Toroidal Effects and Plasma Shifts for a Circular Cross Sectioned Tokamak Using MHD Equilibrium Theory

A.T. Mense

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FUSION TECHNOLOGY INSTITUTE

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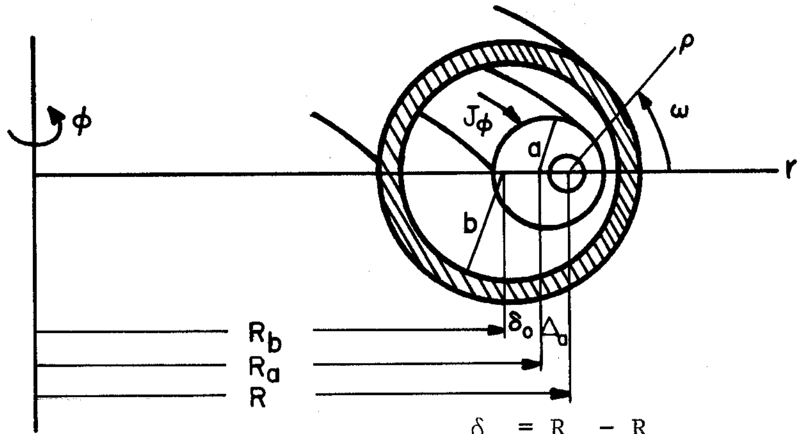
Abstract of Results Directly Applicable to UWMAK-I

With reference to diagram below, the shifts δ_o and



$$\delta_o = R_a - R_b$$

$$\Delta_a = R - R_a$$

Δ_a are found to be given by

$$\Delta_a = \Delta(a) = \int_o^a \frac{\rho'}{R} [\Lambda(\rho') + 1] d\rho'$$

$$\delta_o(t) = \frac{b^2}{2R} \left(\ln \frac{b}{a} + \left(1 - \frac{a^2}{b^2}\right)\left(\Lambda(a) + \frac{1}{2}\right)\right)$$

$$+ \frac{\eta b}{\mu_o R d} \left(\ln 8\frac{R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1 - \frac{B_\perp^e}{B_\perp^o}\right) t$$

$$+ \frac{b^3 \chi}{2\pi R^2} \left(\ln 8\frac{R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1 - \frac{B_\perp^e}{B_\perp^o}\right)\left(1 + \frac{10^{9/2}}{\pi} \sqrt{\frac{\eta t}{hd}}\right)$$

for a circularly cross sectioned chamber and plasma.

Assuming the toroidal (plasma) current $J_\phi = -J_o$ = constant one can find explicit expressions for Δ_a, δ_o depending on pressure profiles.

For a pressure profile which goes as the square root of a parabola

$$p = p_o\left(1 - \frac{\rho^2}{a^2}\right)^{1/2}$$

one finds that

$$\Delta_a = \frac{a^2}{4R}\left[\bar{\beta}_p + \frac{1}{2}\right] = \underline{.75 \text{ meters}} \text{ for } \beta_p = \sqrt{R/a}$$
$$a = 5\text{m.}$$
$$R = 13\text{m.}$$

For this small a Δ_a, we find we are assured that the ψ surfaces are not radically deformed from circular. (If $\bar{\beta}_p = R/a$ $\Delta_a \approx 1.5$m. and then ψ surfaces would still only be slightly non-circular.)

$$\delta(t) = .317 + 1.026(1 - 2.032\, B_\perp^e)t$$
$$+ .160N(1 - 2.032\, B_\perp^e)(1 + 45\sqrt{t}) \text{ meters}$$

where N = # poloidal gaps in shell

$B_\perp^e$ = constant (uniform) vertical field applied at t=0 and kept constant in time from thereon.

The other assumptions concerning these formulae are discussed in the text of this report. By a suitable choice of $B_\perp^e$ as a function of time one can, however, keep the plasma column centered anywhere one wants inside the casing. There is less control however over the positioning of the ψ surfaces inside the plasma.

MHD Equilibrium of a Tokamak Plasma

A. Introduction

The purpose of this paper is not so much to derive and discuss all of the results obtainable from an MHD equilibrium analysis on a Tokamak as it is to establish the "working formulae" one might wish to use to arrive at a first cut, so to speak, on the toroidal effects involved in holding a circular cross sectioned, low β, Tokamak plasma in equilibrium. My development shall follow closely that of V. D. Shafranov as outlined in Reviews of Plasma Physics, Vol. II, pg. 103-150, hereafter referred to as Ref. [1]. I will have occasion to draw from other works of Shafranov, namely;

Ref. [2] Plasma Equilibrium in a Tokamak by V. S. Mukhovatov and V. D. Shafranov Nuc. Fus. 11 (1971) 605-633 (Review Paper).

Ref. [3] Equilibrium of a Toroidal Plasma in a Magnetic Field by V. N. Shafranov, Atomnaya Energiya 13 (1962) 521.

also I will quote from the following excellent Review Paper,

Ref. [4] Tokamak Devices by L. A. Artsimovich, Nuc. Fus. 12 (1972) 215-252.

There are typographical errors in all of the references and I will so note them at the proper places in this paper, hoping, of course, that I do not add to the list of errors myself! While not directly used for this report several other papers are of benefit in reviewing Tokamak equilibria and I shall list them below for quick reference.

Ref. [5] MHD Equilibria in Sharply Curved Axisymmetric Devices by J. D. Callen and R. A. Dory ORNL-TM-3420 (POF)

Ref. [6] Toroidal Containment of a Plasma by Harold Grad, POF 10 (1967) 137-154.

I shall also have need to refer to several texts

Ref. [7] Physics of High Temperature Plasmas, G. Schmidt

Ref. [8] Plasma and Controlled Fusion, Rose and Clark

Ref. [9] Electrodynamics of Continuous Media, Landau and Lifshitz.

I shall limit this paper to circular cross sectioned tori and will have need (as Shafranov does) to use two different labeling schemes to denote the magnetic flux (ψ) surfaces in the device. I shall also initially assume a conducting, circular cross sectioned containing chamber with no gaps. I shall relax this gap constraint in later sections of this paper and discuss its consequences in comparison to an ideal conducting casing with no gaps. My intention is in essence to lead one through, by outline and example, Ref. [1] point out where Shafranov has errors, where he changes coordinate systems, and in some cases where hidden assumptions are present, at least ones I have uncovered. I do this in the hope that one will have a somewhat less arduous task in understanding MHD equilibrium (there are really no texts, monographs, or scrolls on the subject) than I myself have had over the past two years. Of equal importance is the determination of analytical results for one or two possible models of a Tokamak fusion reactor at least to the extent of setting the first calculational "bench marks" on equilibrium parameters for the Wisconsin Fusion Design Study Group.

A future paper will treat roughly this same problem of MHD equilibrium, but from a slightly different point of view. (From the paper "Toroidal Equilibrium," by Johnson, Green, and Wiermer, POF 14 (1971) p. 671-683.)

B. General Remarks

I shall initially be concerned with an ideal conducting ($\sigma \to \infty$) tokamak plasma [App. A]. To stay consistent with [1] I shall define the first coordinate system I am going to use with reference to Figure 1 below.

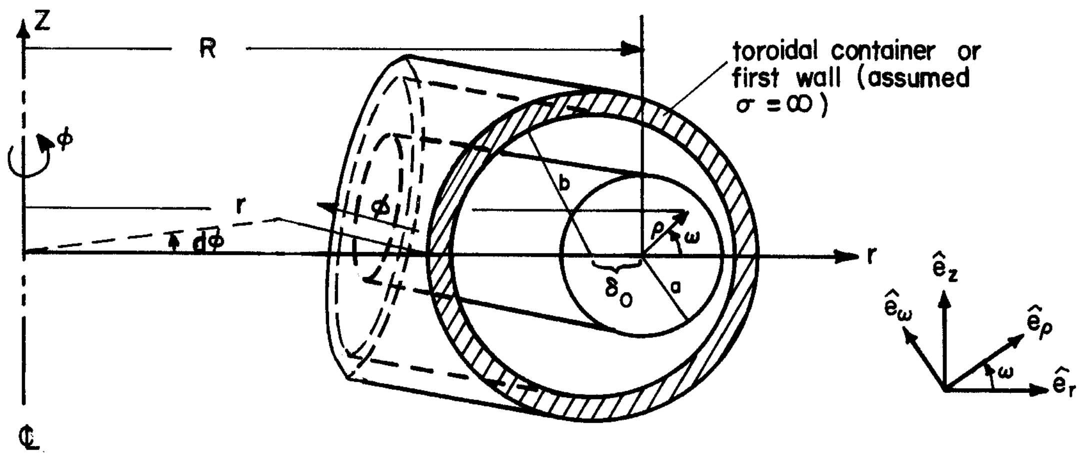


Figure 1

R = radius from ℄ of torus to center of plasma whose circular cross sectioned radius is a.

a = plasma radius

b = radius of casing measured from R - δo

δo = radial shift of plasma cross section (ρ=a) from center of casing. (We shall compute δo)

ρ, ϕ, ω is right-handed (quasi-toroidal) coordinate system

r, z, ϕ is circular cylindrical coordinate system with origin at central axis of machine.

$r = R + \rho \cos \omega$, $z = \rho \sin \omega$, $\phi = \phi$

$\rho = \sqrt{(r-R)^2 + z^2}$, $\omega = \tan^{-1} \frac{z}{r-R}$

The equations needed to give MHD equilibrium for a plasma are listed below.

1) $-\nabla p + \vec{J} \times \vec{B} = 0$ — Eqn. of motion neglecting $\vec{v} \cdot \nabla \vec{v}$, p is isotropic.

2) $\nabla \times \vec{B} = \mu_o \vec{J}$ — Ampere's Law (neglecting displacement current)

3) $\nabla \cdot \vec{B} = 0$ — Gauss's Law

These equations are suitable for describing a plasma in a quasi-stationary electromagnetic field where one may allow relatively slow (diffusion and drift) motions with velocities smaller than the inertial velocities V_{th}, V_A. (i.e. $V << \sqrt{T/m}$, $V << B/\sqrt{mn\mu_o}$).

Equations 1, 2 and 3 may be combined into the following more compact representation. (Ref. 1)

4) $\nabla \cdot \vec{\vec{T}} = 0$ for equilibrium

where

5) $T_{ik} \equiv \{p + \frac{B^2}{2\mu_o}\} (\delta_{ik} - \frac{B_i B_k}{B^2}) + \{p - \frac{B^2}{2\mu_o}\} \frac{B_i B_k}{B^2}$

Since T_{ik} is in tensor form (it is by the way a tensor), I will be concerned with terms such as $T_{\phi\phi}, T_{\omega\omega}, T_{\rho\omega}$, etc. using the ρ, ϕ, ω

system. To find the forces on the plasma, let me consider a section of the plasma as shown in the figure below (geometrical properties are proved in Appendix 1).

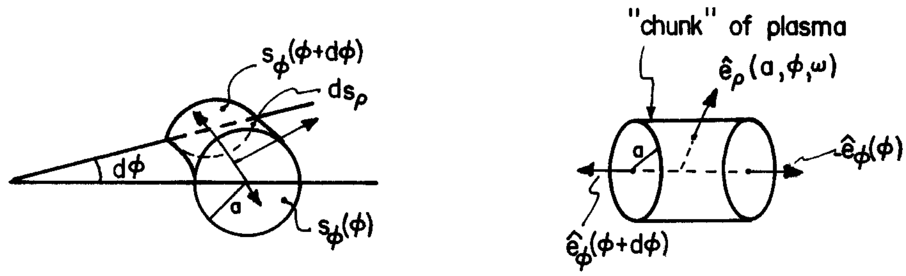


Figure 2

where

$$ds_\phi = \rho d\rho d\omega, \; ds_\rho = (R + \rho\cos\omega)d_\phi\rho d\omega$$

$$\hat{e}_\phi(\phi+d\phi) = \hat{e}_\phi(\phi) + \frac{\partial \hat{e}_\phi}{\partial\phi} d\phi = -\hat{e}_r = -\cos\omega\,\hat{e}_\rho + \sin\omega\,\hat{e}_\omega$$

To keep this slice in equilibrium, one must have the summation of forces on the slice equal to zero. There are two degrees of freedom of interest in an equilibrium analysis. The first is an expansion-contraction of the minor radius of the plasma (see figure below).

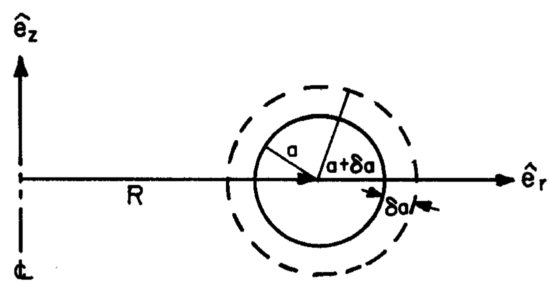


Figure 3

The forces involved in determining the minor radius are the average plasma pressure, toroidal diamagnetic effects, the pinching force due to the plasma current (J_ϕ), and any external particle pressure (p_a). Before I derive any formulae, I should like to discuss the second degree of freedom, i.e. a force balance in the $\hat{e}_r$ direction. This requires a major radius force balance, and one sees that motion with reference to the figure below.

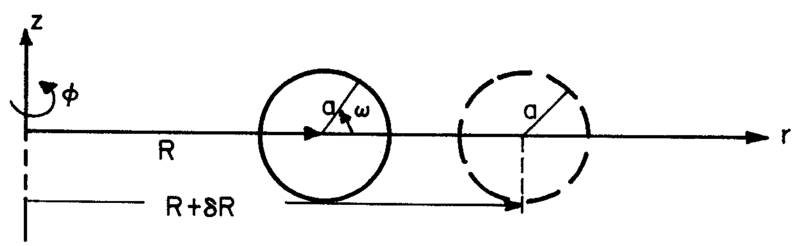


Figure 4

There are basically three distinct physical mechanisms responsible for the plasma wishing to expand in major radius ($R \rightarrow R + \delta R$). They are

1) The plasma "diamagnetic" (J_ω) currents crossed with the toroidal (B_ϕ) field. Since $B_\phi \sim \frac{1}{r}$, it is stronger on the inside of the torus than on the outside, this results in a net force in $\pm\hat{e}_r$ direction. These currents, composed of $\nabla \times M$, (J_ω)g.c. = guiding center current due to drifts, and the component of $J_\parallel$ in the $\hat{e}_\omega$ direction (due to $\nabla \cdot J=0$ and finite $\iota/2\pi$), may produce either a net decrease in toroidal field inside the plasma (diamagnetic) or a net increase (paramagnetic) depending on β poloidal.

2) The plasma conduction (J_ϕ) current crossed with the poloidal (self) field (B_ω). As B_ω is stronger on the inside (R-a) of the plasma loop than on the outside (R+a), there is a net force in the $+\hat{e}_r$ direction. This difference is not only due to the fact that we are

dealing with a ring current, it also comes from having a distorted J_ϕ distribution in an equilibrium plasma configuration. We will see somewhat the extent if this nonuniformity later.

3) The third force is in essence the pressure force (actually the centrifugal force $\frac{mv_\parallel^2}{R}$). This force may most easily be pictured by looking down on a wedge of plasma and noting the $p_\parallel$ has a x and a y component

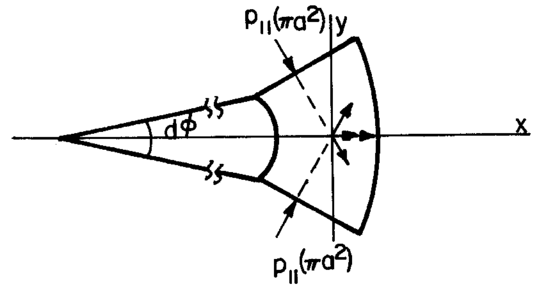


Figure 5

and the y components of the paralled pressure cancel, but the x components add together to produce a net force in the x direction.

To balance these 3 forces, one applies a vertical ($B_{\perp o}$) field such that the plasma current J_ϕ crossed with this B_z field produce an inward $(-\hat{e}_r)$ force on the plasma column. In actuality, we will solve for $B_{\perp o}$ by using this requirement. I now must come up with equations to find these necessary fields to keep the plasma in equilibrium. (See Appendix B for further discussion of this point.)

C. Derivation of Equations

I can find at least one of the equations needed by integrating $\nabla\cdot\vec{\vec{T}}=0$ over the volume of a plasma wedge (as shown below in Figure 6). Doing this one would arrive at a vector equation, each component of this (Force) vector must equal zero for the plasma to be in equilibrium. This is exactly what Shafranov does in Ref. [3].

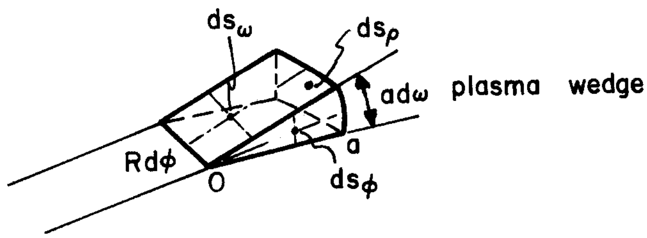


Figure 6

For my purposes, I do not need this general a wedge, I shall be content to use a wedge as shown below in Figure 7.

