\vec{R}_{22}^* = \vec{R}_{22} + \frac{h}{V}[(2\,N\,{}^2\Gamma_{12} + 2\,M\,{}^1\Gamma_{12} - \hat{N}\cdot\vec{R}_{221})\vec{F}_1 + (2\,N\,{}^2\Gamma_{22} + 2\,M\,{}^1\Gamma_{22} - \hat{N}\cdot\vec{R}_{222})\vec{F}_2 - \frac{EN^2 - 2\,FMN + GM^2}{V}\hat{N}]\ . \tag{22}$$

Equations (17)-(22), when substituted in Eq. (16), yield the Christoffel symbols for the parallel surface; final expressions are not given here because of space limitations. Integration of the geodesic equations on the parallel surface, for each pair of values $s_0$, h on the starting curves, yields a series of values of p and q along the geodesic solution curves $p = \alpha(s_0,h,w)$, $q = \beta(s_0,h,w)$ (see Fig. 1). As a by-product, for each iteration values for p' and q' are determined. These are used to find the unit vector $\partial\vec{P}/\partial w$ as follows:

$$\frac{\partial \vec{P}}{\partial w} = (\vec{R}_1 + h\hat{N}_1)p' + (\vec{R}_2 + h\hat{N}_2)q' \ . \tag{23}$$

Expressions for $\partial\vec{P}/\partial h$ and $\partial\vec{P}/\partial s_0$ could be obtained from the geodesic equations by constructing the set of linear differential equations in the unknown derivatives of the solution curves with respect to initial values and integrating them along with the geodesic equations themselves. Instead, it is simpler to approximate the above partial derivatives using increments in $\vec{P}$ at successive values of $s_0$ and h in the generation of the winding pack parameterization.

At this point it is instructive to point out a fundamental difference between flat and twisted windings. Assuming for the moment turns of square cross section, for flat windings turn locations and the winding pack envelope are the same regardless of whether the winding pack is made up of n layers with m turns per layer, or by m pancakes or discs with n turns per pancake. For twisted windings this is not generally true. This can be seen from the fact that while turns in a layer form a family of geodesic parallels, the curves of intersection of successive layers with the starting turn surface do not, unless, for example, the starting turn surface is the set of normals to the starting turn surface along the starting turn in the first layer (in this case the orthogonal geodesics in question are the normals, which are straight lines in space). Even if the above particular starting curve surface is chosen, the surfaces parallel to it do not in general intersect the starting layer surface along geodesic parallels, as would be required if layer winding against the starting layer surface and pancake winding against the starting turn surface were to give the same result.

Also, in a flat winding, a plane orthogonal to any turn intersects all of the other turns orthogonally; the total ampere-turns of the windings for constant current density is just the area of the normal section times the current density. For twisted windings, it is impossible in general to find a family of surfaces that intersect all of the current filaments in a winding pack orthogonally. This is a result of the fact that the infinitesimal angle between geodesics orthogonal to starting curves in layers differing by an infinitesimal depth dh is not necessarily zero; therefore, a turn in one layer may be rotated with respect to nearby turns in the layer below.

USE OF DEVELOPABLES FOR WINDING SURFACES

Fabrication of surfaces for winding can be considerably simplified if a particular class of surfaces, called developables, is used. Developables are surfaces that can be formed out of plane pieces by bending without stretching and are a subset of the class of ruled surfaces, or surfaces formed by sweeping a straight line through space. Ruled surfaces have the parametric form

$$\vec{R}(s_0,u) = \vec{r}_0(s_0) + u\hat{p}(s_0) \tag{24}$$

where $s_0$ is the arc length along the directrix curve $\vec{r}_0$ and u is the length along the ruling intersecting $\vec{r}_0$ at $s_0$.

Developables are those ruled surfaces for which the unit vector $\hat{p}$, its derivative $\hat{p}'$, and the tangent vector to the directrix $\hat{t}$, are coplanar, i.e.

$$\hat{p}'\cdot(\hat{p} \times \hat{t}) = 0 \ . \tag{25}$$

The unit normal $\hat{N}$ is

$$\hat{N} = (\hat{p} \times \hat{t})/\sin\phi \tag{26}$$

where $\phi$ is the angle between $\hat{p}$ and $\hat{t}$. $\hat{N}$ is the same for all values of u along a particular ruling by virtue of Eq. (25). A third vector $\hat{q}$ orthogonal to $\hat{N}$ and $\hat{p}$ can be formed by defining

$$\hat{q} = \hat{N} \times \hat{p} \; . \tag{27}$$

One can now write

$$\hat{p}' = a\hat{q} \tag{28}$$

thereby defining a function a of $s_o$. The remaining two derivatives are given by the expressions

$$\hat{N}' = - \frac{K_n^o}{\sin \phi} \hat{q} \tag{29}$$

and

$$\hat{q}' = \frac{K_n^o}{\sin \phi} \hat{N} - a\hat{p} \; . \tag{30}$$

The quantity $K_n^o$ is the normal curvature, or curvature in the $\hat{N}$-$\hat{t}$ plane of the directrix curve on the developable surface itself; it is given by the formula $K_n^o = \kappa\hat{n} \cdot \hat{N}$, where $\kappa$ is the total curvature and $\hat{n}$ is the principal normal to $\vec{r}_o$. The unit vectors $\hat{p}$, $\hat{q}$, and $\hat{N}$ form a rotating frame field in which any vector may be represented; Eqs. (28)-(30) allow derivatives of vectors to be expressed in terms of the frame field.

The parallel to a developable surface $\vec{R}^* = \vec{r}_o + u\hat{p} + h\hat{N}$ is itself developable, as can be seen by substitution in Eq. (25) with $\vec{r}_o + h\hat{N}$ replacing $\vec{r}_o$, and use of Eq. (29) to evaluate $\hat{N}'$.

The equations for geodesics reduce to a particularly simple form on a developable surface, since they are straight lines on the flattened surface. Equations (12) are equivalent to the equation $K_g = 0$, where $K_g$ is the geodesic curvature of a geodesic curve. The geodesic curvature of the directrix curve is

$$K_g^o = \frac{\hat{n}\cdot\hat{p}}{\sin\phi} = -(a + \phi') \ . \tag{31}$$

The latter equality is easily proved by differentiation of the equality $\hat{t}\cdot\hat{p}$ = cos $\phi$ and use of Eq. (28). But any other curve besides $\vec{r}_0$, including a geodesic, can also be considered to be the directrix curve (the effect of changing directrix curves is to reparameterize the surface). Writing w now for arc length along a geodesic, one has

$$\frac{d\theta}{dw} = -\frac{d\hat{p}}{dw}\cdot\hat{q} = -a\,\frac{ds}{dw} \tag{32}$$

or

$$\frac{d\theta}{ds} = -a \tag{33}$$

where $\theta$ is the angle that the geodesic makes with a ruling; s is the arc length parameter where the ruling intersects the original directrix curve.

On the parallel to the surface of Eq. (24) the directrix curve can be taken to be $\vec{r} = \vec{r}_0 + \hat{N}_0 h$. If a geodesic on the parallel surface is parameterized by s, the arc length along $\vec{r}_0$, one can write for the geodesic

$$\vec{r}_g = \vec{r}_0(s) + h\hat{N}_0(s) + U(s)\ \hat{p}(s) \tag{34}$$

where U(s) is an unknown function of s. Differentiating the above equation with respect to s using Eqs. (28) and (29) yields

$$\vec{r}_g' = \hat{t}_o - \frac{hK_n^o}{\sin\phi}\hat{q}_o + U'\hat{p} + Ua\hat{q} . \tag{35}$$

Taking scalar products with $\hat{p}$ and $\hat{q}$ yields the equation

$$\cos\theta \left(\sin\phi - \frac{hK_n^o}{\sin\phi} + Ua\right) = \sin\theta\,(\cos\phi + U') . \tag{36}$$