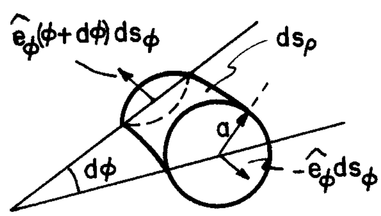


Figure 7

Instead of using all of the components of $\int_v \nabla\cdot\vec{\vec{T}}$, I shall instead use only the projection on the meridonal (z=0) plane. This I get by dotting with $\hat{e}_r$. Doing so gives the force

$$F_r = \hat{e}_r \cdot \int_{v_{wedge}} \nabla \cdot \vec{\vec{T}}\, dv = e_r \cdot \oint_{s_{wedge}} \vec{\vec{T}} \cdot d\vec{s} \qquad (6)$$

This force must of course be zero for equilibrium. The second equation shall be found from using

$$\int_V T_r(\vec{\vec{T}})dv = \int_S (\vec{\vec{T}}\cdot\vec{r})\cdot d\vec{s} \tag{7}$$

which is the Virial Theorem. I shall now describe both in more detail. Writing the equilibrium equation $\nabla\cdot\vec{\vec{T}}=0$ in indicial notation one has*

$$\frac{\partial}{\partial X_i} T_{ik} = 0 \tag{8}$$

where T_{ik} has the meaning that it is the stress (Force/Area) in the k^{th} direction on the face (surface) of a volume element of plasma whose normal is in the i^{th} direction. By integrating over the volume of the element, one obtains the net force in the k^{th} direction on that element.

$$F_K = \int_V \frac{\partial}{\partial X_i} T_{ik} dv = \oint_S T_{ik} ds_i \tag{9}$$

where I have used the divergence theorem and I will integrate over the elemental volume as shown in Figure 7. Now

$$T_{ik} \equiv P_\perp(\delta_{ik} - \frac{B_i B_k}{B^2}) + P_\parallel \frac{B_i B_k}{B^2} \tag{10}$$

where I have made the identification

$$P_\perp = p + B^2/2\mu_o \tag{11}$$

$$P_\parallel = p - B^2/2\mu_o \tag{12}$$

and clearly with this representation one can see that the magnetic field represents a pressure transverse to the field lines (+ sign is (11)), but represents a tension along the field lines (- sign in (12)). Equation 9 may be shown to be equivalent to

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*Many EtM and physics texts take the divergence of a tensor to be $\frac{\partial}{\partial X_i} \cdot T_{ki}$ where the divergence operation is implied over the 2nd index and thus their T_{ik} has just the reverse meaning of my T_{ik}. The convection I use is consistent with tensor calculus texts and also Schmidt's book. Since T_{ik} is symmetric here, it makes no difference but when dealing with antisymmetric tensors it of course does make a difference and one must be careful.

$$\oint_s \{(p + \frac{B^2}{2\mu_o})d\vec{s} - \frac{B(B\cdot ds)}{\mu_o}\} = 0 \qquad (13)$$

By doting this equation with $\hat{e}_r$, one will arrive at the first of the equations I shall need.

$$\frac{F_r}{Rd\phi} = \int_o^a \int_o^{2\pi} (p + \frac{B^2}{2\mu_o} - \frac{B_\phi^2}{\mu_o}) \frac{\rho}{R} d\rho d\omega - \int_o^{2\pi} (p + \frac{B^2}{2\mu_o})(R+\rho\cos\omega)\frac{\rho}{R} \cos\omega d\omega = 0 \qquad (14)$$

Shafranov's Ref. [1] is in error here, his equation (6.3), p. 125 should read (he uses cgs units)

$$d\phi \int (p + \frac{B^2}{8\pi} - \frac{B_\phi^2}{4\pi})ds_\phi = \int (p + \frac{B^2}{8\pi}) \hat{e}_r \cdot d\vec{s}$$

Equation 14 is derived in Appendix C. The second of the equations I wish to use comes from the virial theorem (time independent case)[Schmidt's book p. 74]. Using the vector identity

$$\frac{\partial}{\partial X_i} (X_i T_{ik}) = T_{ik} \delta_{ik} + X_k \frac{\partial T_{ik}}{\partial X_i} \qquad (15)$$

and the equilibrium equation 8, one has after integrating over the entire volume of the plasma torus (not just Figure 7 as was done for equation 14).

$$\int_v T_{ii} dv = \oint_s T_{ik} X_k ds_i \qquad (16)$$

which can be shown (Appendix C) to become

$$\int_v (3p + \frac{B^2}{2\mu_o})dv = \oint\{(p + \frac{B^2}{2\mu_o})\vec{r}\cdot d\vec{s} - \frac{(\vec{B}\cdot\vec{r})(\vec{B}\cdot d\vec{s})}{4\pi}\} \qquad (17)$$

where $\vec{r} \equiv r\,\hat{e}_r + z\,\hat{e}_z = (R\cos\omega+\rho)\hat{e}_\rho - R\sin\omega\,\hat{e}_\omega = R\,\hat{e}_r + \rho\hat{e}_\rho$

Since I will integrate this equation over the entire torus volume, I can take advantage of the fact that the plasma surface (ρ=a) is defined to be the surface containing $\vec{B}$ (and of course $\vec{J}$). Thus $\vec{B}\cdot d\vec{s}=0$ Equation 17 becomes

$$\int_o^a \int_o^{2\pi} \int_o^{2\pi} (3p + \frac{B^2}{2\mu_o})(R + \rho\cos\omega)d\phi\rho d\omega d\rho$$

$$= \int_o^{2\pi} \int_o^{2\pi} (p + \frac{B^2}{2\mu_o})(R + \rho\cos\omega)(R\cos\omega + \rho)\rho d\omega d\phi$$

performing the dϕ integration (all quantities are assumed axisymmetric ∴ they are not functions of ϕ) one has

$$\int_o^a \int_o^{2\pi} (3p + \frac{B^2}{2\mu_o})(R + \rho\cos\omega)\rho d\omega d\rho = \int_o^{2\pi} (p + \frac{B^2}{2\mu_o})(R + p\cos\omega)(R\cos + p)\rho d\omega \quad (18)$$

and this is the second (and last!) of the equations I wish to use. This is Shafranov's (Ref. 1) equation 6.1, page 125.

It is at this point that one must say something about the B fields present (i.e. needed) to have equilibrium. Let me note that for low β plasmas, one can assume the toroidal field will remain essentially 1/r even across the plasma.* On the surface of our circularly cross sectional plasma one can represent $B_\phi(a,\omega)$ as 1/r.

$$B_\phi(a,\omega) = \text{toroidal field on plasma surface}$$

$$= \frac{B_e(a)}{1 + \frac{a}{R}\cos\omega} \approx B_e(a)(1 - \frac{a}{R}\cos\omega) \qquad \frac{a}{R} << 1 \quad (19)$$

*This will be substantiated by calculations at end of report.

where $B_e(a)$ is vacuum toroidal field at $\rho=a$, $\omega=\pi/2$. I will also mention that even inside the plasma one may take $B_\phi(\rho,\omega)$ to have a similar form

$$B_\phi(\rho,\omega) \simeq B_i(\rho)(1 - \frac{\rho}{R}\cos\omega) \quad (20)$$

where any "diamagnetic"(toroidal field) affects are contained in $B_i(\rho)$, whatever it may be. We shall not have need of the explicit formula for $B_i(\rho)$ at this time. I need also to say some words concerning $B_\omega(\rho,\omega)$. To do this, and with reference to Appendix D, I shall choose $B_\omega(\rho,\omega)$ and $B_\rho(\rho,\omega)$ to have analygous functional forms to those fields one finds close to the loop of a circular filamentary current loop (as in Jackson, Chap. 5, p. 142). If one does an asymptotic expansion of the Elliptic integrals involved, one can show that the fields close to a filamentary loop are given by

$$B_\omega \sim B_\omega^\circ(\rho)(1 - \Lambda_\omega^{(\rho)}\frac{\rho}{R}\cos\omega),\ B_\rho \sim -B_\rho\Lambda_\rho\frac{\rho}{R}\sin\omega \quad (21)$$

$$\text{where} \quad \Lambda_\omega^{(\rho)} = \frac{1}{2}\ln\frac{8R_o}{\rho}, \quad \Lambda_\rho^{(\rho)}\ \frac{3}{2}(\ln\frac{8R}{\rho} - 1). \quad (22)$$

Therefore, I choose, as does Shafranov (but he does so without explanation) to take

$$B_\omega(\rho,\omega) \approx B_\omega^\circ(\rho)\ (1 + \Lambda(\rho)\ \frac{\rho}{R}\cos\omega) \quad (23)$$

where at $\rho=a$ I identify (Shaf. eqn. (6.7))

$$B_\omega(a,\omega) = B_a(1 + \Lambda_a\frac{a}{R}\cos\omega) \quad (24)$$

$$\text{where} \quad B_a \equiv \frac{\mu_o I_p}{2\pi a},\ I_p \equiv \int_o J_\phi^\circ(\rho')2\pi\rho' d\rho' \quad (25)$$

Thus one may presume, for low β plasmas, that

$$B^\circ_\omega(\rho) = \frac{\mu_o \int_o^\rho J^\circ_\phi(\rho')2\pi\rho' d\rho'}{2\pi\rho}$$ is the correct extrapolation for $\rho<a$.

I should point out that this form of B_ω in (23) must contain, through Λ, the fields from both the plasma and any external control coils which produce the vertical equilibrium field. The basic limitation I have applied to this time are that the plasma have a circular cross section and that $\frac{a}{R} << 1$. With (19), (20), (23) and (24) one can perform the integrations in (14) and (18). From these two equations I shall (1) ascertain the form of $\Lambda(a)$ and (2) obtain the minor radius pressure balance. Proceeding, one can find that the Virial Theorem (Equation 18) applied to the entire torus ($\rho=a$) becomes

$$3(\bar{p} - p_a) = 3\frac{B_a^2}{2\mu_o} + \frac{B_e^2 - \overline{B_i^2}}{2\mu_o} - \frac{\overline{B_\omega^2}}{2\mu_o} + 2\Lambda(a)\frac{B_a^2}{2\mu_o} \tag{26}$$

The meridonal component of the equilibrium equation (applied to wedge of Figure 7) becomes [when integrated, and noting that $B_\rho^2 \sim \mathcal{O}((\rho/R)^2)$] to $\mathcal{O}((\rho/R))$,

$$\bar{p} - p_a = \frac{B_a^2}{2\mu_o} - \frac{B_e^2 - \overline{B_i^2}}{2\mu_o} - \frac{\overline{B_\omega^2}}{2\mu_o} + 2\Lambda(a)\frac{B_a^2}{2\mu_o} \tag{27}$$

In deriving (26) and (27) I have made use of the following definitions

$$\bar{g} \equiv \frac{1}{\pi a^2}\int_0^a \int_0^{2\pi} g(\rho,\omega)\rho d\omega d\rho$$ = average value of any dynamical quantity g over volume of plasma

p_a = kinetic pressure at $\rho = a$, usually set equal to zero in most analysis.

For example,

$$\frac{\overline{B_i^2}}{2\mu_o} = \text{average toroidal magnetic field internal to plasma}$$

so quite obviously $B_e^2 - \overline{B_i^2}$ is a measure of the diamagnetic (or paramagnetic) effects in the plasma.

Subtracting (27) from (26) produces the minor radius pressure balance

$$\boxed{\bar{p} + \frac{\overline{B_i^2}}{2\mu_o} = p_a + \frac{B_e^2}{2\mu_o} + \frac{B_a^2}{2\mu_o}} \; . \qquad (28)$$

This equation may be rewritten in a handier form by using $\bar{\beta}_p \equiv 2\mu_o\bar{p}/B_a^2$ = Beta Poloidal, (28) becomes

$$\bar{\beta}_p = 1 + \frac{B_e^2 - \overline{B_i^2}}{B_a^2} \quad (\text{taking } p_a = 0) \qquad (29)$$

$\bar{\beta}_p > 1 \Longrightarrow B_e^2 > \overline{B_i^2} \Longrightarrow$ Net diamagnetic J_ω currents in plasma

$\bar{\beta}_p < 1 \Longrightarrow B_e^2 < \overline{B_i^2} \Longrightarrow$ Net paramagnetic J_ω currents in plasma.

Adding (26) and (27) produces an equation for $\Lambda(a)$, remembering that in $\Lambda(a)$ one has the first order contributions to B_ω from both the plasma current J_ϕ and any needed external fields.

$$\boxed{\Lambda(a) = \frac{2\mu_o(\bar{p} - p_a)}{B_a^2} + \frac{l_i}{2} - 1} \qquad (30)$$

where

$$l_i \equiv \frac{\overline{B_\omega^2}}{B_a^2} = \frac{\frac{1}{\pi a^2}\int_0^a\int_0^{2\pi}[B_\omega(\rho,\omega)]^2\rho d\omega d\rho}{B_a^2} = \frac{4\pi}{\mu_o} \text{ x } \left\{\begin{matrix}\text{internal self-}\\ \text{inductance of}\\ \text{plasma per unit}\\ \text{length of torus}\end{matrix}\right\} \qquad (31)$$

Thus, the internal self inductance of the entire plasma is

$$L_i = 2\pi R \frac{\mu_o}{4\pi} l_i = \frac{\mu_o R}{2} l_i \tag{32}$$

It is through l_i that the current distribution inside the plasma enters the equilibrium calculation. This I note is true only for $(a/R) \ll 1$ type of expansion.

As one knows for J_ϕ = constant across circular conductor for $a/R \ll 1$ one has $l_i = 1/2$ [easily computed from (31)] and thus $L_i = \mu_o R/4$ which agrees with the texts [Smythe, Chap. 8, p. 340]. One does not as yet know, and cannot in fact find out from the type of analysis so far used, how much of

$$B_\omega = B_a(1 + \frac{a}{R} \Lambda(a)\cos\omega)$$

is due to the plasma and how much is due to external maintaining field i.e. vertical equilibrium field ($B_\perp$). We shall determine the value of $B_\perp$ from a slightly different analysis and as such will find it to be $\propto \frac{a}{R} B_a$. This I will do in a moment, first let me digress briefly and discuss what is meant by magnetic surfaces or "ψ surfaces" as they are called, and for that matter how one might try to define exactly how to denote the "edge" of the plasma $\rho=a$. To do this, one must remember that the essence of what I am doing rests upon there being a region of low electrical conductivity between the plasma and the wall in which the current density J_ϕ may be taken to be zero. If this region exists then one can give a more physical (but not necessarily unique) meaning to the words "plasma surface" , "plasma boundary," or "plasma radius" in the case of a circular cross sectioned plasma. My forcing the cross section to be circular will be reflected in the requirement that the external vertical maintaining field which is used to balance the three expansion terms discussed earlier to be exactly vertical (B_z only) to 1st order in a/R. One might think that we are putting the cart before the horse in dictating plasma shape but the philosophy for doing this is as follows. One can follow one of two options, [Ref. 3]

Option 1: If one specifies the geometric configuration of the plasma than solving the set of equations (1) - (3) will dictate the required fields and, therefore, currents at infinity (or at external conductors).

Option 2: If one specifies the fields (or currents) at boundaries (or external coils) then the shape of the plasma becomes determined, at least formally, by solving equations (1) - (3).

In practice, it is obviously more convenient to use Option 1 and assume the cross section of the pinch in the r-z plane while using the coordinate system most convenient for the solution of the problem. To solve the system of equations (1) - (3) for equilibria one finds it convenient to exploit the axisymmetry of the magnetic field configuration* and in so doing I will describe what is meant by ψ and ψ surfaces.

*One must depend heavily on axisymmetry to obtain mathematically exact equilibria, for discussion of fine points see H. Grad. Ref. 6.

D. Determining an Equation for ψ

Consider the equilibrium equations

$$\nabla p = \vec{J} \times \vec{B}$$

$$\nabla \times \vec{B} = \mu_o \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Using the geometrical properties of the quasi-toroidal coordinates we have been using [Appendix A] I note that

$$B_p = B_p \hat{e}_\rho + B_\omega \hat{e}_\omega = \text{poloidal field}$$

$$= \nabla \times \vec{A} = \frac{1}{\rho(R+\rho\cos\omega)} [\frac{\partial}{\partial\phi} (\rho A\omega) - \frac{\partial}{\partial\omega} ((R + \rho\cos\omega)A_\phi)]\hat{e}_\rho$$

$$+ \frac{1}{R+\rho\cos\omega} [\frac{\partial}{\partial\rho} ((R + \rho\cos\omega)A_\phi) - \frac{\partial}{\partial\phi} A_\rho]\hat{e}_\omega \qquad (33)$$

and I shall choose $A_\rho = A_\omega = 0$ as the most convenient mathematical formulation. Thus remembering $r = R + \rho\cos\omega$

$$B_\rho = \frac{-1}{r} \frac{1}{\rho} \frac{\partial}{\partial\omega} (rA\phi) \qquad (34)$$

and

$$B_\omega = \frac{1}{r} \frac{\partial}{\partial\rho}(rA\phi). \qquad (35)$$

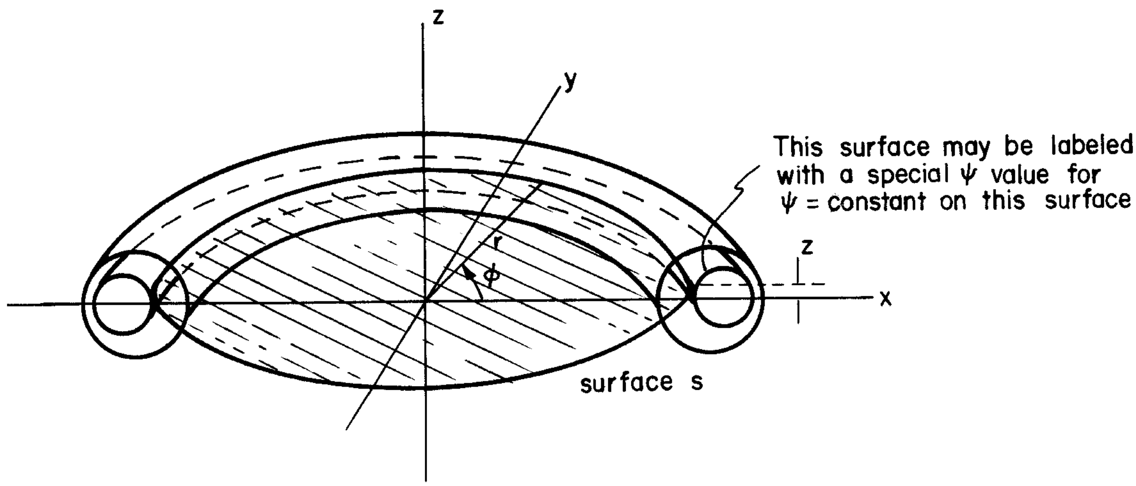
Let me now define ψ.

$$\psi(r,z) \equiv 2\pi r\, A_\phi(r,z) = \psi(\rho,\omega) = 2\pi r\, A_\phi(\rho,\omega) \qquad (36)$$

where one notes that $\oint A_\phi r d\phi = \int_0^r (\nabla x A_\phi) \cdot \vec{ds} = \int_0^{2\pi} \int_0^r B_z r d\phi dr$

which is the flux of magnetic field through a disk of radius r. Thus, <u>$\psi(r,z)$ has been "normalized" to have the physical property of being the</u>

"poloidal" flux through this disk which is parallel to the x-y plane a height z above it and has radius r. This is not a unique definition of ψ however.



One has

$$B_\rho = -\frac{1}{2\pi r}\frac{1}{\rho}\frac{\partial\psi}{\partial\omega}\ , \quad B_\omega = \frac{1}{2\pi r}\frac{\partial\psi}{\partial\rho} \qquad (37)$$

and thus $\vec{B}_p$ in vector notation becomes

$$\vec{B}_p = \nabla \times \left(\frac{\psi}{2\pi r}\right) = \frac{\nabla\psi \times \hat{e}_\phi}{2\pi r} \qquad (38)$$

where one gets $\nabla \cdot B_p = \nabla \cdot \left(\nabla \times \left(\frac{\psi}{2\pi r}\right)\right) = 0$. From (38) I shall note the property of ψ I am interested in using. i.e.

$$\vec{B}_p \cdot \nabla\psi = \left(\frac{\nabla\psi \times \hat{e}_\phi}{2\pi r}\right) \cdot \nabla\psi = 0 \text{ identically.}$$

This statement implies that if I followed $\vec{B}$ poloidal field lines and looked at how ψ changes along any of these lines, it would not!

$$\vec{B}_p \cdot \nabla\psi = 0 \Rightarrow \psi = \text{constant along } \vec{B}_p \text{ lines.} \qquad (39)$$

This is just the requirement for a "streamline" in fluid mechanics. ψ sufaces therefore contain lines of $\vec{B}_p$. How may I incorporate the toroidal field into this formalism? Well, the best way is to use the axisymmetry property that $\frac{\partial}{\partial\phi} = 0$. Thus,

$$B_\phi e_\phi \cdot \nabla\psi = B_\phi \frac{1}{r}\frac{\partial\psi}{\partial\phi} = 0$$

because by supposition $\psi \neq f(\phi)$.

Thus

$$\vec{B} \cdot \nabla\psi = (B_\phi \hat{e}_\phi + \vec{B}_p) \cdot \nabla\psi = 0 \qquad (40)$$

and so in general, if I follow <u>any $\vec{B}$</u> field line around in a tokamak it will lie on some specific surface of constant ψ. Different values of ψ obviously have different field lines on their surfaces. This is a property which requires exact axisymmetry, thus any asymmetric effects must be handled in some other manner than the one I shall follow here. This may indeed limit the "practical" application of this formalism to real plasma devices which have as part of their construction some asymmetric perturbations.) Going back to the equation $\nabla p = \vec{J} \times \vec{B}$ one sees that

$$\vec{B} \cdot \nabla p = \vec{B} \cdot (\vec{J} \times \vec{B}) = 0 \qquad (41)$$

and

$$\vec{J} \cdot \nabla p = \vec{J} \cdot (\vec{J} \times \vec{B}) = 0 \qquad (42)$$

so that $\vec{B} \cdot \nabla p = \vec{B} \cdot \nabla\psi = 0$ and so one may take

$$p = p(\psi) \qquad (43)$$

Again this is a direct consequence of axisymmetry. One may argue this point even more strongly by taking the $\hat{e}_\phi$ component of the equilibrium equation, and impose axisymmetry on p first and not say anything about ψ. Doing this one has

$$\frac{1}{r}\frac{\partial p}{\partial \phi} = 0 = J_\omega B_\rho - J_\rho B_\omega \tag{44}$$

using

$$B_\rho = -\frac{1}{2\pi r}\frac{1}{\rho}\frac{\partial \psi}{\partial \omega}, \quad B_\omega = \frac{1}{2\pi r}\frac{\partial \psi}{\partial \rho}$$

and Maxwell's equation

$$\mu_o J_\omega = \frac{1}{r}\left[\frac{\partial}{\partial \rho}(rB\phi)\right]$$

$$\mu_o J_\rho = -\frac{1}{r}\frac{1}{\rho}\frac{\partial}{\partial \omega}(rB\phi)$$

one obtains from (44) the equation

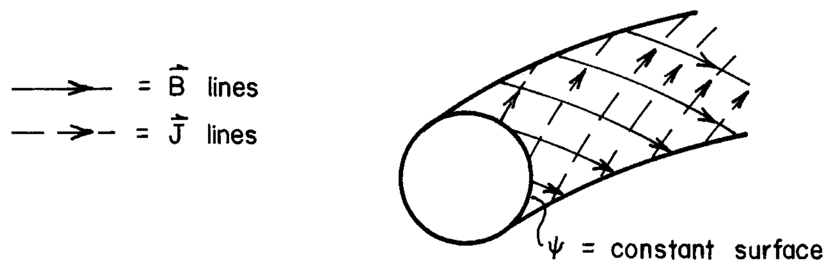
$$\left(\frac{1}{\mu_o r}\frac{\partial}{\partial \rho}(rB\phi)\right)\left(-\frac{1}{2\pi r}\frac{1}{\rho}\frac{\partial \psi}{\partial \omega}\right) - \left(-\frac{1}{\mu_o r}\frac{1}{\rho}\frac{\partial}{\partial \omega}(rB_\phi)\right)\left(\frac{1}{2\pi r}\frac{\partial \psi}{\partial \rho}\right) = 0$$

or

$$\frac{\partial}{\partial \rho}(rB_\phi)\frac{1}{\rho}\frac{\partial \psi}{\partial \omega} - \frac{1}{\rho}\frac{\partial}{\partial \omega}(rB_\phi)\frac{\partial \psi}{\partial \rho} = 0. \tag{45}$$

But the left hand side of (45) is merely the Jacobian of the transformation from $[rB_\phi, \psi]$ space to $[\rho, \omega]$ space and since it is zero this implies that rB_ϕ and ψ are not linearly independent. Thus, $rB_\phi = f(\psi)$. If this $f(\psi)$ = constant then one recovers the vacuum $\frac{1}{r}$ toroidal field. Since axisymmetry lets $\vec{B}$ be a function of ψ and equilibrium force balance [equation (1)] says $\vec{B}$ is a function of p, one can formally solve $\vec{B}(\psi) = \vec{B}(p)$ from $p = p(\psi)$ or if we wished $\psi = \psi(p)$. I shall use the former. We have a specific form for functionally how $\vec{B}_p$ varies with ψ equation (38). We now need one for B_ϕ.

Since $p = p(\psi)$ and according to equation (42) $\vec{J} = \vec{J}(p)$ one has that $\vec{J} = \vec{J}(p(\psi)) = \vec{J}(\psi)$ i.e. $\vec{J}$ lines also lie on surfaces of constant ψ. See Figure 9 below



Define $I(\psi)$ to be an integral over the same surface as in Figure 8

$$I(\psi) = \int_s \vec{J}\cdot d\vec{s} \quad , \; d\vec{s} = ds\, \hat{e}_z$$

Now from Maxwell's equation one has

$$\mu_o J_z = \frac{1}{r} \frac{\partial}{\partial r} (rB\phi)$$

$$\int d(rB_\phi) = \frac{\mu_o}{2\pi} \int J_z 2\pi r dr = \frac{\mu_o}{2\pi} I(\psi)$$

$$B_\phi = \frac{\mu_o I(\psi)}{2\pi r} \qquad (46) \qquad \text{and repeating } \vec{B}_p \qquad \vec{B}_p = \frac{\nabla\psi \times \hat{e}_\phi}{2\pi r} \quad .$$

Using (46) and (38) for B_ϕ and $\vec{B}_p$ one can find an equation which one must solve to find $\psi(r,z)$ (or $\psi(\rho,\omega)$ if you prefer).

E. Finding Equation for ψ

Using circular cylindrical coordinates one notes that

$$\nabla \times (B_\phi e_\phi) = \frac{\mu_o}{2\pi}\left[-\frac{1}{r}\frac{\partial I}{\partial z}\hat{e}_r + \frac{1}{r}\frac{\partial I}{\partial z}\hat{e}_z\right]$$

$$= \frac{\mu_o}{2\pi r}\frac{dI}{d\psi}\left[-\frac{\partial\psi}{\partial z}\hat{e}_r + \frac{\partial\psi}{\partial r}\hat{e}_z\right]$$

however we have shown that

$$\vec{B}_p = \frac{\nabla\psi \times \hat{e}_\phi}{2\pi r} = \frac{1}{2\pi r}\left[-\frac{\partial\psi}{\partial z}\hat{e}_r + \frac{\partial\psi}{\partial r}\hat{e}_z\right]$$

therefore

$$\boxed{\nabla \times (B_\phi \hat{e}_\phi) = \mu_o \vec{B}_p \frac{dI}{d\psi}} \qquad (47)$$

$$\nabla \times (B_\phi \hat{e}_\phi + B_p) = \mu_o \vec{J} = \mu_o\left[\frac{\vec{J}\cdot\vec{B}}{B^2}\vec{B} + \vec{J}_\perp\right]$$

$$2\mu_o\left[\frac{\vec{J}\cdot\vec{B}}{B^2}\vec{B} + \left\{\frac{\vec{B}_p \times \nabla\psi}{B_p^2 + B_\phi^2} + \frac{B_\phi\hat{e}_\phi \times \nabla\psi}{B_p^2 + B_\phi^2}\right\}\frac{dp}{d\psi}\right] \qquad (48)$$

where I have made use of

$$\vec{B} \times \nabla p = \vec{B} \times (\vec{J} \times \vec{B}) = B^2\vec{J} - (\vec{J}\cdot\vec{B})\vec{B}$$

$$\vec{J} = \frac{\vec{J}\cdot\vec{B}}{B^2} B + \frac{\vec{B} \times \nabla p}{B^2} = \vec{J}_\parallel + \vec{J}_\perp .$$

Using (47) and (48) one has

$$\mu_o \vec{B}_p \frac{dI}{d\psi} + \nabla \times \vec{B}_p = \mu_o [\alpha \vec{B} + \left\{ \frac{\vec{B}_p \times \nabla\psi}{B_p^2 + B_\phi^2} + \frac{B_\phi \hat{e}_\phi \times \nabla\psi}{B_p^2 + B_\phi^2} \right\} \frac{dp}{d\psi}] \quad (49)$$

Using the following identities

$$\mu_o B_p \frac{dI}{d\psi} = \mu_o \frac{dI}{d\psi} \frac{\nabla\psi \times \hat{e}_\phi}{2\pi r}$$

$$\nabla \times \vec{B}_p = \nabla \times \left(\frac{\nabla\psi \times \hat{e}_\phi}{2\pi r}\right) = - \frac{1}{2\pi r} \quad \frac{\partial^2\psi}{\partial z^2} + r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial\psi}{\partial r}\right)]\hat{e}_\phi$$

$$\vec{B}_p \times \nabla\psi = \frac{(\nabla\psi \times \hat{e}_\phi)}{2\pi r} \times \nabla\psi = \frac{(\nabla\psi)^2 \hat{e}_\phi}{2\pi r} - \frac{\overbrace{(\hat{e}_\phi \cdot \nabla\psi)}^{0}\nabla\psi}{2\pi r}$$

$$= \frac{B_p |\nabla\psi| \hat{e}_\phi}{2\pi r}$$

$$B_\phi e_\phi \times \nabla\psi = -2\pi r \vec{B}_p B_\phi$$

equation (49) becomes, with $\alpha \equiv \frac{\vec{J} \cdot \vec{B}}{B^2}$

$$\mu_o \frac{dI}{d\psi} \vec{B}_p - \frac{1}{2\pi r} \left[\frac{\partial^2\psi}{\partial z^2} + r \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial\psi}{\partial r}\right)\right]\hat{e}_\phi = \alpha\mu_o \vec{B} + \frac{\mu_o p'}{B^2} [B_p |\nabla\psi| \hat{e}_\phi - 2\pi r \vec{B}_p B\phi]$$

separating into vector components one has

$$\mu_o \frac{dI}{d\psi} B_p = \alpha\mu_o \vec{B}_p + \frac{\mu_o p'}{B^2} (-2\pi r \vec{B}_p B_\phi)$$

and

$$- \frac{1}{2\pi r} \left[\frac{\partial^2\psi}{\partial z^2} + r \frac{\partial}{\partial r}\left(\frac{1}{r} \frac{\partial\psi}{\partial r}\right)\right] = \alpha\mu_o B\phi + \frac{\mu_o p' B_p}{B^2} |\nabla\psi|$$

simplifying these two equations somewhat, using $|\nabla\psi| = 2\pi r\, B_p$ and multiplying the first equation by B_ϕ one has

$$\mu_o B_\phi \frac{dI}{d\psi} + \frac{\mu_o (2\pi r) B_\phi^2}{B^2} \frac{dp}{d\psi} = \mu_o B_\phi \frac{(\vec{J} \cdot \vec{B})}{B^2}$$

and

$$\frac{1}{2\pi r}\left[\frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] = -\mu_o \frac{(\vec{J} \cdot \vec{B})}{B^2} B_\phi - \frac{\mu_o\ 2\pi r B_p^2}{B^2}\frac{dp}{d\psi}$$

adding these two equations one has

$$\left[\frac{\partial^2\psi}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] = -\mu_o (2\pi r)^2 \frac{dp}{d\psi} - \mu_o\ 2\pi r\ B_\phi \frac{dI}{d\psi}$$

$$= -2\pi r\ \mu_o \left[2\pi r \frac{dp}{d\psi} + B_\phi \frac{dI}{d\psi}\right]$$

and this is the famous (elliptic) magnetic differential equation which one must eventually solve to find out the pressure distribution, current density, etc. inside** the plasma. This equation can be put into several forms and I shall demonstrate.

$$B_\phi = \frac{\mu_o I(\psi)}{2\pi r}$$

$$\therefore \quad B_\phi \frac{dI}{d\psi} = \frac{\mu_o I(\psi)}{2\pi r}\frac{dI}{d\psi} = \frac{\mu_o}{4\pi r}\frac{dI^2}{d\psi}$$

**One can solve for ψ and $\therefore \vec{B}$ outside the plasma also by setting R.H.S. equal to zero. One could also use the scalar magnetic potential for outside the plasma.

or alternately

$$B_\phi \frac{dI}{d\psi} = B_\phi \frac{d}{d\psi}\left(\frac{2\pi r}{\mu_o} B_\phi\right) = \frac{2\pi r}{2\mu_o} \frac{d}{d\psi} B_\phi^2 .$$

Therefore, one has defining (as do many authors)

$$\Delta^* \equiv \frac{\partial^2}{\partial z^2} + r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) \tag{50}$$

$$\Delta^*\psi = -2\pi r\, \mu_o\, J_\phi \tag{51}$$

$$= -2\pi r\, \mu_o \left[2\pi r \frac{dp}{d\psi} + B_\phi \frac{dI}{d\psi}\right] \tag{51a}$$

$$= -2\pi r\, \mu_o \left[2\pi r \frac{dp}{d\psi} + \frac{\mu_o}{4\pi r} \frac{dI^2}{d\psi}\right] \tag{51b}$$

$$= -2\pi r\, \mu_o \left[2\pi r \frac{dp}{d\psi} + \frac{\pi r}{\mu_o} \frac{dB_\phi^2}{d\psi}\right] \tag{51c}$$

Thus one usually tries to solve

$$\Delta^*\psi = -2\pi r\, \mu_o \left[A(\psi) r + \frac{C(\psi)}{r}\right] \tag{52}$$

$$A(\psi) = 2\pi \frac{dp}{d\psi} \quad \text{and} \quad C(\psi) = \frac{\mu_o}{4\pi} \frac{dI^2}{d\psi}$$

after making an appropriate choice of the functional form of $P(\psi)$and $I(\psi)$. Shafranov has done a great deal with setting $A(\psi) = A =$ constant and $C(\psi) = C =$ constant [ref.] which implies $p = \frac{p_o}{\psi_o}\psi$ and $I^2 = \frac{4\pi}{\mu_o} C = \left(\frac{2\pi r}{\mu_o} B_\phi\right)^2$ which implies $B_\phi = \frac{B_o R_o}{r} \sim \frac{1}{r}$ vacuum toroidal field. This is not then such a bad choice for low β plasmas where there are negligible plasma diamagnetic effects. After choosing a functional form for $A(\psi)$ and $C(\psi)$ one attempts to solve for $\psi(r,z)$. In most practical problems one must resort to the use of a computer. The case of a circular cross sectioned tokamak plasma can be handled analytically [ref. 3), however ψ can only be put into manageable form when $a/R << 1$ so one may do an expansion. The result becomes [ref. 3] for $\rho > a$

$$\psi = -\mu_o R I_p (\ln \frac{8R}{\rho} - 2) + [\frac{\mu_o I_p}{2} (\ln \frac{8R}{\rho} - 1)\rho + \frac{C_1}{\rho} + C_2\rho]\cos\omega \quad (53)$$

where C_1, C_2 are constants which are determined by boundary conditions. $\psi = \psi_p + \psi_e$ where $\psi_e = C_2\rho \cos\omega$ corresponds to the external field produced by sources other than the plasma and ψ_p is the flux outside the plasma due to the plasma currents themselves. Applying the boundary conditions at $\rho = a$ that $\bar{B} \cdot \hat{n}_2 o$ and that $B^i_\omega(a,\omega)=B^e_\omega(a,\omega)=B_a(1 + \Delta(a)\frac{a}{R} \cos\omega)$ one finds that for $\rho \geq a$

$$\psi = -\mu_o R I_p (\ln \frac{8R}{\rho} - 2) + \frac{\mu_o I_p}{2} [\ln\frac{\rho}{a} + (\Lambda + \frac{1}{2})(1 - \frac{a^2}{\rho^2})]\rho\cos\omega \quad (54)$$

where

$$\psi_e = \frac{\mu_o I_p}{2} (\ln \frac{8R}{a} + \Lambda - \frac{1}{2})\rho\cos\omega, \quad \rho\cos\omega = r-R \quad (55)$$

and

$$\psi_p = -\mu_o R I_p (\ln \frac{8R}{\rho} - 2) + \frac{\mu_o I_p}{2} [- \ln\frac{8R}{\rho} + 1 - \frac{a^2}{\rho^2}(\Lambda + \frac{1}{2})]\rho\cos\omega \quad (56)$$

Therefore, the external field necessary to hold the plasma in equilibrium is found from

$$B_r = -\frac{1}{2\pi r}\frac{\partial\psi_e}{\partial z} = 0$$

$$B_z = \frac{1}{2\pi r}\frac{\partial\psi_e}{\partial r} = \frac{\mu_o I_p}{4\pi r} (\ln 8\frac{R}{a} + \Lambda - \frac{1}{2}) \quad (57)$$

to the lowest order in $(\frac{a}{R})$. Note this uniform vertical field has been derived assuming J_ϕ is flowing in the plasma in the $-\hat{e}_\phi$ direction, so that it produces a $B_\omega > 0$.

$$B_{\perp_o} = B_z = \frac{\mu_o I_p}{4\pi r} \left(\ln \frac{8R}{a} + \Lambda - \frac{1}{2}\right)$$

$$= B_a \frac{a}{2R} \left(\ln \frac{8R}{a} + \Lambda - \frac{1}{2}\right) < B_a \qquad (58)$$

Therefore, we can see that $B_\perp^2$ would not show up in a pressure balance taken only to (a/R) order.

F. Partial Summary

Let me summarize what we have accomplished so far. First, we know the fields outside the plasma, and on the plasma surface $\rho = a$, or at least we could find them from ψ. Let me list what we know and will have future use for in this paper.

Toroidal field

$$B_\phi(\rho,\omega) = B_e\left(1 - \frac{\rho}{R}\cos\omega\right) \qquad \rho \geq a \qquad \frac{\rho}{R} << 1$$

$$B_\phi(\rho,\omega) = B_i(\rho)\left(1 - \frac{\rho}{R}\cos\omega\right) \qquad \rho \leq a \qquad \frac{\rho}{R} << 1$$

B_e = toroidal field at $\rho = a$, $\omega = \pi/2$.

Poloidal field

$$\vec{B}_p = \frac{\nabla\psi \times \hat{e}_\phi}{2\pi r}$$

$$\left.\begin{aligned} B_\rho(\rho,\omega) &= -\frac{1}{2\pi r}\frac{1}{\rho}\frac{\partial\psi}{\partial\omega} \\ B_\omega(\rho,\omega) &= \frac{1}{2\pi r}\frac{\partial\psi}{\partial\rho} \end{aligned}\right\} \quad \text{valid for all } \rho$$

For $\rho > a$ and $\rho/R << 1$ one has [assuming A=C=0 in (52)]

$$\psi = \psi_e + \psi_p$$

$$\psi_e = \frac{\mu_o I_p}{2}\left(\ln \frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\rho\cos\omega$$

$$\psi_p = -\mu_o R I_p (\ln \frac{8R}{\rho} - 2) + \frac{\mu_o I_p}{2}[-\ln \frac{8R}{\rho} + 1 - \frac{a^2}{\rho^2}(\Lambda(a) + \frac{1}{2})]$$

we have not as yet found ψ for $\rho < a$.

<u>External vertical field</u>

$$B_{\perp_o} = \frac{a}{R} B_a (\ln 8 \frac{R}{a} + \Lambda(a) - \frac{1}{2}) < B_a \qquad \frac{a}{R} << 1$$

<u>Minor Radius Pressure Balance</u>

$$\overline{\beta_p} = 1 + \frac{B_e^2 - \overline{B_i^2}}{B_a^2} \equiv \frac{2\mu_o \bar{p}}{B_a^2}$$

<u>B_ω on surface of circular plasma column</u>

$$B_\omega(a,\omega) = B_a(1 + \Lambda \frac{a}{R} \cos\omega) \qquad \frac{a}{R} << 1$$

$$B_a \equiv \frac{\mu_o I_p}{2\pi a}$$

<u>Equational form of $\Lambda(a)$</u>

$$\Lambda(a) = \overline{\beta_p} + \frac{li}{2} - 1$$

$$l_i \equiv \frac{\overline{B_\omega^2}}{B_a^2} = \frac{4\pi}{\mu_o} \times \begin{Bmatrix}\text{internal self inductance of torus} \\ \text{per unit length of torus}\end{Bmatrix}$$

<u>Equation for ψ</u>

$$\Delta^*\psi = -2\pi r \mu_o J_\phi(r,z) = -2\pi r \mu_o [A(\psi) r + \frac{C(\psi)}{r}]$$

$$A(\psi) = 2\pi \frac{dp}{d\psi} \quad , \quad C(\psi) = \frac{\mu_o}{4\pi} \frac{d}{d\psi} I^2$$

$p = p(\psi)$ only for axisymetric equilibrium configurations

$I = I(\psi) = \frac{2\pi r}{\mu_o} B_\phi$ = current which produces toroidal field.