With the help of Eq. (33) integration of Eq. (36) is reduced to quadratures and one has

$$U \sin\theta = \int_{s_1}^{s} \sin(\phi - \theta)\, ds' - h \int_{s_1}^{s} \frac{K_n^o \cos\theta}{\sin\phi}\, ds' + u_1 \sin\theta_1 \tag{37}$$

and

$$\theta = \theta_1 - \int_{s_1}^{s} a\, ds . \tag{38}$$

In the above, $s_1$ and $u_1$ are the starting parameter values of the geodesic and $\theta_1$ is its initial angle with respect to $\hat{p}(s_1)$. In practice, Eqs. (37) and (38) are not very useful for determination of geodesic parallels because they do not involve arc length w along the geodesic explicitly. For numerical calculations, as was previously the case with determination of the starting turn curves, it is better to integrate a system of first order differential equations in arc length w. These are

$$\frac{du}{dw} = \cos\theta \tag{39}$$

$$\frac{ds}{dw} = \frac{\sin\theta\cos\theta}{\sin(\phi - \theta) - \left(\frac{hK_n^o}{\sin\phi} - Ua\right)\cos\theta} . \tag{40}$$

For determination of the geodesic parallels of the starting turn in an arbitrary layer, it is necessary to express the initial parameters $s_1$, $u_1$, $\theta_1$ of the geodesics in terms of the starting turn surface. Assuming this surface also to be developable, it must have the parametric form

$$\vec{R}_s = \vec{r}_o + l\hat{s} \ . \tag{41}$$

Introducing the unit binormal vector $\hat{B} = \hat{t} \times \hat{N}$, the most general developable surface with $\vec{r}_o$ as a directrix is given by

$$\hat{s} = [(K_n^o \cot\phi + \psi')\hat{t} + \sin\psi\ (K_g^o \sin\psi + K_n^o \cos\psi)\hat{N} + \cos\psi\ (\sin\psi\ K_g^o + K_n^o \cos\psi)\hat{B}]/[(K_n^o \cot\phi + \psi')^2 + (K_g^o \sin\psi + K_n^o \cos\psi)^2]^{1/2} \tag{42}$$

where $\psi$, the angle between the normals to the two developables along $\vec{r}_o$, is an arbitrary continuous differentiable function of arc length along $\vec{r}_o$. For a surface that intersects the starting layer surface orthogonally, $\psi = \pi/2$, $\psi' = 0$ and Eq. (42) reduces to

$$\hat{s} = \frac{K_n^o \cot\phi\hat{t} + K_g^o\hat{N}}{[K_n^{o2} \cot^2\phi + K_g^{o2}]} \ . \tag{43}$$

In Eq. (43), the quantity $K_n^o \cot\phi$ can be identified as the geodesic torsion $\tau_g^o$ of the directrix curve. Determination of the intersection curves and starting parameters for geodesics is accomplished by the numerical method outlined previously for the general case (Eqs. 5-8).

From the foregoing, it is clear that at least the top and bottom layer surfaces and one side surface (the starting turn surface) can be made to be developable; the resultant fourth surface is, in general, not developable.

DEVELOPABLES THAT ARE DERIVED FROM A GENERAL CURVED SURFACE

A developable can be used to approximate locally a general curved surface. Given a curve on a general surface $\vec{R}(p,q)$, and expressions $p = f(s)$, $q = g(s)$ where s is arc length, the envelope of tangent planes to the general surface along the curve forms a developable surface; the distance between the two surfaces is second order in distance from the curve along the surface. The normal curvature $K_n^o$, geodesic curvature $K_g^o$, and geodesic torsion $\tau_g$ are the same for the tangent plane envelope surface and the original surface along the above curve. If the above quantities are known, the ruling vector $\hat{p}$ for the tangent plane envelope surface can be obtained by substituting $\psi = 0$ into Eq. (42); the result is

$$\hat{p} = \frac{K_n^o \cot\phi \hat{t} + K_n^o \hat{B}}{[\cot^2\phi K_n^{o2} + K_n^2]^{1/2}} \tag{44}$$

or

$$\hat{p} = \frac{\tau_g^o \hat{t} + K_n^o \hat{B}}{[\tau_g^{o2} + K_n^{o2}]^{1/2}}. \tag{45}$$

The quantities $K_n$, $K_g$, and $\tau_g$ for the original surface are given by the well-known expressions

$$K_n = \frac{Lf'^2 + 2\,Mf'g' + Ng'^2}{D^2} \tag{46}$$

$$K_g = \frac{V}{D^3}\,[f'^2\,{}^2\Gamma_{11} - g'^3\,{}^1\Gamma_{22} + (2\,{}^1\Gamma_{12} - {}^1\Gamma_{11})f'^2g'$$
$$- (2\,{}^1\Gamma_{12} - {}^2\Gamma_{22})f'g'^2 + f'g'' - f''g'] \qquad (47)$$

$$\tau_g = \frac{(FL - EM)f'^2 + (GL - EN)f'g' + (GM - FN)g'^2}{VD^2} \qquad (48)$$

with $D = |\vec{R}_1 f' + \vec{R}_2 g'|$. If desired, a more general surface that intersects the original surface at an angle can then be found by use of Eq. (42).