$\Delta^* \equiv \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r}$ (not the Laplacian)

G. Finding B from $\nabla\Phi$

Before I begin to study how ψ and therefore p, $\vec{B}$, and $\vec{J}$ varies inside the plasma, it may do some good to exploit two further points. First, one could find the fields for p>a by solving $\nabla^2\Phi = 0$ and letting $\vec{B} = \nabla\Phi$, this is done in [Ref. 3] and is worth at least quoting.

$$B_\rho = \frac{\partial\Phi}{\partial\rho} \quad , \quad B_\omega = \frac{1}{\rho}\frac{\partial\Phi}{\partial\omega} \tag{59}$$

where

$$\Phi = \Phi_o\omega + \Phi_1\sin\omega + \Phi_2\sin 2\omega + \ldots$$

$$\Phi_o \equiv B_a a \tag{60}$$

$$\Phi_1 \equiv \frac{a}{2R} B_a a[(\Lambda(a) + \frac{1}{2})(-\frac{a}{\rho} + \frac{\rho}{a}) + \frac{\rho}{a}(\ln\frac{\rho}{a} - 1)]$$

and Φ_2 will not be needed for our calculations since it appears to $\theta((a/R)^2)$. Performing the required derivatives one has for the $\rho \geq a$ fields which obey the correct continuity conditions

$$B_\omega(\rho,\omega) = B_a\frac{a}{\rho} + B_a\frac{a}{2R}[(1 + \frac{a^2}{\rho^2})(\Lambda(a) + \frac{1}{2}) + \ln\frac{\rho}{a} - 1]\cos\omega \tag{61*}$$

and

$$B_\rho(\rho,\omega) = \frac{a}{2R}B_a[(1 - \frac{a^2}{p^2})(\Lambda(a) + \frac{1}{2}) + \ln\frac{\rho}{a}]\sin\omega \tag{62}$$

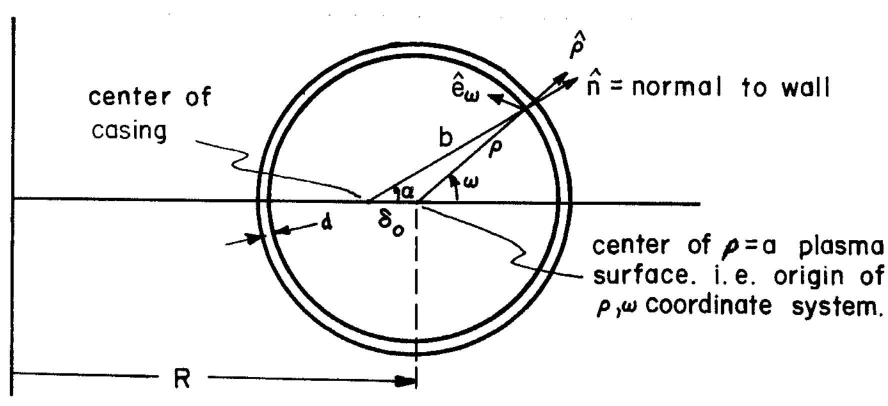
---

*Ref. [4] has an error in this formula for B_ω, formula (3.3), pg. 128 should have $H_\phi(a)\frac{a}{r}$ not $H_\phi(a)\frac{a}{R}$.

It may be the source of some confusion at this point as to why I applied two boundary conditions at (ρ=a) the plasma surface and therefore "used up" my two constants, when I know good and well that I must satisfy $\vec{B} \cdot \hat{n} = 0$ at the inside wall of the casing (since I have assumed it to be $\sigma = \infty$ also). To answer this let me say that when I asked that the entire poloidal field B_ω which is composed of the field from the plasma current plus the equilibrium field due to the image currents on the surface of the perfect conducting wall to be equal to $B_a(1 + \Lambda\frac{a}{R}\cos\omega)$ at $\rho = a$ I have in effect determined where, relative to the center of the casing, the plasma column must reside in order to fill this prescription on B_ω. What remains is to find this shift δ_o of the plasma column center from the center of the casing. This is the next calculation I now wish to make.

H. Computing Shift δ_o

One notes that with just the formulae calculated so far one can compute the shift of the center of He(ρ=a) plasma cross section from the center of the perfectly conducting circular cross-sectioned casing. This shift* δ_o can be computed by using (61) and (62) along with a little geometry. Consider Figure 10 below, for $\delta_o << b$, then



*Ref [1] uses Δ instead of δ_o.

the angle $\alpha \approx \omega$, and $b^2 = p^2 + \delta_o + 2_\rho\, \delta_o \cos\omega$. To first order in $(\frac{\delta_o}{b})$ one has an equation for the wall

$$\rho = b - \delta_o \cos\omega \tag{63}$$

and the unit normal vector to the wall (pointing into casing)

$$\hat{n} = (1, 0, -\frac{\delta_o}{b}\sin\omega). \tag{64}$$

To determine δ_o, I shall require that the normal components of the plasma magnetic field equal zero at the wall i.e. $\vec{B}\cdot\hat{n} = 0$. Keeping only terms that are first order in $(\frac{b}{R})$, $(\frac{a}{R})$, and $(\frac{\delta_o}{b})$ one has

$$B \cdot n = B_\phi^{(1)}(b) - B_\omega^{o}(b)\,\frac{\delta_o}{b}\sin\omega = 0 \tag{65}$$

where the superscripts explicitly show the <u>order</u> of the component envolved ($B_\rho^{(1)}$ = (62), but $B_\omega^{o} = B_a \frac{a}{b}$ only). Solving for δ_o gives

$$\delta_o = \frac{b^2}{2R}\{\ln\frac{b}{a} + (\Lambda(a) + \frac{1}{2})(1 - \frac{a^2}{b^2})\} \geq 0 \tag{66*}$$

Note I have made no approximations as far as the ratio of a/b is concerned. If, in addition to this perfectly conducting casing there was also present a uniform vertical field $(B_\perp)$ perpendicular to the z = 0 (meridonal) plane. The effective boundary condition on the wall $(\rho = b - \delta_o\cos\omega)$ then becomes

$$B_\rho^{(1)} - B_\omega^{o}\,\frac{\delta_o}{b}\sin\omega = B_\perp \sin\omega \tag{67}$$

---

*Ref. [1] has a typographical error here, his formula (6.15), p. 127, has a $\Lambda_{,}$ and it should be Λ.

and thus δ_o becomes (explicitly writing in $\Lambda(a)$)

$$\delta_o = \frac{b^2}{2R}\{\ln\frac{b}{a} + (1 - \frac{a^2}{b^2})(\bar{\beta}_p + \frac{li}{2} - \frac{1}{2})\} - b\,\frac{B_\perp}{B_b} \tag{68}$$

where $B_b \equiv \frac{\mu_o I}{2\pi b}$ and $B_\perp$ is considered positive if its direction is the same as the B_ω field on the outer (r>R) side of the torus. Thus, $B_\perp > 0$ enhances the flux in this outside space between the plasma while weakening it once inside of the torus. This, of course, results in a pushing force on the column towards the center of the casing. This makes δ_o smaller -- as indeed it should!

I. <u>Validity of $\sigma=\infty$ casing</u>

I have also said nothing of the time span over which the approximation of a perfect conducting wall is valid. One can estimate this to be essentially the skin time (τ_s) for the field to penetrate the thickness (d) of the casing which has conductivity σ_ω. Thus for times

$$t << \pi d^2 \mu_o\ \sigma_\omega \equiv \tau_s \tag{69}$$

one can assume the wall is a perfect conductor. (For example: d = 1cm, Cu @ 1000°F, $\tau_s \approx 7$ millisec). If t considerably exceeds this time then the casing does not provide equilibrium. (i.e. the correct $B_\perp$ field at the plasma surface). In this case equilibrium can be maintained if we apply an external vertical field of the value derived in formula (58). The needed field is

$$B_{\perp_o} = \frac{a}{2R} B_a(\ln\frac{8R}{a} + \bar{\beta}_p + \frac{li-3}{2}).$$

This equation is valid for times

$$t >> 2\pi bd\mu\sigma_\omega \equiv \tau_{\frac{L}{R}} = 2\,\frac{b}{d}\,\tau_s \qquad (70)$$

where $\tau_{\frac{L}{R}}$ is an effective L/R time for the casing and to understand the basic difference between τ_s and $\tau_{L/R}$ let me present the following qualitative picture which, due to the lack of a good quantitative (yet simple) example, I cannot defend with rigor. This will be only a short digression.

J. Field Penetration

With the appropriate combination of Maxwell's equation, and the constitutive relation of $\eta\vec{J} = \vec{E}$ one can derive an equation valid for the propagation of $\vec{B}$ in a medium of resistivity η, permeability μ, and permitivity t. The equation in vector form is

$$\nabla \times (\nabla \times \vec{B}) - \frac{\mu}{\eta}\frac{\partial\vec{B}}{\partial t} - \varepsilon\mu\,\frac{\partial^2 B}{\partial t^2} = 0 \qquad (71)$$

where the medium is assumed isotropic, homogeneous, linear, and time independent so that ε, μ, η are all constants. Identical equations for $\vec{E}, \vec{A}$, and ϕ may be derived if the change density is zero. In a vacuum $\eta=\infty$, $\mu=\mu_o$, $\varepsilon=\varepsilon_o$, and one has the wave equation.

The case of very small resistivity is of interest here, and one easily notes that the second term of (71) becomes very important. If the resistivity is zero, as in a perfect conductor then we see the only solution is

$$\frac{\partial\vec{B}}{\partial t} = 0 \qquad (72)$$

for any $\vec{B}$ field that could exist physically inside the perfect conductor.

This is merely a rephrasing of the point that the flux ($\int B \cdot d\vec{A}$) enclosed by a perfect conductor must remain constant. One might then jusk frankly ask over what time scale may one assume this perfect conductivity? There are a couple of possible physical parameters envolved in this determination. First, the plasma frequency (ω_{pe}) characteristic of a metal (Fermi Gas) is approximately $10^{16} sec^{-1}$, where I have assumed $n_e \approx 8.5 \times 10^{28}$ electrons/m^3 which corresponds to copper. (Incidentally, this corresponds to a m.f.p. for an electron at 300 Å assuming T = 300°K, $\nu_f = 1.56 \times 10^8$ cm/sec). Secondly, the collision frequency $\nu = \frac{ne^2}{\sigma m} \approx 4.7 \times 10^{13} sec^{-1}$ which corresponds to a $\tau_{coll.} \approx 2 \times 10^{-14}$ sec. Both of these time scales are extremely short composed to the times of interest in this problem. One may then ask when is it justified to ignore the third term (displacement current) in (71).

Consider

$$a)\ \nabla \times (\nabla \times \vec{B}) \approx \frac{B}{L^2}$$

$$b)\ \frac{\mu}{\eta} \frac{\partial B}{\partial \psi} \approx \frac{\mu}{\eta} \frac{B}{T}$$

$$c)\ \varepsilon\mu \frac{\partial^2 B}{\partial t^2} \approx \frac{1}{C^2} \frac{B}{T^2}$$

$$\frac{c}{a} = \frac{L^2}{C^2 T^2} \quad ; \quad \frac{c}{b} = \frac{\eta\varepsilon}{T} \quad ; \quad \frac{b}{a} = \frac{\mu}{\eta} \frac{L^2}{T}$$

for L ≈ 1cm, for Cu $\eta \approx 5 \times 10^{-8} \Omega$-m

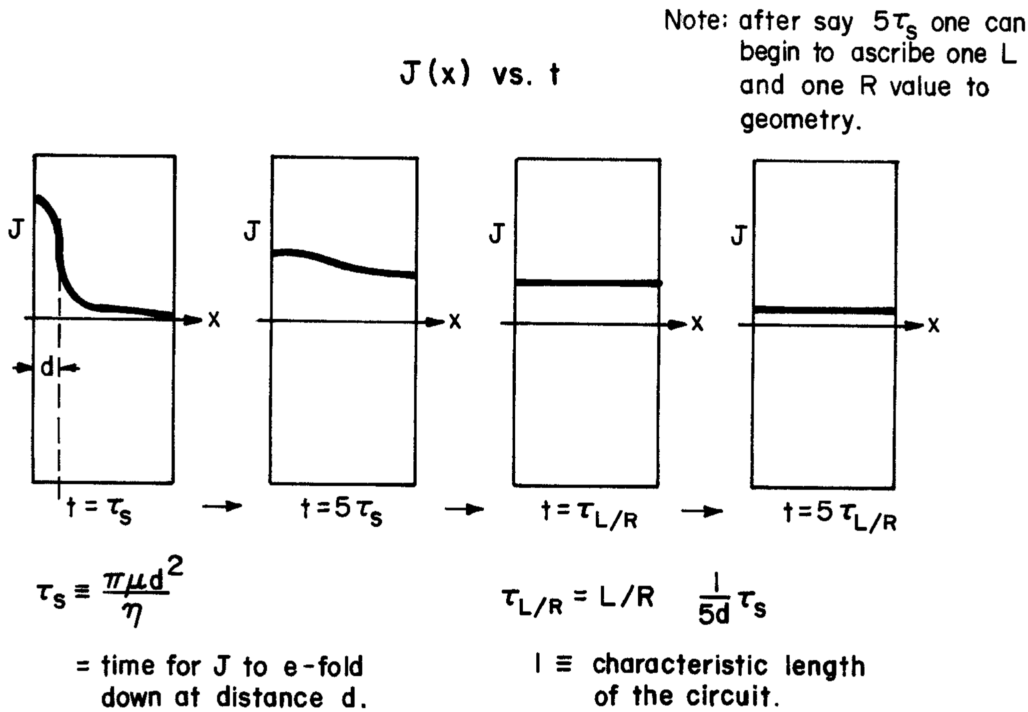
$\frac{c}{a} << 1 = \underline{T >> \frac{1}{3} \times 10^{-10} \text{sec}} =$ For phenomena whose time variations are much slower than 10^{-10} seconds one can ignore displacement current.

$\frac{b}{a} \approx 1 = \underline{T \approx \frac{\mu}{\eta} \approx 20 \text{ sec.}}$

$\frac{c}{b} << 1 = \underline{T >> \eta t \approx 10^{-20} \text{ sec.}}$ independent of scale lengths L.

Thus for times longer than 10^{-10} seconds, (say a long microsecond !) one is justified in ignoring the effects of the third term. I might point out that by ignoring $\frac{\partial^2 B}{\partial t^2}$ I have changed my equations from a hyperbolic to a parabolic partial differential equation. I have not concerned myself, however with what effects one then might have in trying to solve a boundary value problem where one wishes to match a diffusion equation solution to a wave equation (say at a metal vacuum interface). I personally believe that one would have to keep the third term if one wished to do this type of solution fitting at a boundary.

Thus, for the first and second terms to be equivalent, one has $\frac{\mu}{\eta} \frac{L^2}{T} \approx 1$, where $T \approx \frac{\mu L^2}{\eta} \approx \tau_s$ which is the so called "skin time" for field (and current) penetration into a conducting medium to a depth $\approx$ L. After several skin times, the current has diffused fairly well through the conductor. One now has (provided there are no gaps in my casing) a complete circuit around the torus which may be characterized by some lumped parameters such as L and R. The time scale over which the entire current $I = \int_0^d J \cdot d\vec{A}$ decays in then $\tau_{L/R} \approx L/R$ and should in fact represent the smallest (time) eigenvalue to the diffusion equation problem solved in its generality for the particular geometry of interest. [See Landau & Lifshitz, "Electrodynamics of Continuous Media," Chap. VII p. 188-201]. Thus with reference to the figures below, one may qualitatively see the effects of field diffusion for the two time scales, τ and $\tau_{L/R}$.



It would be pedagodically beneficial if one could come up with a simple analytical example to illustrate these two time scales. I have been unsuccessful in my attempts to do so! With the previous ideas in mind, then we can return to the problem of containing a toroidal plasma in a resistive casing with no gaps.

K. Plasma Shift for $\tau_{L/R} > t > \tau_s$

Since unfortunately most present day experiments last for times between τ_s and $\tau_{L/R}$ one must go to a little more trouble to find $\delta(\tau)$, the shift of the $(\rho 2\alpha)$ plasma center from center of casing. To find

$\delta(t)$ one must solve the following equation [Ref. 4]

$$\frac{d}{dt}\{I(t)[\delta(t) - \delta_o]\} = \frac{\eta b I(t)}{\mu_o R d}\left(\ln\frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1 - \frac{B_\perp(t)}{B_{\perp_o}}\right) \quad (73)$$

$$= \frac{b^2}{\tau_{L/R}}\,\frac{I(t)}{2R}\left(\ln\frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1 - \frac{B_\perp(t)}{B_{\perp_o}}\right)$$

where

$I(t)$ = plasma (toroidal) current

$\tau_{L/R} = \mu_o bd/2\eta = (b/2\pi d)\tau_s$

b = radius of containing wall

$B_{\perp_0}$ = field computed from (58)

δ_o = shift for perfect conducting casing, formula (68)

$B_\perp(t)$ = any additional vertical field acting on the plasma loop at instant t.

B(t), whether programmed from external sources or not can usually be measured as a function of t as can I(t), Therefore, in principle, one can find $\delta(t)$. The effects of both transverse (poloidal) gaps and longitudinal (toroidal) gaps can markedly affect $\delta(t)$. These effects will be discussed later.

L. Computation of Magnetic surfaces inside (ρ=a) plasma

A consideration of equilibrium conditions would not be complete without considering the effects of the plasma pressure and current distritution on the positioning of the magnetic (ψ) surfaces in the plasma. Up to this time I have only delt with fields outside the plasma and the shift δ_0 of the $\rho = a$ plasma surface from the center of the casing. In order to discuss the shift of the ψ surfaces inside the plasma it will be convenient to shift my coordinate system from that shown in figure #1, where the origin of the (ρ,θ,ω) system was at the center of the (ρ=a) plasma surface, to the set of coordinates shown below (Fig. 12) where the origin of the ρ,θ,ω coordinate system is the magnetic axis not the center of the plasma.

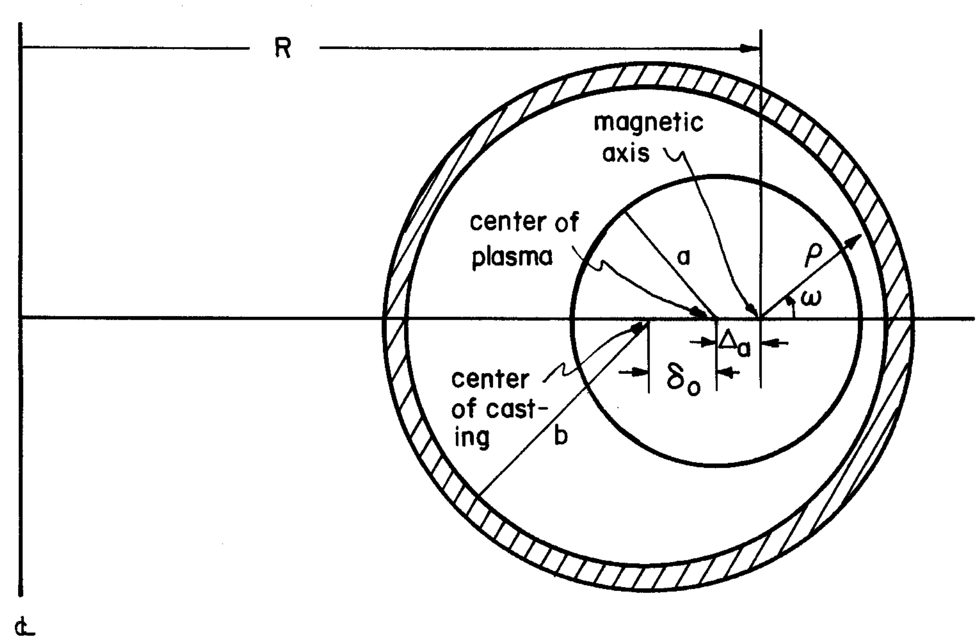


Figure #12

R= radius from ℄ of torus to magnetic axis (not to be confused with R used previously which was from ℄ to center of ρ = a surface)

a = radius of plasma surface

δ_o = shift of center of plasma surface from center of container

Δ_a = shift of center of plasma surface from magnetic axis.

(ρ,θ,ω) = quasi-toroidal coordinate system whose orgin is magnetic axis (B_ω=0 at magnetic axis.)

In this coordinate system the equation for a magnetic surface of circular cross sectional radius ρ_o vs given by

$$\rho = \rho o - \Delta(\rho o)\cos\omega \qquad (74*$$

Thus the equation of the plasma surface (ρ_o = a) is $\rho = a-\Delta_a\cos\omega$. One must now compute $\Delta(\rho_o)$. Definity $B_w^{(1)}(\rho,w)$ to the first order, $\theta(\rho/R)$, toroidal correction to the poloidal field in our coordinate system, then one can find what the toroidal correction to the azimuthal field on the magnetic surface ($\rho=\rho o - \Delta\cos\omega$) should be. Call this correction δB_ω.

$$\delta B_\omega \equiv B_\omega^{(1)}(\rho,\omega) + \frac{dB^o_\omega}{dp}\delta\rho \approx B_\omega^{(1)}(\rho o,\omega) + \frac{dB^o_\omega}{dp}\Big|_{\rho o}\delta\rho \qquad (75$$

where $\delta\rho \equiv \rho-\rho o = -\Delta(\rho o)\cos\omega$, $B^o_\omega(\rho) \equiv B_a\frac{\rho}{a}$ (76

$$\delta B_\omega = B_\omega^{(1)}(\rho o,\omega) - \Delta(\rho o)\cos\omega\frac{dB^o_\omega}{dp}\Big|_{\rho o}$$

Since, in equilibrium, this ρo surface must be held in by fields of the same form as the (ρ=a) plasma surface, one knows that on the plasma surface $\rho = \rho o \quad \Delta(\rho o)\cos\omega$

* The next order correction to this formula is $\rho = \rho o - \Delta(\rho o)\cos\omega - \frac{\Delta^2(\rho_o)}{2\rho_o}\sin^2\omega$

$$B_\omega(\rho,\omega) = B^\circ_\omega(\rho)\ \left(1 + \frac{\rho}{R}\,\Lambda(\rho)\ \cos\omega\ \right)$$

$$\approx B^\circ_\omega(\rho o)\ \left(1 + \frac{\rho o}{R}\,\Lambda(\rho o)\cos\omega\right) \qquad \text{assuming } \frac{\Delta}{R} << \frac{\rho o}{R} \tag{78}$$

and $$\Lambda(\rho) \equiv \frac{2\mu o(\overline{p}(\rho)-p(\rho))}{(B^\circ_\omega)^2} + \frac{li(\rho)}{2} - 1$$

Thus $\delta B_\omega = B_\omega(\rho,\omega) - B^\circ_\omega(\rho o) = B^\circ_\omega(\rho o)\ \frac{\rho o}{R}\ \Lambda(\rho o)\cos\omega$ and

therefore using this equation and (77 one finds that to $\theta(\frac{\rho}{R})$

$$B^{(1)}_\omega(\rho,\omega) - B^{(1)}_\omega(\rho o,\omega) = [B^\circ_\omega(\rho o)\ \frac{\rho o}{R}\ \Lambda(\rho o) + \Delta(\rho o)\ \frac{dB^\circ_\omega}{d\rho}\Big|_{\rho o}]\cos\omega$$

$$B^{(1)}_\omega(\rho,\omega) \approx [B^\circ_\omega(\rho)\ \frac{\rho}{R}\ \Lambda(\rho) + \Delta(\rho)\ \frac{dB^\circ_\omega}{d\rho}\ ]\ \cos\ \omega \tag{79}$$

Equation (79 is one of the factors one wishes to find from a MHD equilibrium analysis. Continuing now, I wish to find the toroidal corrections to ψ and ρ in the coordinate system used here. Obviously on magnetic surfaces (ψ = constant and $p = p(\psi)$ only => p = constant) $\delta\psi$ and δp are both zero on the surface. Thus

$$\delta\psi = 0 = \psi^{(1)}(\rho,\omega) + \frac{d\psi^\circ}{d\rho}\ \delta\rho \approx \psi^{(1)}(\rho o,\omega) + \frac{d\psi^\circ}{d\rho}\ \Big|_{\rho o}\ \delta\rho$$

$$\Rightarrow\ \psi^{(1)}(\rho,\omega) \approx\ -\frac{d\psi^\circ}{d\rho}\ \delta\rho = \Delta(\rho)\ \frac{d\psi^\circ}{d\rho}\ \cos\omega \tag{80}$$

since $B_\omega = \frac{1}{2\pi r}\,\frac{\delta\psi}{\delta\rho}$ then $B^\circ_\omega(\rho) = \frac{1}{2\pi R}\,\frac{\delta\psi^\circ}{\delta\rho}$ therefore

$$\boxed{\psi^{(1)}(\rho,\omega) \approx 2\pi R B^\circ_\omega(\rho)\ \Delta(\rho)\cos\omega} \tag{81}$$

and $\delta p = 0 = p^{(1)}(\rho,\omega) + \frac{dp^\circ}{d\rho}\ \delta\rho.$ This gives

$$\boxed{p^{(1)}(\rho,\omega) = \Delta(\rho)\ \frac{dp^\circ}{d\rho}\ \cos\omega} \tag{82}$$

It is to be noted, for example, that one needs only the zero order pressure and fields to compute the toroidal corrections. The magnetic surfaces to zero order are not only circular they are also concentric. Thus one can specify (easily) pressure profiles $P° = po(1-\frac{\rho^2}{a^2})$ and in the zero order profile the magnetic axis is assumed to coincide with the center of all the magnetic surfaces, even $\rho = a$! Some of this will become clearer when I work several examples later. In addition to $B_\omega^{(1)}$, $\psi^{(1)}$, and $p^{(1)}$ one need to know $B_\rho^{(1)}$, $J_\rho^{(1)}$, $J_a^{(1)}$, $B_\phi^{(1)}$, $J_\phi^{(1)}$ and eventually $\Delta(\rho)$. I proceed to find these. Knowledge of $\psi=\psi° +\psi^{(1)}$ allows one to find $B_\rho^{(1)}$ since

$$B_\rho = \frac{-1}{2\pi r} \frac{1}{\rho} \frac{\delta\psi}{\delta\omega}, \quad \frac{\delta\psi°}{\delta\omega} = 0 \quad \text{so that } B\rho = B_\rho^{(1)}(\rho,\omega)$$

and

$$\boxed{B_\rho^{(1)}(\rho,\omega) = -\frac{1}{2\pi R} \frac{1}{\rho} \frac{\delta\psi^{(1)}}{\delta\omega} = \frac{\Delta(\rho)}{\rho} B_\omega^\circ(\rho) \sin\omega} \qquad (83$$

to order $(\frac{\rho}{R})$, where I have used (81 for $\psi^{(1)}$. In an analogous manner to finding $B_\rho^{(1)}$ and $B_\omega^{(1)}$ I can find $J_\rho^{(1)}$ and $J_\omega^{(1)}$. $\vec{J} = \vec{J}(\psi)$ so that to $\theta(\frac{\rho}{R})$, and assuming $\frac{\Delta}{R} << \frac{\rho}{R}$ one has

$$J_\omega(\rho,\omega) = \frac{1}{2\pi r} \frac{\delta I_T}{\delta\rho}$$

$$J_\omega^\circ(\rho) = \frac{1}{2\pi r} \frac{\delta I_{T°}}{\delta\rho}$$

$$\delta J_\omega = J_\omega^{(1)}(\rho,\omega) + \frac{dJ_\omega^\circ}{d\rho}\delta\rho = J_\omega^\circ(\rho o) \frac{\rho o}{R} \Lambda(\rho)\cos\omega$$

$$\Rightarrow \quad \boxed{J_\omega^{(1)}(\rho,\omega) = [J_\omega^\circ(\rho) \frac{\rho}{R} \Lambda(\rho) + \Delta(\rho) \frac{dJ_\omega^\circ}{d\rho}] \cos\omega} \qquad (84$$

and

$$\boxed{J_\rho^{(1)}(\rho,\omega) = \frac{\Delta(\rho)}{\rho} J_\omega^\circ(\rho) \sin\omega} \qquad (85$$

To obtain the corrections to the lonitudinal (toroidal) plasma current dinsity (J_ϕ) and magnetic field (B_ϕ) one must first write the corrections over the plasma surface ($\rho = \rho o - \Delta\cos\omega$).

Since by equation (19 and (20 one has

$$B_\phi = B_i(\rho)\ (1 - \frac{\rho}{R}\cos\omega\ ) \equiv B_\phi^\circ(\rho)(1 - \frac{\rho}{R}\cos\omega)$$

one has

$$\delta B_\phi \equiv B_\phi(\rho,\omega) - B_\phi^\circ(\rho) = - B_\phi^\circ(\rho)\ \frac{\rho}{R}\cos\omega\ . \tag{86}$$

The toroidal correction to J_ϕ can only be obtained from inspections the general form of $J_\phi(\psi)$. From (51 and (52 one has

$$\Delta^*\psi = -2\pi r\ \mu o\ J_\phi = -2\pi r\ \mu o\ [A(\psi)r + \frac{C(\psi)}{r}\ ]$$

which implies

$$J_\phi = A(\psi)\ r + \frac{C(\psi)}{r}\ , \quad A(\psi) = 2\pi\frac{dp}{d\psi},\ C(\psi) = \frac{\mu o}{4\pi}\frac{dI^2}{d\psi}$$

adding and subtracting $\dfrac{R^2A(\psi)}{r}$ one has

$$J_\phi = (\frac{R^2A(\psi) + C(\psi)}{r}) + A(\psi)\ (\ r - \frac{R^2}{r}\ ) \tag{87}$$

letting $r = R + \rho\cos\omega$ one has to first order ρ/R

$$J_\phi(\rho,\omega)\ (\ \frac{R^2A(\psi^\circ) + C\ (\psi^\circ)}{R})(\ 1 - \frac{\rho}{R}\cos\omega\ ) + A\ (\psi^\circ)(2\rho\cos\ \omega)$$

$$\approx J_\phi^\circ\ (\rho)\ (\ 1 - \frac{\rho}{R}\cos\omega) + 2\ \frac{dp^\circ}{d\psi}\ (\ 2\rho\cos\omega) \tag{88}$$

where I have made the identification

$$J_\phi^\circ\ (\rho) \equiv A\ (\psi^\circ)\ R + \frac{C(\psi^\circ)}{R} \tag{89}$$

which as one can easily see logically comes from the form of (52.

Thus using

$$2\pi \frac{dp^\circ}{d\psi} (2\rho \cos\omega) = 2\pi \frac{dp^\circ}{d\rho} (2\rho \cos\omega) \left(\frac{d\psi^\circ}{d\rho} \right)^{-1}$$

$$= 4\pi \rho \cos\omega \frac{dp^\circ}{d\rho} \left(\frac{1}{2\pi R\, B^\circ_\omega(\rho)} \right)$$

$$= \frac{2\rho}{R\, B^\circ_\omega(\rho)} \frac{dp^\circ}{d\rho} \cos\omega \qquad (90*$$

one has

$$\delta J_a(\rho,\omega) = J_\phi(\rho,\omega) - J^\circ_\phi(\rho)$$

$$= -\frac{\rho}{R} J^\circ_\phi(\rho)\cos\omega + \frac{2\rho}{R\, B^\circ_\omega(\rho)} \frac{dp^\circ}{d\rho} \cos\omega$$

$$= \frac{\rho}{R} \left[\frac{2}{B^\circ_\omega(\rho)} \frac{dp^\circ(\rho)}{d\rho} - J_\phi{}^\circ(\rho) \right] \cos\omega \qquad (91$$

Using (86 and (91 one can find that

$$\boxed{B^{(1)}_\phi(\rho,\omega) = \left[- B^\circ_\phi(\rho) \frac{\rho}{R} + \Delta(\rho) \frac{dB^\circ_\phi(\rho)}{d\rho} \right] \cos\omega} \qquad (92**$$

and

$$\boxed{J^{(1)}_\phi(\rho,\omega) = \left\{ \left[\frac{2}{B^\circ_\omega(\rho)} \frac{dp^\circ(\rho)}{d\rho} - J^\circ_\phi(\rho) \right] \frac{\rho}{R} + \Delta(\rho) \frac{dJ^\circ_\phi(\rho)}{d\rho} \right\} \cos\omega} \qquad (93$$

*Eqn. (90 have corresponds to the equation just above (6.28) in Ref.[1] and Ref [1] is in error because it has left out the $\cos\omega$ on the right hand side.

**Error in (6.29) of Ref. [1]. It has $\Delta\rho$ instead of $\Delta(\rho)$.

One should note that all toroidal corrections can be expressed in terms of $\Delta(\rho)$. I shall show directly that $\Delta(\rho)$ can be expressed as an integral over the plasma cross sections of $\Lambda(\rho)$, which is called the "asymmetry coefficient" of the poloidal (azimuthal) field. To see this relationship one must return to the basic definition of $\vec{B}$ interms of $\nabla\psi$.

$$B_\omega = \frac{1}{2\pi r} \frac{\delta\psi}{\delta\rho} = \frac{1}{2\pi r} \left(\frac{\delta\psi^o}{\delta\rho} + \frac{\delta\psi'}{\delta\rho} \right)$$

$$= \frac{1}{2\pi r} \left[2\pi R\, B^\circ_\omega(\rho) + \frac{d}{d\rho} (2\pi R\, B^\circ_\omega(\rho)\; \Delta(\rho)\cos\omega) \right]$$

where I have used (81 for $\psi^{(1)}$. Expanding $\frac{1}{r} \approx 1 - \frac{\rho}{R} \cos\omega$ one has to $\theta(\rho/R)$

$$B^{(1)}_\omega(\rho,\omega) = - B^\circ_\omega(\rho)\, \frac{\rho}{R} \cos\omega + \frac{d}{d\rho} (B^\circ_\omega(\rho)\; \Delta(\rho))\cos\omega \qquad (94$$

but equation (79 gives $B^{(1)}_\omega(\rho,\omega)$ as

$$B^{(1)}_\omega(\rho,\omega) = B^\circ_\omega(\rho)\, \frac{\rho}{R}\, \Lambda(\rho)\cos\omega + \Delta(\rho)\, \frac{dB^\circ_\omega}{d\rho} \cos\omega \qquad (79$$

Comparing these two equations one sees that

$$B^\circ_\omega(\rho)\, \frac{\rho}{R}\, \Lambda(\rho) + \Delta(\rho)\, \frac{dB\omega^\circ}{d\rho} = - B^\circ_\omega(\rho)\, \frac{\rho}{R} + \Delta(\rho)\, \frac{dB^\circ_\omega}{d\rho} + B^\circ_\omega(\rho)\, \frac{d\Delta}{d\rho}$$

and therefore, for $\Lambda(\rho) = \dfrac{2\mu o(\overline{p}(\rho) - p^\circ(\rho))}{(B^\circ_W(\rho))^2} + \dfrac{li(\rho)}{2} - 1$

$$\frac{d\,\Delta}{d\,\rho} = \frac{\rho}{R} \left[\Lambda(\rho) + 1\right]$$

$$\boxed{\Delta(\rho) = \int_o^\rho \frac{\rho'}{R} \left[\Lambda(\rho') + 1 \right] d\rho'}$$

and the distance between the centers of the cross sections of any two ψ surfaces of radii a and b is given by

$$\Delta(a,b) = \int_a^b \frac{\rho}{R} \; [\; \Lambda(\rho) + 1 \;] \; d\rho \qquad (96$$

One can even use this formula to give the displacement of the vacuum surfaces, since $\Lambda=0$ outside the plasma.

To tie all of what I have done together, I shall work completely two examples, the last of which shall demonstrates the types of problems one may encounter in choosing $p(\rho)$ arbitrarily.

M. Example #1, Assume a (Zero order) pressure distribution and current density profile.

$$P^\circ \approx Po\ (1-\frac{\rho^2}{a^2})$$

$$J_\phi^\circ = J_o = \text{constant} = I/\pi a^2$$

(across plasma only)

This choice will make some of the math easier.

Step 1 Compute $\bar{p}_o(\rho)$

$$\bar{p}_o(\rho) \equiv \frac{1}{\pi\rho^2}\int_o^p p^\circ(\rho)\ 2\pi\rho d\rho = \frac{P_o}{\rho^2}\int_o^\rho (1-\frac{\rho^2}{a^2})2\rho d\rho$$

$$= P_o\frac{a^2}{\rho^2}\int_o^{(\rho/a)^2}(1-u)du = p_o[1-1/2(\frac{\rho}{a})^2]$$

$$\bar{p} \equiv \bar{p}(a) = \frac{P_o}{2}$$

$$\underline{\bar{p}_o(\rho) = 2\bar{p}\ [1-1/2(\frac{\rho}{a})^2]}$$

Step 2 Compute $li(\rho)$

$$li(\rho) = \frac{\frac{1}{\pi\rho^2}\int_o^\rho [B_\omega^o(\rho')]^2\ 2\pi\rho' d\rho'}{[B_\omega^\circ(\rho)]^2}$$

$$B_\omega^o\ (\rho') = \frac{\mu_o I}{2\pi a^2}\ \rho'$$

$$li(\rho) = \frac{1}{\pi\rho^4}\left[\int_o^\rho (\rho')^2\ 2\pi\rho' d\rho'\right]$$

$$= \frac{1}{2}\ \text{(independent of } \rho)$$

$li \equiv li(a) = li(\rho) = 1/2$ for this special case!

Step 3 Compute $\Lambda(\rho)$ from

$$\frac{2\mu_o\,(\bar{p}_o(\rho)-p^\circ(\rho))}{[B^o_\omega(\rho)]^2} + \frac{li(\rho)}{2} - 1$$

$$\Lambda(\rho) = \frac{2\mu_o\,[\,2\,\bar{p}(1-1/2(\frac{\rho}{a})^2) - 2\,\bar{p}(1-\frac{\rho^2}{a^2})\,]}{(\frac{\mu_o I}{2\pi a^2})^2\,\rho^2} + 1/2\,(1/2) - 1$$

Using $B_a = \frac{\mu_o I}{2\pi a}$ which is zero order B_ω field for a uniform current at plasma surface, one has

$$\Lambda(\rho) = \frac{2\mu_o}{B_a^2}\,\frac{a^2}{\rho^2}\,2\bar{p}\,[1-1/2(\frac{\rho}{a})^2 - 1 + (\frac{\rho}{a})^2\,] - \frac{3}{4}$$

$$= \frac{2\mu_o\bar{p}}{B_a^2}\,\frac{2a^2}{\rho^2}\,[\,\frac{1}{2}\,\frac{\rho^2}{a^2}\,] - \frac{3}{4}$$

and this gives the very simple result*

$$\Lambda(\rho) = \bar{\bar{\beta}}_p - \frac{3}{4} = \Lambda(a)$$

where $\bar{\bar{\beta}}_p \equiv \bar{\bar{\beta}}_p(a) = \frac{2\mu_o\bar{p}}{B_a^2}$, $B_a = \frac{\mu_o I}{2\pi a}$

Step 4 Compute $\Delta(\rho)$ from (95

$$\Delta(\rho) = \int_0^\rho \frac{\rho'}{R}\,[\bar{\bar{\beta}}_p - \frac{3}{4} + 1]d\rho' = \frac{1}{R}\,[\bar{\bar{\beta}}_p + \frac{1}{4}\,] \int_0^\rho \rho' d\rho'$$

$$\boxed{\Delta(\rho) = \frac{\rho^2}{2R}\,[\bar{\bar{\beta}}_p + \frac{1}{4}\,]}$$

* Ref. [1] has error for $\Lambda(\rho)$. ρ^2 in derominator of eqn. just above (6.3) should not be there.