THE RECTIFYING DEVELOPABLE

Another developable that has been considered by magnet system designers[(2)] is the rectifying developable, or envelope of rectifying planes of a space curve. This surface is most conveniently represented in terms of the Frenet-Serret frame $\hat{t}$, $\hat{n}$, $\hat{b}$, where $\hat{t}$ and $\hat{n}$ are the curve tangent and principal normal, and $\hat{b} = \hat{t} \times \hat{n}$ is the binormal. In this frame of reference, the ruling vector for the most general possible developable has the form

$$\hat{p} = \frac{(\psi' + \tau)\hat{t} + \kappa \sin\psi \cos\psi \hat{n} + \kappa \cos^2\psi \hat{b}}{[(\psi' + \tau)^2 + \kappa^2 \cos^2\psi]^{1/2}} \qquad (49)$$

where $\psi$ is now the angle between the normal to the surface and the principal normal to the space curve $\hat{n}$; the new quantity $\tau$ is the torsion of the space curve; it contains three derivatives of the space curve. The choice $\psi = 0$ now yields the ruling vector for the rectifying developable:

$$\hat{p} = \frac{\tau\hat{t} + \kappa\hat{b}}{[\tau^2 + \kappa^2]^{1/2}} \,. \qquad (50)$$

If desired, $\hat{p}$ may be expressed in terms of a curve $p = f(s)$, $q = g(s)$ and a defining surface $\vec{R}(p,q)$ by repeated differentiation and use of the Frenet-Serret formulas; the result contains first, second, and third derivatives of the parameter functions and of the defining surface.

The surface with the ruling vector of Eq. (50) has the remarkable property that the directrix curve and its geodesic parallels are themselves geodesics on the surface and of equal length. They therefore don't tend to slip sideways under winding tension -- a desirable aspect for coil winders. (There will be problems, however; if the principal curvature changes sign.) When the surface is developed or flattened to a plane, the directrix curve and its parallels become straight lines, hence the term "rectifying". Moreover, since the windings are geodesics, their geodesic curvature is zero. This means they do not bend in their tangent plane and that wide, flat ribbon-like conductors can be used to maximize transport current while minimizing bending strains; this feature should be especially useful in winding coils with brittle superconductors such as $Nb_3Sn$.

On the rectifying developable (unlike the general developable) an analytic expression for geodesic parallels can be easily found;[3] it is

$$U(s_0) = \frac{w}{\sin \phi} \; . \tag{51}$$

A parametric formula for geodesic windings on the surfaces parallel to the rectifying developable can also be derived in analytic form; it is

$$U(s_0 h) = \frac{1}{\sin (\phi - \alpha)} \left[ \sin \alpha \int_s^{s_0} (1 + \kappa h) \, ds + \cos \alpha \int_s^{s_0} h\tau \, ds + w \right] . \tag{52}$$

The angle $\alpha$ represents a skewing in the layers required to make the ends of a turn meet and is given by the formula

$$\tan \alpha = - \frac{h \int_0^{L_0} \tau(s)\, ds}{L_0 + h \int_0^{L_0} \kappa(s)\, ds} \tag{53}$$

where $L_0$ is the length of the directrix curve in the starting layer.