Step 5   Compute $B_{\perp o}$ from

$$B_{\perp o} = \frac{a}{2R} B_a \left(\ln \frac{8R}{a} + \bar{\beta}_p^- + \frac{li-3}{2}\right) = \frac{a}{2R} B_a \left(\ln \frac{8R}{a} + \bar{\beta}_p^- - 1.25\right)$$

Using $\Delta(\rho)$ now one can complete the calculations for the toroidal corrections. The computations are straight forward and so I shall state just the result and indicate the formulae used.

$$B_\omega^{(1)} = \frac{a}{R} B_a \left[\frac{3}{2} \left(\overline{\beta_p} + \frac{1}{4}\right) - 1 \right] \frac{\rho^2}{a^2} \cos\omega \qquad \text{Formula 79,76}$$

$$B_\omega(\rho,\omega) = B_a \frac{\rho}{a} \left[1 + \frac{\rho}{R} \left[\frac{3}{2} \left(\overline{\beta_p} + \frac{1}{4}\right) - 1\right] \cos \omega \right] \qquad 80$$

$$\psi^{(1)}(\rho,\omega) = B_a \pi\rho^2 \left[\overline{\beta_p} + \frac{1}{4} \right] \left(\frac{\rho}{a}\right) \cos\omega \qquad 81$$

$$\psi^o = 2\pi R \int_o^\rho B_\omega^o(\rho')d\rho' = \pi\rho^2 B_a\left(\frac{R}{a}\right) + \psi_o, \text{ where } \psi_o = \psi \text{ at } \rho = 0,$$

$\psi_o = 0$ for convenience.

$$\underline{\psi = B_a \pi\rho^2 \frac{R}{a} \left[1 + \frac{\rho}{R} \left[\overline{\beta_p} + \frac{1}{4} \right] \cos\omega \right]}$$

$$= \frac{\mu_o}{4\pi} J_o \pi\rho^2 \, 2\pi R \left[1 + \frac{\rho}{R} \left[\overline{\beta_p} + \frac{1}{4} \right] \cos\omega \right]$$

$$P^{(1)}(\rho,\omega) = -\,2\,\frac{\rho}{R}\,\bar{P}\,[\overline{\beta_p} + \frac{1}{4}]\,(\frac{\rho}{a})^2\cos\omega \qquad 82$$

$$= -\,\frac{\rho}{R}\,P_o\,[\,\overline{\beta_p} + \frac{1}{4}\,]\,(\frac{\rho}{a})^2\cos\omega$$

$$P = P_o\,[\,1 - \frac{\rho^2}{a^2}\,(1 + \frac{\rho}{R}\,[\overline{\beta_p} + \frac{1}{4}\,]\cos\omega\,)\,]$$

$$B_\rho = B_\rho^{(1)}(\rho,\omega) = B_a\,\frac{a}{2R}\,[\overline{\beta_p} + \frac{1}{4}\,]\,(\frac{\rho}{a})^2\sin\omega \qquad 83$$

[Ref. [1], has an error for B_ρ, his formula (6.34) does not have $(\frac{\rho}{a})^2$] One must now do a little algebraie juggling to proceed further. Going back to the equilibrium equation

$$\Delta^*\psi = -\,2\pi r\,\mu_o\,J_\phi = -\,2\pi r\mu_o\,[A(\psi)r + \frac{c(\psi)}{r}\,]$$

One notes from formulae (87,(88, and especially (89 that

$$J_\phi^o = A(\psi^\circ)R + \frac{c(\psi^\circ)}{R}$$

where

$$A(\psi^\circ) = 2\pi \frac{dp^\circ}{d\psi} = 2\pi \frac{dp^\circ/d\rho}{d\psi^\circ/d\rho}$$

$$C(\psi^\circ) = \frac{\mu_o}{4\pi} \frac{d}{d\psi} (I_T^\circ)^2 = \frac{\mu_o}{4\pi} (\frac{2\pi}{\mu_o})^2 R^2 \frac{d}{d\psi} (B_\phi^\circ(\rho))^2$$

$$= \frac{\mu o}{4\pi} (\frac{2\pi}{\mu o})^2 R^2 \frac{d(B_\phi^\circ(\rho))^2/d\rho}{d\psi^\circ/d\rho}$$

and we already have assumed $J_\phi^\circ = -J_o = -\frac{I}{\pi a^2}$ = constant. The minus sign is due to fact I wish to have B_ω° due to the plasma current J_ϕ° in the $+\hat{e}_\phi$ direction. Using the formula for $d\psi^\circ/d\rho$, $d\rho^\circ/d\rho$, and $J_\phi^\circ = -J_o = -I/\pi a^2$ one finds

$$\frac{\frac{d}{d\rho}(B_\phi^\circ(\rho))^2}{2\mu_o} = \frac{4\rho}{a^2} \frac{Ba^2}{2\mu_o} [\overline{\beta p} - 1]$$

and using definition $B_e \equiv B_\phi^\circ(a)$, i.e. B_ϕ at $\rho=a$, $\omega=\pi/2$. One has

$$\frac{(B_\phi^\circ(\rho))^2}{2\mu_o} = \frac{B_e^2}{2\mu_o} - 2\frac{B_a^2}{2\mu_o} [\overline{\beta p} - 1] [1 - \frac{\rho^2}{a^2}]$$

and thus

$$B_\phi^\circ(\rho) = B_e [1 - 2 (\frac{B_a}{B_e})^2 [\overline{\beta p} - 1] (1 - \frac{\rho^2}{a^2})]^{1/2}$$

$$\approx B_e (1 - (\frac{Ba}{Be})^2 [\overline{\beta p} - 1] (1 - \frac{\rho^2}{a^2})) \quad \text{for } (\frac{Ba}{Be})^2 << 1$$

Using this formula for $B_\phi^\circ(\rho)$ one can compute $J_\omega^\circ(\rho)$ from its definition

$$J_\omega^\circ = \frac{1}{2\pi r}\frac{d}{d\rho} I_T = \frac{1}{2\pi r}\frac{d}{dp}\left[\frac{2\pi r\, B_\phi^\circ(\rho)}{\mu_o}\right]$$

$$\approx \frac{1}{\mu_o}\frac{dB_\phi^\circ}{dp} \qquad \text{to zero order}$$

$$\approx 2\frac{\rho}{a} J_o \frac{B_a}{B_e}[\overline{\beta p} - 1]$$

$$\frac{dJ\omega^\circ}{dp} \approx \frac{J\omega^\circ(\rho)}{\rho}$$

And now one can proceed to obtain the rest of the toroidal corrections.

$$J_\omega^{(1)}(\rho,\omega) \approx J_\omega^\circ(\rho)\frac{\rho}{R}\left[\frac{3}{2}[\overline{\beta p} + \frac{1}{4}] - 1\right]\cos\omega \qquad 84$$

$$\approx 2\left(\frac{a}{R}\frac{B_a}{B_e}\right)\left(\frac{\rho^2}{a^2}\right) J_o [\overline{\beta p} - 1]\left[\frac{3}{2}(\overline{\beta p} + \frac{1}{4}) - 1\right]\cos\omega$$

∴

$$\underline{J_\omega(\rho,\omega) = 2 J_o \frac{\rho}{a}\frac{B_a}{B_e}[\overline{\beta p} - 1]\left[1 + \frac{\rho}{R}\left[\frac{3}{2}(\overline{\beta p} + \frac{1}{4}) - 1\right]\cos\omega\right]}$$

In most present Tokamak reactors Ba/Be ~ θ (a/R) and so

$J_\omega \sim \theta((a/R)^2)\, I_T$.

$$B_\phi^{(1)}(\rho,\omega) = -B_\phi^\circ(\rho)\frac{\rho}{R}\left[1 - 2\left(\frac{Ba}{Be}\right)^2\left(\frac{\rho}{a}\right)^2[\overline{\beta p} - 1][\overline{\beta p} + \frac{1}{4}]\right] \qquad 92$$

$$\approx B_\phi^\circ(\rho)\frac{\rho}{R} \qquad \text{if } \left(\frac{Ba}{Be}\right)^2 \leqslant \frac{\rho}{R}$$

∴ $\underline{B_\phi(\rho,\omega) \quad B_\phi^\circ(\rho)\left(1 - \frac{\rho}{R}\cos\omega\right)}$ = vacuum toroidal 1/r field.

$$J_\phi^{(1)}(\rho,\omega) = - J_o \; [\; \overline{\beta p} + \frac{\rho}{a}\frac{Ba}{Be} \; [\overline{\beta p} - 1] \;] \; \frac{2\rho}{R} \cos\omega \qquad 93$$

$$\underline{J_\phi(\rho,\omega) = - J_o \; [\; 1 + 2 \frac{\rho}{a} \{\overline{\beta p} + \frac{\rho}{a}\frac{Ba}{Be} \; [\overline{\beta p} - 1 \;]\} \cos\omega]}$$

Note again minus sign denotes J_ϕ in $-\hat{e}_\phi$ direction.

Now that I have computed all of the first order (ρ/R) toroidal corrections for one model I shall proceed to do the same except with a more realistic (?) set of pressure and current profiles. I will only give final results of this calculation, and leave the verification as "labor of love" for the interested reader.

Example #2, (UWMAK-1)

$$P = p_o \; (\; 1 - \frac{\rho^2}{a^2})^{1|2}$$

$$J_\phi^o = -J_o = \text{constant}$$

In this latter choice I have assumed $J \sim T^{3/2}$ and due to the divertor the T profile will be fairly flat.

$$\overline{P}(\rho) = \overline{P} \; (\frac{a^2}{\rho^2}) [\; 1 - (\; 1 - \frac{\rho^2}{a^2} \;)^{3/2} \;]$$

$$\overline{P} \equiv \frac{2}{3} P_o$$

$$li(\rho) = \frac{1}{2} = li(a)$$

$$B_\omega^\circ(\rho) = B_a(\frac{\rho}{a}), \rho \leq a \quad , \quad Ba = \frac{\mu_o I}{2\pi a}$$

Summary of Toroidal Corrections in Example #1.

Assuming $\quad P^\circ = P_o(\; 1 - \frac{\rho^2}{a^2} \;) = \overline{P} = P_o/2$

$$J_\phi^\circ = -J_o = -I/\pi a^2$$

one finds

$$\overline{P_o}(\rho) = 2\overline{p} \; (1 - \frac{1}{2} \; (\frac{\rho}{a})^2)$$

$$li(\rho) = li(a) = \frac{1}{2}$$

$$\Lambda(\rho) = \Lambda(a) = \overline{\beta p} - \frac{3}{4}$$

$$\overline{\beta p} \equiv \frac{2\mu o \bar{p}}{Ba^2} = \frac{1}{2} \beta p \max$$

$$\Delta(\rho) = \frac{\rho^2}{2R} [\overline{\beta p} + \frac{1}{4}]$$

$$B_\omega(\rho,\omega) = Ba \frac{\rho}{a} + Ba \frac{a}{R} (\frac{\rho}{a})^2 \quad (\frac{3}{2} [\overline{\beta p} + \frac{1}{4}] -1) \cos\omega \quad \beta\rho(\rho,\omega) = \frac{a}{2R} B_a (\frac{\rho}{a})^2 [\beta\rho + \frac{1}{4}]\sin\omega$$

$$B_\phi(\rho,\omega) = B^\circ_\phi(\rho) \;[1 - \frac{\rho}{R} (1 - 2 (\frac{Ba}{Be} \frac{\rho}{a})^2 \quad [\overline{Bp} + \frac{1}{4}][\overline{Bp} - 1] \cos\omega]$$

$$B^\circ_\phi(\rho) = Be[1 - 2 (\frac{Ba}{Be})^2 \; [\overline{\beta p} -1](1 - \frac{\rho^2}{a^2}) \;]^{1/2}$$

Be = Toroidal field at $\rho = a$, $\omega = \pi/2$

$$J_\omega(\rho,\omega) = J^\circ_\omega(\rho) \;[\; 1 + \frac{\rho}{R} (\frac{3}{2} (\overline{\beta p} + \frac{1}{4}) - 1) \cos\omega \;]$$

$$J^\circ_\omega(\rho) = 2 \frac{\rho}{a} \;\; J_o \frac{Ba}{Be} [\overline{\beta p} - 1]$$

$J_o = J^\circ_\phi$ = zero order plasma current density

$$J^{(1)}_\rho(\rho,\omega) = J_o(\frac{\rho}{a})^2(\frac{a}{R} \frac{Ba}{Be}) \;[\overline{\beta p} - 1][\overline{\beta p} + \frac{1}{4}] \sin\omega$$

$$J_\phi(\rho,\omega) = J_o \;[1 - 2 \frac{\rho}{R} [\overline{\beta p} \;(1 + \frac{\rho}{a} \frac{Ba}{Be}) - \frac{\rho}{a} \frac{Ba}{Be} \;] \cos\omega \;]$$

$$p(\rho,\omega) = p_o \;[1 - (\frac{\rho}{a})^2 (1 + \frac{\rho}{R} [\overline{\beta p} + \frac{1}{4} \;] \cos\omega \;) \;]$$

$$\psi(\rho,\omega) = \frac{\mu_o \pi R J_o}{2} \rho^2[1 + \frac{\rho}{R} [\overline{\beta p} + \frac{1}{4} \;] \cos\omega]$$

$$\Lambda(\rho) = \overline{\beta_p} \;[\; \frac{a^4}{\rho^4} (1 - (1-\frac{\rho^2}{a^2})^{3/2}) - \frac{3a^2}{2\rho^2}(1 - \frac{\rho^2}{a^2})^{1/2} \;] - \frac{3}{4}$$

$$\Lambda(a) = \overline{\beta p} - \frac{3}{4} \qquad , \; \overline{\beta p} \equiv \frac{2\mu_o \bar{P}}{Ba^2}$$

$$\Delta(\rho) = \frac{a^2}{2R} [\overline{\beta p} \left\{ \frac{3}{2} - \frac{a^2}{\rho^2} (1 - (1 - (\frac{\rho^2}{a}))^{3/2}) \right\} + \frac{1}{4} \frac{\rho^2}{a^2}]$$

$$\Delta(a) = \frac{a^2}{4R} [\overline{Bp} + \frac{1}{2}] < \frac{a^2}{2R} [\overline{Bp} + \frac{1}{4}] = \Delta(a) \text{ for } p = po(1 - \frac{\rho^2}{a^2})$$

$$\Delta(5m_i) = \frac{(5)^2}{4(13)} [\frac{2}{3} \sqrt{13/5} + \frac{1}{2}] \approx .75 \text{ meters.}$$

where I have assumed $\overline{\beta p} = \frac{2}{3} (\beta p) \max = \frac{2}{3} \sqrt{R/a}$.

$$B^{(1)}_\omega(\rho,\omega) = B_a \frac{a}{2R} \left\{ \overline{\beta p} [3 [1 - (1 - (\frac{\rho}{a})^2)^{1/2}] + \frac{a^2}{\rho^2} [1 - (1 - \frac{\rho^2}{a^2})^{3/2}] - \frac{3}{2}] - \frac{5}{4} \frac{\rho^2}{a^2} \right\} \cos\omega$$

$$B_\omega(\rho,\omega) = B_a \frac{\rho}{a} + B^{(1)}_\omega (\rho,\omega)$$

$$\psi^{(1)}(\rho,\omega) = B_a \pi\rho^2 [\overline{\beta p} \left\{ \frac{3}{2} - \frac{a^2}{\rho^2} (1 - (1 - \frac{\rho^2}{a^2})^{3/2}) \right\} + \frac{1}{4} \frac{\rho^2}{a^2}] \frac{a}{\rho} \cos\omega$$

$$\psi(\rho,\omega) = B_a \pi\rho^2 \frac{R}{a} + \psi^{(1)}(\rho,\omega)$$

$$P^{(1)}(\rho,\omega) = - \frac{Po}{2} \frac{\frac{\rho}{R} [\overline{\beta p} \left\{ \frac{3}{2} - \frac{a^2}{\rho^2} (1-(1- \frac{\rho^2}{a^2})^{3/2}) \right\} + \frac{1}{4} \frac{\rho^2}{a^2}] \cos \omega}{(1 - \frac{\rho^2}{a^2})^{1/2}}$$

$$p = p_o (1 - \frac{\rho^2}{a^2})^{1/2} + P^{(1)}(\rho,\omega)$$

$$B^{(1)}_\rho(\rho,\omega) = B_a \frac{a}{2R} [\overline{\beta p} \left\{ \frac{3}{2} - \frac{a^2}{\rho^2}(1-(1-(\frac{\rho}{a})^2)^{3/2}) \right\} + \frac{1}{4} \frac{\rho^2}{a^2}] \sin\omega$$

Find $B^o_\phi(\rho)$, $J^o_\omega(\rho)$

$$B^o_\phi(\rho) = Be [1 - 2 (\frac{Ba}{Be})^2 [\overline{\beta p} \frac{3}{2} (1-(\frac{\rho}{a})^2)^{1/2} + 1 - \frac{\rho^2}{a^2}]]^{1/2}$$

$$Be \equiv B^o_\phi(\rho= a, \omega = \pi/2)$$

$$J^{o}_{\omega}(\rho) = \frac{Jo}{2}\,\frac{\rho}{a}\,\frac{Ba}{B^{o}_{\phi}(\rho)}\left[\frac{\frac{3}{2}\,\overline{\beta p}}{\left(1 - \frac{\rho^2}{a^2}\right)^{1/2}} + 2\right]$$

That J^{o}_{ω} goes to ∞ at $\rho = a$ comes from the unphysical nature of the pressure distribution $p(\rho)$ at $\rho = a$.

NOTE: $$\frac{dp^{o}}{d\rho} = P_o\,\frac{1}{2}\,\frac{\left(-\,2\,\frac{\rho}{a^2}\right)}{\sqrt{1 - \left(\frac{\rho}{\alpha}\right)^2}} = -\,\frac{P_o(p/a)}{\sqrt{a^2 - p^2}}$$

$$\longrightarrow 0 \text{ at } p = 0$$

$$\longrightarrow -\infty \text{ at } \rho = a$$

Since B^{o}_{ϕ} is finite at $\rho = a$, as is J^{o}_{ϕ} and B^{o}_{ω} one must, near the plasma edge, hold ∇p exclusively with $J^{o}_{\omega} \times B^{o}_{\phi}$ and for B^{o}_{ϕ} finite and $\nabla p \rightarrow \infty$, J^{o}_{ω} must of necessity be infinite. Thus this is an unrealistic distribution near the plasma "edge".

$$J^{(1)}_{\omega}(\rho, \omega) = \text{unrealistically complicated expression}$$

$$= \left[J^{o}_{\omega}(\rho)\,\frac{\rho}{R}\,\Lambda(\rho) + \Delta(\rho)\,\frac{dJ^{o}_{\omega}}{d\rho}\right]\cos\omega$$

where $$\frac{dJ^{o}_{\omega}}{d\rho} \approx \frac{J^{o}_{\omega}}{\rho} + \frac{J_o\ \rho\ Ba}{2a\ Be}\left[\frac{\frac{3}{2}\,\overline{\beta p}\ \ \rho/a^2}{\left(1 - \frac{\rho^2}{a^2}\right)^{3/2}}\right]$$

but one can deduce limiting properties for J^{1}_{ω} as follows:

$$\lim_{\rho\to o} J^{o}_{\omega} \to o \qquad \lim_{\rho\to o} \frac{dJ^{o}_{\omega}}{d\rho} \to Jo\,\frac{Ba}{Be}\left[\frac{3}{2}\,\overline{\beta p} + 2\right]$$

$$\lim_{\rho\to o} \Delta(\rho) \to o \Rightarrow \lim_{\rho\to o} J^{1}_{\omega} \to o\ , \quad \lim_{\rho\to \alpha} J^{(1)}_{\omega} \to \infty$$

also therefore perturbation technique breaks down

$$B_\omega^{(1)}(\rho,\omega) = [- B_\phi^\circ(\rho) \frac{\rho}{R} + \Delta(\rho) \frac{dB_\omega^\circ(\rho)}{d\rho}] \cos\omega$$

and $\lim_{\rho\to a} \frac{d B_\phi^\circ}{d\rho} \to \infty$ => breakdown is expension procedure.

However one may still look at $\rho \to o$ and $B_\phi^{(1)}$ should be valid.

$$\frac{dB_\phi^\circ}{d\rho} \approx (\frac{Ba}{Be})^2 \frac{\rho}{a^2} [\frac{\overline{\beta p} \ \frac{3}{2}}{(1 - (\frac{\rho}{a})^2)^{1/2}} + 2]$$

$$J_\phi^{(1)}(\rho,\omega) = Jo [1 - \frac{3}{4} \frac{\overline{\beta p}}{\sqrt{1 - \frac{\rho^2}{a^2}}}] \frac{\rho}{R} \cos\omega$$

and this too $\longrightarrow -\infty$ as $\rho \to a$, as we might have suspected from $J \times B \simeq \nabla p$ force balance. (one begins to see this break down in the expansion for $\rho \approx 2\sqrt{a/R} \approx 1.2$ for UWMAK-1).

$$J_\phi(\rho,\omega) = - Jo [1 - (1 - \frac{3}{4} \frac{\overline{\beta p}}{\sqrt{1 - \frac{\rho^2}{a^2}}}) \frac{\rho}{R} \cos\omega]$$

(valid for $\rho \leq 1.2$ meters when a = 5 m., R = 13m., $\overline{\beta p} = \frac{2}{3}\sqrt{R/a}$)

I have thus computed the toroidal corrections applicable for a flatter pressure distribution. During the course of the calculations one has discovered that due to the infinite pressure gradient at $\rho = a$ for $p \sim \sqrt{\text{parabola}}$ the expansion procedure (in p/R) breaks down so that unreliable estimates exist for $\rho > 3R/\overline{\beta p}$ or so.

There are two further topics of discussion necessary at this point and they shall be treated in the following where I have assumed $B_\perp^e$ may vary with time. If $B_\perp^e$ were a constant then

$$\delta_2(t) = \delta o + \frac{\eta a}{\mu_o y R d} (\ln(8 \frac{R}{a}) + \Lambda(a) - \frac{1}{2})(1 - \frac{B_\perp^e}{B_\perp o}) t \quad \text{m.} \quad (99$$

and η = resistivity of shell (Ω-m) (for cu, typically $10^{-8}\Omega$-m)

$y = b/a$ (typically .8 →.91)

d = thickness of casing

R = radius from ℄ to center of plasma column.

Effect of poloidal gaps

When one takes into account poloidal (azimuthal) gaps as shown below in Figure #13 then in addition to $\delta_2(t)$

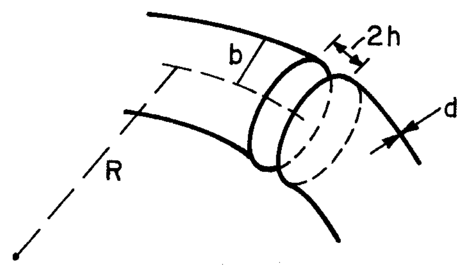


Figure #13

one must account for a contribution due to, if you will, the flux leaking out of the gaps. This is found to be

$$\delta\eta c \simeq \frac{b^3\chi}{2\pi R^2} \left(\ln \frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1 - \frac{B_\perp^e}{B_\perp o}\right)\left(1 + \frac{10^{9/2}}{\pi} \sqrt{\frac{\eta t}{hd}}\right) \text{ m.} \quad (100.$$

and must be added to $\delta_2(t)$.

Computation of δo, shift of plasma column with respect to casing center.

Ideal conducting casing.

$$\delta o = \frac{b^2}{2R} \left(\ln \frac{b}{a} + \left(1 - \frac{a^2}{b^2}\right) \left(\Lambda(a) + \frac{1}{2}\right) \right) \qquad \text{m.}$$

Letting $y \equiv a/b$ one has for ideal casing ω/o any external (uniform) vertical field present that

$$\delta o = \frac{1}{y^2} \frac{a^2}{2R} \left\{ \ln\left(\frac{1}{y}\right) + \left(1 - y^2\right) \left(\Lambda(a) + \frac{1}{2}\right) \right\} \text{ m.} \qquad (96.$$

where $\Lambda(a) = \overline{\beta p} + \frac{li}{2} - 1.$

If one assumes that inside this perfectly conducting casing (with no gaps) there is also present a uniform vertical field, $B_{\perp}^{\infty}$ say, then using (6 one has

$$\delta_1 = \delta o - b \frac{B_{\perp}\infty}{Bb} \qquad \text{where} \quad B_b = \frac{\mu_o I}{2\pi b} \qquad (97.$$

Finite conducting of casing.

Including finite conductivity effects, and assuming for simplicity that the plasma current does not charge much during an expansion towards the wall one can calculate the total shift of the plasma δ_2. If $B_{\perp}^{\infty} = 0$ and $B_{\perp}^{e}$ is switched on at $t = 0$, one has

$$\delta_2(t) = \delta o + \frac{\eta b}{\mu oRd} \left(\ln \frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right) \int_o^{\tau} \left(1 - \frac{B_{\perp}^{e}(t)}{B_{\perp}o}\right) dt \qquad (98.$$

A derivation of the above equations (99 & 100) will be the topic of a future paper, but for completeness I wished to include the results here. In (100 the function χ is defined by

$$\chi = \frac{N\pi}{2 \ln(b/h)} \qquad (101$$

where N = # of gaps (assumed equally spaced around torus)

2h = gap width

Formula (100 also assumes $B_{\perp}^{e}$ is not a function of time after t = o Thus the total shift of the (ρ=a) plasma column with respect to the center of a (ρ=b) circularly cross-sectioned, finite resistivity, containing shell, with N equally spaced poloidal gaps* is given by

$$\delta(t) = \frac{b^2}{2R}\left(\ln\frac{b}{a} + \left(1-\frac{a^2}{b^2}\right)\left(\Lambda(a)+\frac{1}{2}\right)\right)$$

$$+ \frac{\eta b}{\mu_o R d}\left(\ln\frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1-\frac{B_\perp^e}{B_\perp^o}\right) t$$

$$+ \frac{b^3\chi}{2\pi R^2}\left(\ln\frac{8R}{a} + \Lambda(a) - \frac{1}{2}\right)\left(1-\frac{B_\perp^e}{B_\perp^o}\right)\left(1 + \frac{10^{9/2}}{\pi}\sqrt{\frac{\eta t}{hd}}\right) \quad (102.$$

(meters)

* I will also discuss the effect of toroidal (or longitudinal) gaps in subsequent paper.

For the two examples worked earlier I can compute explicitly $\delta(t)$, it becomes; using the following data

$a = 5m.$, $b = 5.5m$, $R = 13.0m$, $\beta p = \sqrt{R/a} \neq \overline{Bp}$, $Ip = 21.0 \times 10^6 A.$

$2h = .1$, $d = .05$, $\eta = 5 \times 10^{-8}\ \Omega\text{-m}$ (Cu at 1000°F)

and remembering that I have assumed $B_\perp^e$ is switched on at t=o and remains a constant from there on out.

Example #1

$p = p_o(1-\rho^2/a^2) \quad J^\circ = -Jo = \text{const.}$

$\overline{\beta p} = \frac{1}{2}\beta p = \frac{1}{2}\sqrt{\frac{R}{a}} = .806$

$\Lambda(a) = .056$

$B_\perp^\circ = .418$ Tesla, $Ba = .84$ Tesla

$$\delta(t) = .184$$
$$+ .871\,(1-2.392\,B_\perp^e)\,t$$
$$+ .135N(1-2.392\,B_\perp^e)\,(1 + 45.0\sqrt{t}\,) \qquad \text{meters}$$

For $B_\perp^e = 0$, $N = 1$

$$\delta(t) = .184 + .871 + .135\,(1 + 45.0\sqrt{t}\,) \qquad \text{meters}$$

$\Rightarrow \delta(t) = .5$ at t= 6.68 milliseconds.

Example #2

$$p = po(1-\rho^2/a^2)^{1/2}, \quad J_\phi{}^o = J_o = \text{const.}$$

$$\overline{\beta p} = \frac{2}{3}\beta p = \frac{2}{3}\sqrt{\frac{R}{a}} = 1.27$$

$$\Lambda(a) = .52$$

$B^\circ_\perp$ = .492 Tesla Ba = .84 Tesla

$$\delta(t) = .317$$

$$+ 1.026\ (1-2.032\ B^e_\perp)\ t$$

$$+ .160N\ (1-2.032\ B^e_\perp)(1 + 45\sqrt{t}) \qquad \text{meters}$$

For $B^e_\perp = o$, $N = 1$ $\delta(t) = .5m$ at $t = 10$ μsec.

If $B^e_\perp$ had been present prior to t=0 and had the constant value B∞ then $B^e_\perp(t) = B_\perp{}^\infty + \Delta B_\perp{}^e$. Usually $B_\perp{}^\infty$ may be small ($\frac{B_\perp\infty}{B_\perp o} < .1$) to cancel out δo shift. This implies that the zero order shift is not δo but is δ_1. If $\Delta B^e_\perp(t)$ is assumed zero than the above examples need be corrected only by subtracting from $\delta(t)$ the quantity $5\ b^2 B^\infty_\perp/Ip$(Meg amps).

For example #1 $\frac{5b^2}{I_p} B^\infty_\perp = .184$ one only needs

$$B^\infty_\perp = .0255 \text{ Tesla} << .42\text{ T} = B^\circ_\perp$$

for an Ip = 21.0 Meg amps

Example #2 for $\frac{5b^2}{Ip} B^\infty_\perp = .317$ one only needs

$$B^\infty_\perp = .044 \text{ Tesla} << .49\text{ T} = B^\circ_\perp$$

for Ip = 21.0 Meg amps.

It may be noted however that considerable technological effort must be expelled in order to produce this uniform field $B^\infty_\perp$ in the presence of the conducting walls with their image currents present. A possibility may exist however for a startup with $B^\infty_\perp$ already present in the chamber since it is $<<B^\circ_\phi$ and therefore may not spiral field lines into chamber walls "fast enough" to prevent breakdown of gas and establishment of Ip. These points are treated in more detail in Ref. [2].

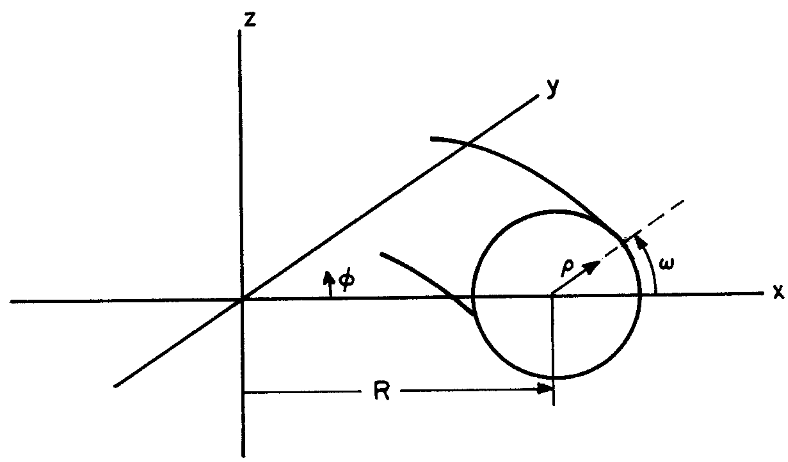
CONCLUSION

It has been the purpose of this report to develop MHD equlibrium theory for a circular cross sectioned Tokamak. This has been accomplished by following an expansion procedure in (p/R) and parallels the works of V. D. Shafranov. These calculations are not meant to be used as exact formulas for computing the equilibrium fields in as "non-round" a shaped design as proposed by the Wisconsin Fusion Design Group (UWMAK-1). It is intended however to fulfill (other than pedagogical purposes) the need for a set of bench mark calculations on toroidal corrections and plasma shifts present in any Tokamak.

The complications brought about by an other than uniform maintaining field and the presence of nulls at the plasma bounary are not considered. The adjustments necessary to handle the problems associated with a very thick conducting blanket are also not discussed. These topics hopefully will be discussed in later papers as MHD equilibrium theory is sophisticated to meet these needs.

Appendix A

Quasi-Toroidal Geometry



If (r,ϕ,z) is the position of a point in circular cylindrical coordinates then the corresponding point is (ρ,ϕ,ω) in quasi-toroidal coordinates. The two systems are related by

$$r = R + \rho \cos\omega$$

$$z = \rho \sin\omega$$

$$\phi = \phi$$

$$\rho = + \left[(r-R)^2 + z^2 \right]^{1/2}$$

$$\omega = \sin^{-1}\left(\frac{z}{\sqrt{(r-R)^2+z^2}} \right) = \tan^{-1}\left(\frac{z}{r-R} \right)$$

Relating to rectangular coordinates one has

$$x = (R + \rho \cos\omega) \cos\phi$$

$$y = (R + \rho \cos\omega) \sin\phi$$

$$z = \rho\sin\omega$$

<u>Unit Vectors</u>

$$\hat{e}_\rho = \cos\omega \cos\phi \, \hat{e}_x + \cos\omega \sin\phi \, \hat{e}_y + \sin\omega \, \hat{e}_z$$

$$\hat{e}_\phi = -\sin\phi \, \hat{e}_x + \cos\phi \, \hat{e}_y$$

$$\hat{e}_\omega = -\sin\omega \cos\phi \, \hat{e}_x - \sin\omega \sin\phi \, \hat{e}_y + \cos\omega \, \hat{e}_z$$

and conversely

$$\hat{e}_x = \hat{e}_\rho \cos\omega \cos\phi - \hat{e}_\omega \sin\omega \ \cos\phi - \hat{e}_\phi \sin\phi$$

$$\hat{e}_y = \hat{e}_\rho \cos\omega \sin\phi - \hat{e}_\omega \sin \ \sin\phi + \hat{e}_\phi \cos\phi$$

$$\hat{e}_z = \hat{e}_\rho \sin\omega + \hat{e}_\phi \cos\omega$$

where obviously $\hat{e}_\rho \times \hat{e}_\phi = \hat{e}_\omega$ and $(\rho,\dot{\omega},\phi)$ defines the triple for a right handed coordinate system. (Not all authors use this set, some define $\hat{e}_\phi$ to be in the opposite direction to this example. Others define ω to be the complement of the ω I use i.e. $r = R-\rho\cos\omega$. Be careful!)

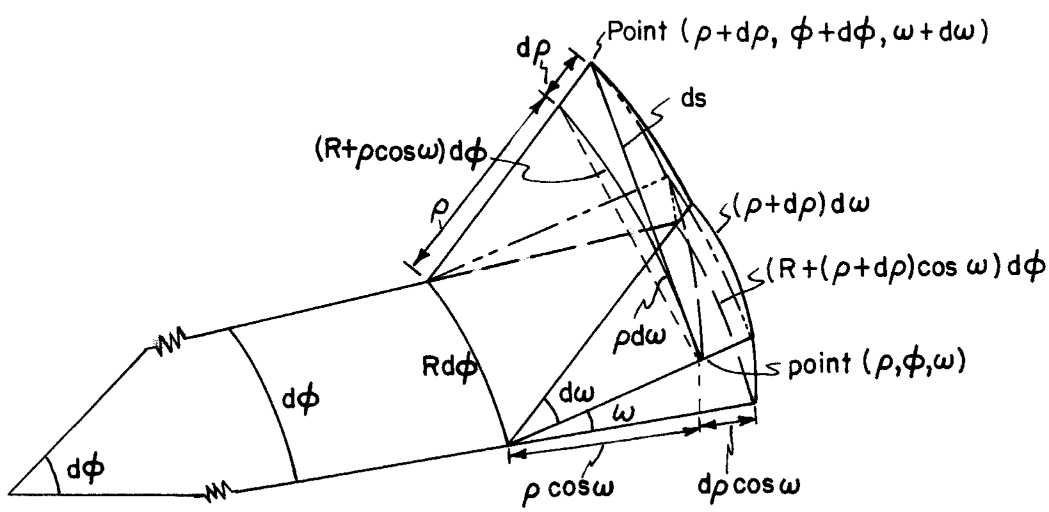
Derivatives of Unit vectors

$$\frac{\delta\hat{e}_\rho}{\partial\phi} = \hat{e}_\phi \cos\omega \ ; \quad \frac{\delta\hat{e}_\rho}{\partial\omega} = \hat{e}_\omega \ ; \quad \frac{\delta\hat{e}_\rho}{\partial\rho} = 0$$

$$\frac{\delta\hat{e}_\phi}{\partial\phi} = \hat{e}_\phi \sin\omega - \hat{e}_\rho \cos\omega = \hat{e}_r \ ; \quad \frac{\delta\hat{e}_\phi}{\partial\omega} = 0 \ ; \quad \frac{\delta\hat{e}_\phi}{\partial\rho} = 0$$

$$\frac{\delta\hat{e}_\omega}{\partial\phi} = -\sin\omega \ \hat{e}_\phi \ ; \quad \frac{\delta\hat{e}_\omega}{\partial\omega} = -\hat{e}_\rho \ ; \quad \frac{\delta\hat{e}_\omega}{\partial\rho} = 0$$

Infinitesimals and metric



$$(ds)^2 = (d\rho)^2 + \rho^2(d\omega)^2 + (R + \rho\cos\omega)^2(d\phi)^2 = g_{ij}\ d\ x^i dx^j$$

$$dv = (R+\rho\cos\omega)\ d\phi\ \rho d\omega\ d\rho\ ,\ [J] = (R+\rho\cos\omega)\ \rho$$

$$g_{ij} = \begin{pmatrix} 1 & o & o \\ o & (R+\rho\cos\omega)^2 & o \\ o & o & \rho^2 \end{pmatrix} \quad X^i = \begin{pmatrix} \rho \\ \phi \\ \omega \end{pmatrix}$$

<u>Vector Relations</u>

Define: $h = 1 + \frac{\rho}{R}\cos\omega \;\; =) \;\; r = Rh$

Gradient:

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{e}_\rho + \frac{1}{Rh}\frac{\partial\psi}{\partial\phi}\hat{e}_\phi + \frac{1}{\rho}\frac{\partial\psi}{\partial\omega}\hat{e}_\omega$$

Divergence of Vector:

$$\nabla\cdot\vec{B} = \frac{1}{\rho Rh}\left[\frac{\partial}{\partial\rho}(\rho Rh B\rho) + \frac{\partial}{\partial\phi}(\rho B\phi) + \frac{\partial}{\partial\omega}(RhB\omega)\right]$$

Curl of Vector

$$(\nabla x\vec{A})_\rho = \frac{1}{\rho Rh}\left[\frac{\partial}{\partial\phi}(\rho A\omega) - \frac{\partial}{\partial\omega}(RhA_\phi)\right]$$

$$(\nabla x\vec{A})_\phi = \frac{1}{\rho}\left[\frac{\partial}{\partial\omega}A_\rho - \frac{\partial}{\partial\rho}(\rho A_\omega)\right]$$

$$(\nabla x\vec{A})_\omega = \frac{1}{Rh}\left[\frac{\partial}{\partial\rho}(RhA_\phi) - \frac{\partial}{\partial\rho}A_\rho\right]$$

Laplacian on a scalar

$$\nabla^2\psi = \frac{1}{\rho Rh}\left[\frac{\partial}{\partial\rho}\left(\rho Rh\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{\rho}{Rh}\frac{\partial\psi}{\partial\phi}\right) + \frac{\partial}{\partial\omega}\left(\frac{Rh}{\rho}\frac{\partial\psi}{\partial\omega}\right)\right]$$

Equilibrium operator

$$\Delta^*\psi = \nabla^2\psi - \frac{2}{Rh}\left(\cos\omega\frac{\partial\psi}{\partial\rho} - \frac{\sin\omega}{\rho}\frac{\partial\psi}{\partial\omega}\right) = A(\psi)\ (Rh)^2 + B(\psi)$$

Appendix B

There is a "Quick and Dirty" method of obtaining the equilibrium force balance and Shafranov, Ref [1], demonstrates its utility. Consider the following definitions:

$$Q \equiv \int \left(\frac{B^2}{2\mu_o} + p\right) dV$$

$$\equiv Q_2^B + Q_3^B + Q^P$$

where

Q_2^B = energy in poloidal and external "holding" fields

$$= \frac{1}{2} L_2 I_2^2 + M_2 I_2 I_{2e} + \frac{1}{2} L_{2e} I_{2e}^2$$

Q_3^B = energy in Toroidal field (includes (dia/para) magnetic effects in plasma)

$$= \frac{1}{2} L_3 I_3^2 + M_3 I_3 I_{3e} + \frac{1}{2} L_{3e} I_{3e}^2$$

L_2 = self inductance of plasma loop

$$= \mu_o R \left(\ln \frac{BR}{a} - 2 + \frac{li}{2} \right)$$

I_{2e} = current of primary loop (transformer, divertor coils, control loops, etc.)