FABRICATION OF DEVELOPABLE SURFACES

The process of bending a flat surface into a curved surface is an isometric transformation or isometry; arc lengths and angles between intersecting curves are invariant under the transformation. Another invariant quantity is the geodesic curvature $K_g$; when a developable is flattened, the plane curvature of a flattened curve is just the geodesic curvature of the unflattened curve. This means that the parametric formula for any flattened curve can be obtained from that for the unflattened curve by quadratures. The Cartesian coordinates of a point on the directrix curve are

$$x = x_0 + \int_0^s \cos \psi(s')\, ds' \tag{54}$$

$$y = y_0 + \int_0^s \sin \psi(s')\, ds' \tag{55}$$

$$\psi = \psi_0 - \int_0^s K_g\, ds' = \psi_0 - \phi_0 + \phi + \int_0^s a\, ds' \tag{56}$$

where $x_0$, $y_0$, and $\psi_0$ are constants of integration and are determined by the choice of the fixed coordinate system. The rulings remain straight after the transformation and form an angle of $\psi - \phi$ with the x-axis. The above equations provide all of the information needed to find the Cartesian coordinates of any point on the flattened surface, given parameter values $s_0$ and u for it on the curved surface. If the starting curve is the directrix, the orthogonal geodesics intersect the x-axis at an angle $\psi - \pi/2$. For the surface parallel to the starting developable, also developable, one can take the directrix to be $\vec{r} = \vec{r}_0 + h\hat{N}$, with arc length $s_0$ along $\vec{r}_0$ and length along rulings from $\vec{r}$ to be the surface parameters. The tangent to this directrix makes a different angle $\phi^*$ with the rulings at the same value of $s_0$, the angle being given by the expression

$$\tan \phi^* = \left(1 - \frac{hK_n^0}{\sin^2 \phi}\right) \tan \phi \; . \tag{57}$$

As the rulings are the same except for being displaced in space by the vector $h\hat{N}$, the value of a is the same as in the starting surface for the same value of $s_0$ with the above parameterization of the surface, and Eqs. (54)-(56) can be used with $\phi^*$ substituting for $\phi$ to find the directrix curve on the flattened parallel surface.

The above equations provide the information needed to find the shape of flat pieces for fabrication of developables, given the parameter values for their boundaries. The remaining task is to specify the bend radii. The direction of largest normal curvature on a developable is perpendicular to the rulings, or in the $\hat{q}$ direction, the value being

$$K_{max} = \frac{K_n^0}{\sin \phi \, (\sin \phi + ua)} \; . \tag{58}$$

On the parallel surface parameterized by $s_0$, the arc length along the directrix for the starting surface, the direction of largest normal curvature is still given by $\hat{q}$, but the value is

$$K^*_{max} = \frac{K^o_n}{\sin\phi\left(\sin\phi + ua - \frac{hK^o_n}{\sin\phi}\right)} . \qquad (59)$$

The above specification of bend radii is not entirely satisfactory from a manufacturer's point of view, because producing a radius of curvature that varies continuously both along and in the direction perpendicular to the rulings is difficult to achieve in practice. Instead, developables can be approximated either by plane segments or by a combination of plane pieces and segments of right circular cones. In the latter case, winding surfaces with continuous tangents can be produced. If cone segments are used, plane pieces must be inserted between them in the general case in order to avoid discontinuities in the surface tangents. This is a consequence of the fact that the rulings of a general developable do not necessarily intersect at a point as would be the case for a conoid, but rather are tangent to a space curve (the latter property is an alternative to Eq. (25) for the definition of a developable). A detailed procedure for constructing these approximations to the exact developable surface can be found in Ref. 4.

## CONCLUSIONS

In the foregoing, the mathematical machinery needed to find and specify the detailed shape of twisted coil windings has been outlined. Practical implementation requires development of computer programs to solve the differential equations. Fortunately, all of the differential equations are of the initial value type and solution is expected to be straightforward. Output of such programs would best be combined with graphics packages for easy visualization of the results.

For the case of conductors of finite thickness, the results given are only approximately true. The approximation should be sufficiently accurate for most superconducting coil winding packs, because superconductor strain limitations place limits on bend radii. For copper coils with large winding strains, the layer surfaces are not exactly parallel surfaces (as defined by Eq. 1) and turns in a layer are not exactly geodesic parallels, because the cross-sectional dimensions change when the conductor is bent. The final conductor cross section is not even rectangular and cannot be calculated by analytic means. For such winding packs, a semi-empirical approach in which approximate formulas for cross-sectional dimensions of deformed conductors are found by bending experiments can be contemplated. The approximate formulas would then be used to find layer envelopes and positions of parallel turns by finite methods.

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