L_{2e} = self inductance of primary loop.

$\underline{\overline{\Psi}} \equiv M_2 I_{2e}$ = externally produced Flux passing through plane (z=0) of plasma loop.

$$= \int_o^R B_z \, 2\pi r dr \quad \text{for } a \ll R$$

$$\frac{\partial \overline{\Psi}}{\partial R} = 2\pi R B_z(R) \ , \quad \frac{\partial \overline{\Psi}}{\partial a} = 0 \ , \quad \frac{\partial L2}{\partial a} \simeq -\frac{\mu o R}{a}$$

$$\int \frac{B_3^2}{2\mu_o} dV \simeq \int_{\text{plasma}} \frac{B_i^2}{2\mu_o} dV + \int_{\text{ext}} \frac{B_e^2}{2\mu_o} dV = \frac{1}{2} L_3 I_3^2 + M_3 I_3 I_{3e} + \frac{1}{2} L_{3e} I_{3e}^2$$

where I have assumed that most promounced effects of I_3 are inside plasma and $\therefore\ \vec{B}_i = 0$ outside plasma. One must analyze role of I_3. I_3 produces a toroidal (dia or para) magnetic field which is the difference between the total internal field in the plasma and the field that would be there (from external sources) if I_3 were not present. (Assumes I_{3e} stays constant)

$$\int_{plasma} \frac{(\vec{B}_i - \vec{B}_e)^2}{2\mu o}\, dV \equiv \frac{1}{2} L_3 I_3^{\,2} \qquad \text{definition of } L_3$$

$$\int_{plasma} \frac{2\ (\vec{B}_i - \vec{B}_e).\vec{B}_e}{2\ \mu_o}\, dV \equiv M_3 I_3 I_{3e} \qquad \text{definition of } M_3$$

$$\frac{1}{3} L_3 I_3^{\,2} = \int_{plasma} \frac{B_i^{\,2}}{2\mu_o}\, dV + \int_{plasma} \frac{B_e^{\,2}}{2\mu_o}\, dV - 2\int_{plasma} \frac{\vec{B}_i.\vec{B}_e}{2\mu_o}\, dV$$

$$M_3 I_3 I_{3e} = 2\int_{plasma} \frac{\vec{B}_i.\vec{B}_e}{2\mu_o}\, dV - 2\int_{plasma} \frac{B_e^{\,2}}{2\mu_o}\, dV \rightarrow \frac{1}{2} L_{3e} I_{3e}^{\,2} = \int_{plasma} \frac{B_e^{\,2}}{2\mu_o}\, dV$$

since $V = 2\pi^2 Ra$

$$\frac{\partial V}{\partial a} = 2\frac{V}{a} \ ; \ \frac{\partial V}{\partial R} = \frac{V}{R}$$

$Qp \equiv \bar{p}\, V$ where Qp is assumed to be varied at constant pressure (bad assumption of doing adiabatic compression)

$$\therefore\ \frac{\partial Qp}{\partial a} = \bar{p}\, \frac{\partial V}{\partial a} = 2\ \frac{\bar{p}V}{a}$$

$$\frac{\partial Qp}{\partial R} = \bar{p}\, \frac{\partial v}{\partial R} = \frac{\bar{p}V}{R}$$

Now one must know L_3 and M_3. Shafranov says they are the same?

$$L_3 = M_3 = \mu_o\ (R - \sqrt{R^2 - a^2}\,)$$ which assumes surface currents in both cases.

$$= \frac{\mu_o}{2}\ \frac{a^2}{R}\ (1 + (\tfrac{a}{2R})^2) \equiv L_o\ (1+g)$$

$$\frac{\partial Lo}{\partial a} = \frac{2}{a} Lo \qquad \frac{\partial g}{\partial a} = \frac{2}{a} g$$

$$\frac{\partial Lo}{\partial R} = - \frac{Lo}{R} \qquad \frac{\partial g}{\partial R} = - \frac{2g}{R}$$

$$\frac{\partial L_3}{\partial a} = \frac{2}{a} L_3 \frac{(1+2g)}{(1+g)} \simeq \frac{2}{a} L_3$$

$$\frac{\partial M_3}{\partial a} = \frac{2}{a} M_3 \frac{(1+2g)}{(1+g)} \simeq \frac{2}{a} M_3$$

$$\frac{\partial L_3}{\partial R} = - \frac{L_3}{R} \frac{(1+3g)}{(1+g)} \simeq - \frac{L_3}{R}$$

$$\frac{\partial M_3}{\partial R} = - \frac{M_3}{R} \frac{(1+3g)}{(1+g)} \simeq - \frac{M_3}{R}$$

$$\left.\frac{\partial Q_2^B}{\partial a}\right|_{\substack{I_2=(const)\\ \bar{p}=(const)}} = \frac{1}{2} I_2^2 \left(- \frac{\mu_o R}{a} \right) \qquad = - \frac{2v}{a} \frac{B_a^2}{2\mu_o})$$

$$B_a = \frac{\mu_o I}{2\pi a}$$

$$\left.\frac{\partial Q_3^B}{\partial a}\right|_{\substack{I_3=(const)\\ \bar{p}=(const)\\ I_{3e}=(const)}} = \frac{1}{2} I_3^2 \left(\frac{2}{a} L_3 \frac{(1+2g)}{(1+g)}\right) + I_3 I_{3e} \left(\frac{2}{a} M_3 \frac{(1+2g)}{(1+g)}\right)$$

$$= \frac{2V}{a} \left(\frac{\overline{B_i^2}}{2\mu_o} - \frac{B_e^2}{2\mu_o} \right) \frac{(1+2g)}{(1+g)}$$

$$\left.\frac{\partial Q_\rho}{\partial}\right|_{\bar{p}=(const)} = \frac{2V}{a} \bar{p}$$

F_a = forces in minor radius expansion

$$= \frac{\partial Q}{\partial a} = \frac{2V}{a} \left(- \frac{B_a^2}{2\mu_o} + \left(\frac{1+2g}{1+g}\right) \frac{\vec{B}_i - B_e^2}{2\mu_o} + \bar{p} \right)$$

$$Fa = 0 \Rightarrow \overline{p} = \frac{Ba^2}{2\mu_o} + \frac{B_e^{\ 2} - B_i^{\ 2}}{2\mu_o} \left\{ \frac{1+2g}{1+g} \right\}$$

where I have included toroidal corrections through $g = (\frac{a}{2R})^2$

$$\frac{1+2g}{1+g} = 1 - (\frac{a}{2R})^2 - (\frac{a}{2R})^4 \ \ldots$$

so to first order $(\frac{a}{2R})$ one has Eqn. (28 back.

$$Fr = \frac{\partial Q}{\partial R} = \frac{V}{R} \left\{ \frac{Ba^2}{\mu_o} (\ln \frac{8R}{a} - 1 + \frac{li}{2}) + \overline{p} + \frac{B_e^{\ 2} - B_i^{\ 2}}{2\mu_o} + \frac{B_z \ B_a R}{2\pi a \ \mu o} \right\}$$

$$F_R = 0 \Rightarrow B_z \equiv B_{\perp o} = -\frac{a}{2R} B_a \left\{ \ln \frac{8R}{a} - \frac{3}{2} + \frac{li}{2} + \overline{\beta}_p \right\}$$

and we can see that the major radius force balance produces a formula for B_o which is exactly Eqn. (58.

Thus one can arrive at 1) the minor radius pressure balance, and 2) the required value of B_o by essentially doing a "circuit" theory analog at the problem. In fact we have said that at equilibrium (F_a=0, F_R=0) that Q has an extremum. The extremum is a minimum.

Appendix C

Derivation of Equilibrium an virial equations

Starting with the equilibrium equation integrated over slice as shown below, are has

$$\oint (p + \frac{B^2}{2\mu_o})\ d\bar{s} = \int \frac{\vec{B}\ (\vec{B}\cdot d\vec{s})}{\mu_o} \qquad \text{(C1}$$

and
$$\hat{e}_\rho\ ds\rho = \hat{e}_\rho\ (R+\rho\cos\omega)\ d_\phi\ \rho d\omega$$
$$\hat{e}_\phi\ ds\alpha = \hat{e}_\phi\ \rho dw d\rho$$
$$\hat{e}_\phi(d+d\alpha) = \hat{e}_\phi(\phi) + \frac{\partial \hat{e}\phi}{\partial\phi}\ d\phi$$
$$= \hat{e}_\phi(\phi) - \hat{e}_r d_\phi$$
$$\hat{e}_r \cdot \hat{e}_\alpha = 0$$
$$\hat{e}_r \cdot d\vec{s} = \hat{e}_r \cdot \hat{e}_\rho ds_\rho = \cos\omega\ [(R+\ \cos\omega) d\phi\rho dw]$$

$$\vec{B} = B_\phi\ \hat{e}_\phi + B_\rho\ \hat{e}_\rho + B_\omega\ \hat{e}_\omega$$

$$\vec{B}\cdot d\vec{s} = B_\rho ds_\rho + B_\phi ds_\phi \qquad \text{(for slice remember)}$$

$$B_\phi(\phi+d\phi)\ \hat{e}_\phi\ (\phi+d\phi) = B_\phi(\phi)\ (\hat{e}_\phi - \hat{e}_r d_\phi) \qquad (B_\phi \neq f(_\phi)) \text{ by axi-symmetry assumption)}$$

writing out left hand side of (cl one has

$$\int_{S_\phi(\phi)} (p + \frac{B^2}{2\mu_o})\ (-\hat{e}_\phi)\ ds_\phi + \int_{S_\rho} (p + \frac{B^2}{2\mu_o} \hat{e}_\rho ds_\rho + \int_{S_\phi(\phi+d\phi)} (p + \frac{B^2}{2\mu_o}) (\hat{e}_\phi - \hat{e}_r d\phi)\ ds_\phi$$

doting with $\hat{e}_r$ gives

$$-d\phi \int_{s_\phi} (p + \frac{B^2}{2\mu_o})\ ds_\phi + \int_{S_\rho} (p + \frac{B^2}{2\mu_o})\ \hat{e}_r \cdot \hat{e}_\rho\ ds_\rho \qquad \text{(c2}$$

Taking the right hand side of (C1 gives

$$\int_{S_\phi(\phi)} \frac{(B_\phi \hat{e}_\alpha + B_\rho \hat{e}_\rho + B_\omega \hat{e}_\omega)}{\mu_o} (-B_\phi dS_\phi) + \int_{S_\rho} (\ldots\ldots) (\bar{B}\cdot\hat{e}_\rho ds_\rho)$$

$$+ \int_{S_\phi(\phi+d_\phi)} \left[\frac{B_\phi(\hat{e}_\phi - \hat{e}_r d_\phi) + B_\rho(\hat{e}_\rho + \hat{e}_\rho \cos\omega d_\phi) + B_\omega(\hat{e}_\omega - \sin\omega d\phi \hat{e}_\phi)}{\mu_o} \right] (B_\phi ds_\phi)$$

dot with e_r gives

$$-d_\phi \int_{S_\phi} \frac{B_\phi^2}{\mu_o} \, ds\phi \qquad \text{(C3}$$

Thus combining (C2 and (C3 one has

$$-d_\phi \int_{S_\phi} (p + \frac{B^2}{2\mu_o}) \, ds_\phi + \int_{S_\rho} (p + \frac{B^2}{2\mu_o}) \hat{e}_r \cdot \hat{e}_\rho ds_\rho = -d_\phi \int_{S_\phi} \frac{B_\phi^2}{\mu_o} ds_\phi$$

or

$$\int (p + \frac{B^2}{2\mu_o}) \hat{e}_r \cdot \hat{e}_\rho \, ds_\rho = d_\phi \int (p + \frac{B^2}{2\mu_o} - \frac{B_\phi^2}{\mu_o}) \, ds_\phi$$

But $e_r \cdot e_\rho ds_\rho = e_r \cdot \vec{ds}$ and thus one arrives at (14.

$$\int (p + \frac{B^2}{2\mu_o}) (R + \rho\cos\omega) \cos\omega \, \rho \, d\omega = \int (p + \frac{B^2}{2\mu_o} - \frac{B_\phi^2}{\mu_o}) \rho d\rho \, d\omega \qquad \text{Q.E.D.}$$

Virial equation. Biginning with

$$\int T_{ii} \, d_v = \oint_s T_{ik} X_k dS:$$

$$T_{ik} = P_\perp (\delta_k - \frac{B_i B_k}{B^2}) + P_{11} \frac{B_i B_k}{B^2}$$

$$P_\perp = p + \frac{B^2}{2\mu_o}), \quad P_{11} = p - \frac{B^2}{2\mu_o}$$

one has

$$\int P_{\perp} (\delta_{ii} - \frac{B_i B_i}{B^2}) + P_{11} \frac{B_i B_i}{B^2}) \, dV$$

$$= \oint_s P \quad \delta_{ik} X_k dS_i + \oint_s (P_{11} - P_{\perp}) \frac{B_i B_k}{B^2} X_k \, dS_i \qquad \text{(C4}$$

The left hand side becomes, (noting $\delta_{ii} = 3, \frac{B_i B_i}{B^2} = 1$)

$$\int (P_{\perp} (3-1) + P_{11} 1) \, dV = \int [2 \; (P + \frac{B^2}{2\mu_o}) + P - \frac{B^2}{2\mu_o}] \, dV$$

$$= \int \; (3 \; P + \frac{B^2}{2\mu_o}) \; dV \qquad \text{(C5}$$

The right hand side of (C4 becomes

$$\oint_s (P + \frac{B^2}{2\mu_o}) \; (\vec{r}.d\vec{s}) + \oint_s (P - \frac{B^2}{2\mu_o} - p - \frac{B^2}{2\mu_o}) \; \frac{(\vec{B}.\vec{r})(\vec{B}.d\vec{s})}{B^2}$$

$$= \oint_s (p + \frac{B^2}{2\mu_o}) \; \vec{r}.d\vec{s} - \oint_s \frac{(\vec{B}.\vec{r})(\vec{B}.d\vec{s})}{\mu_o} \qquad \text{(C6}$$

where $\vec{r}.d\vec{s} \equiv (r\hat{e}_r + z\hat{e}_z) \; . \; (\hat{e}_\rho ds_\rho) = [(R+\rho\cos\omega)\hat{e}_r.\hat{e}_\rho + \rho\sin\omega \hat{e}_z.\hat{e}_\rho] ds_\rho$

$$= (R+\rho\cos\omega)(R\cos\omega+\rho) \; \rho d\omega d\phi$$

and combining (C5 and (C6 into (C4 one has

$$\int_v (3p + \frac{B^2}{2\mu_o}) \; dv = \oint_s \{p + \frac{B^2}{2\mu_o}) \; \vec{r}.d\vec{s} - \frac{(\vec{B}.\vec{r})(\vec{B}.d\vec{s})}{\mu_o} \}$$

Taking this integral over the <u>entire torus</u> surface one notes that by definition of the surface being a magnetic surface $\vec{B}.d\vec{s} = 0$.

Thus one is left with

$$\int_v \left(3p + \frac{B^2}{2\mu_o}\right) dv = \oint_s \left(p + \frac{B^2}{2\mu_o}\right) \vec{r}.\vec{ds}$$

which is equation (18. Q.E.D.

appendix D

Magnetic fields ($\vec{B}_\omega$ $\vec{B}_\rho$) close to circular current loop.

Noting from appendix A that

$$B_\rho = -\frac{1}{\rho R_o h}\frac{\delta}{\delta\omega}(Rh\ A_\phi)$$

$$B = \frac{1}{R_o h}\frac{\delta}{\delta\rho}(Rh\ A_\phi)$$

$$h = 1 + \frac{\rho}{R_o}\cos\omega$$

For a current loop, in cylindrical coordinates (Smythe, 3rd Ed., chap. 7, p 291)

$$A_\phi(r,z) = \frac{\mu_o I}{\pi k}\sqrt{\frac{R_o}{r}}\left[\left(1 - \tfrac{1}{2}k^2\right) K - E\right]$$

$$k^2 \equiv \frac{4\,R_{o}r}{(R_o+r)^2 + z^2}$$, K,E are elliptic integrals.

convience I include Br,Bz in terms of elliptic integrals.

$$B_\rho = B_r \cos_\omega + B_z \sin_\omega$$

$$B_w = B_r \sin_\omega + B_z \cos_\omega$$

$$B_r = \frac{\mu_o I}{2\pi r}\frac{z}{\left[(R_o+r)^2 + z^2\right]^{1/2}}\left[-K + \frac{R_o^2 + r^2 + z^2}{(R_o-r)^2 + z^2} E\right]$$

$$B_z = \frac{\mu_o I}{2\pi}\frac{1}{\left[(R_o+r)^2 + z^2\right]^{1/2}}\left[K + \frac{R_o^2 - (r^2+z^2)}{(R_o-r)^2 + Z^2} E\right]$$

Let $r = R_o + \rho \cos\omega$, $Z = \rho \sin\omega$, assume $\frac{\rho}{R} << 1$

$k^2 \cong 1 - (\frac{\rho}{2R_o})^2$, Using the asymptotic forms of K and E

$$K(k) = \ln\left(\frac{4}{\sqrt{(\rho/2R_o)^2}}\right) + \frac{1}{4}(\frac{\rho}{2R_o})^2 (\ln \frac{4}{\sqrt{(\rho/2R_o)^2}} - 1) + \ldots$$

$$E(k) = 1 + 1/2 (\frac{\rho}{2K_o})^2 (\ln \frac{4}{\sqrt{(\rho/2R_o)^2}} - \frac{1}{2}) + \ldots$$

Using these asymptotive expansions gives (k = 1)

$$A_\phi(\rho,\omega) \simeq \frac{\mu_o I}{\pi} (1 - \frac{\rho}{2R_o} \cos\omega) \ln \frac{8R_o}{\rho} - 1 \qquad (-\hat{e}_\phi)$$

one finds then that

$$B_\rho = -\frac{\mu_o I}{\pi\rho} \frac{\rho}{R_o} \left[\frac{3}{2} (\ln \frac{8R_o}{\rho} - 1)\right] \sin\omega$$

$$= - B_p^o(\rho) \Lambda_\rho \frac{\rho}{R_o} \sin\omega \quad , \quad \Lambda_\rho \equiv \frac{3}{2} (\ln \frac{8R_o}{\rho} - 1)$$

$$B_p^o(\rho) \equiv \frac{\mu_o I}{\pi\rho}$$

and

$$B_\omega = \frac{\mu_o I}{\pi\rho} \left[1 - \frac{\rho}{2R_o} \left[\ln \frac{8R_o}{\rho}\right] \cos\omega\right]$$

$$= B_p^o(\rho) \left[1 - \frac{\rho}{R_o} \Lambda\omega \cos\omega\right]$$

$$\Lambda_\omega = 1/2 \left[\ln \frac{8R_o}{\rho}\right]$$