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FUSION TECHNOLOGY INSTITUTE

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APPROXIMATION OF GENERAL CURVED SURFACES IN THE FABRICATION OF SHELL STRUCTURES

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SUMMARY

In the manufacture of shell structures of complicated shape, for example, vacuum vessels for stellerators, considerable savings in fabrication costs can be made if the vessels can be fabricated by cutting pieces from sheet or plate stock, bending them to form three-dimensional developable surfaces, and then welding the edges of the surfaces together to approximate a general curved surface. If the sheet stock is thin enough, the bends can be either continuous or made up of small discrete-angle bends with plane sections between bends; for thick stock, the bends must be continuous. In the following, both cases will be treated. Finally, approximation of general curved surfaces by a patchwork of plane surfaces with curved or straight-line intersections will be discussed.

Introduction

The differential geometry of curved surfaces was a central theme of the mathematics of the 18th and 19th centuries and yet is still an area of ongoing research. The purpose of this work is to apply the mathematical machinery of differential geometry to the practical problems of manufacture of shell structures. In the following, established results of differential geometry will be used without proof and mathematical concepts introduced only as needed; the reader is referred to the many textbooks[1,2,3] on the subject for background material.

Continuous Bending of Sheets: Developable Surfaces

The surface which the fabricated structure is intended to approximate (called hereafter the ideal surface) is assumed to be a collection of smoothly varying surface regions joined together along curved lines (or in some cases, points). For example, for a stellerator/torsatron with continuous windings on a toroid, the vacuum vessel wall is made up of the sides and bottoms of troughs containing the helical windings and of the undisturbed toroidal surface connecting the top edges of the troughs. The ideal surfaces in this case are not necessarily developable (that is, can be formed from a plane surface by bending without stretching); certainly the toroidal surface between windings is not. If the ideal surface is not developable, the surface can be approximated by one or more developable surfaces in the following way. Let the ideal surface be represented in a three-dimensional rectangular coordinate system by the vector $\vec{R}$ (p,q), where p and q are surface parameters (e.g. the toroidal and poloidal angles for a toroid). The edges of the region to be approximated are represented by the curves $\vec{r}_1$ and $\vec{r}_2$, represented on the surface by the parametric forms $p = a(t)$, $q = b(t)$ and $p = c(t)$, $q = d(t)$, respectively. A trial intermediate curve $p = f(t)$, $q = g(t)$ lying on the ideal surface between the two boundary curves is then constructed (see Fig. 1). Having chosen the intermediate curve, the next step is to construct the envelope of tangent planes to the ideal surface along the intermediate curve; the resultant surface is known to be developable. (It will be shown later that the envelope of tangent planes is just one of a class of developables associated with the intermediate curve.) The envelope surface contains the intermediate curve and deviates everywhere else from the intermediate curve, the deviation being of second order in the distance along the surface from the curve for small distances. There are no *a priori* restrictions on the intermediate curve; the choice depends on the nature of the boundary constraints placed by the designer on the approximate surface. Finally, a parallel surface can be generated from the envelope surface; the parallel surface is also developable, according to a theorem of differential geometry. The distance of the parallel surface from the envelope surface can then be adjusted to minimize the boundary deviations according to design constraints. In practice, the above procedure would be iterated with different choices of the intermediate curve and various parallel displacements of the envelope surface, until an approximating developable surface meeting the boundary conditions is found.

The envelope of tangent planes $\vec{R}^*$ along $p = f(t)$, $q = g(t)$ has the parametric form of a ruled surface:

$$\vec{R}^* = \vec{r}(t) + u\hat{p}(t) \qquad (1)$$

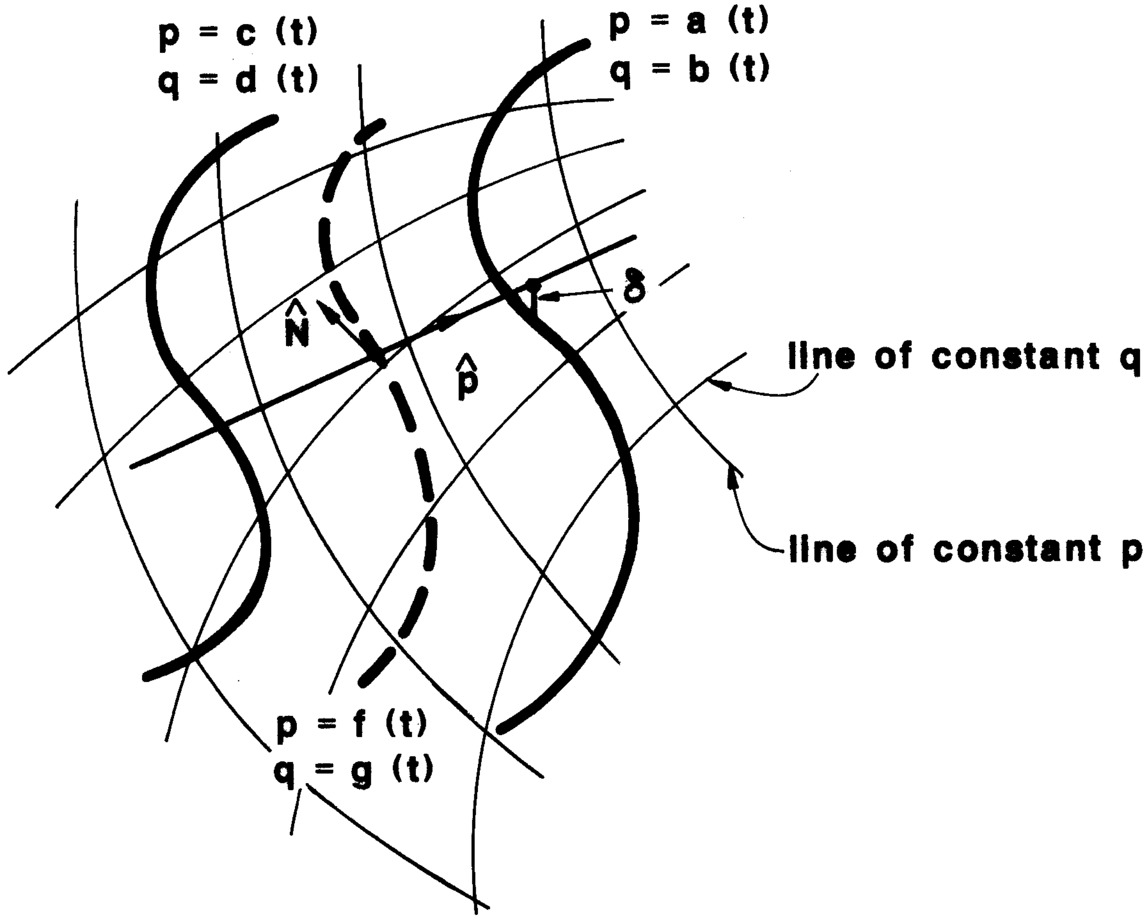

Fig. 1. Approximation of a general surface by the envelope of tangent planes to the surface along the curve $p = f(t)$, $q = g(t)$. The distance δ is a measure of the deviation of the ideal and approximate surfaces at a boundary line.

where $\vec{r}(t) = \vec{R}[f(t), g(t)]$ and $\hat{p}$ is a unit vector called the unit ruling vector. The condition that the surface be developable is equivalent to the condition that the vectors $\vec{r}'$, $\hat{p}$, and $\hat{p}'$ be coplanar; i.e.,

$$(\vec{r}' \times \hat{p}) \cdot \hat{p}' = 0 . \tag{2}$$

In the above, the prime symbol indicates differentiation with respect to the parameter t. In Eq. (1), the space curve $\vec{r}(t)$ is known as the *directrix*; the straight line formed when t is fixed and u is varied is called a *ruling* or *generator*; the surface is thus swept out by the ruling as t is varied along $\vec{r}(t)$; u is the distance along the ruling between a point on the surface and the directrix $\vec{r}(t)$. In the case of a developable surface, the rulings can (1) intersect at a point (in which case the surface is a conoid), (2) all be parallel (in which case the surface is a general cylinder), or (3) be tangent to a space curve. For the envelope of tangent planes along $\vec{r}(t)$, the unit vector $\hat{p}$ is given by the expression

$$\hat{p}(t) = \frac{\hat{N}(t) \times \hat{N}'(t)}{|\hat{N}'(t)|} \tag{3}$$

where $\hat{N}(t)$ is the surface normal restricted to $\vec{r}(t)$. The surface normal is

$$\hat{N} = \frac{\vec{R}_1 \times \vec{R}_2}{|\vec{R}_1 \times \vec{R}_2|} \tag{4}$$

where the subscripts refer to partial differentiation with respect to p and q, respectively. The derivative of the surface normal along $\vec{r}(t)$ can be written in the form

$$\hat{N}' = \hat{N}_1 f' + \hat{N}_2 g' . \tag{5}$$

Use of Eqs. (3), (4) and (5) yields the following expression for $\hat{p}(t)$, the unit ruling vector:

$$\hat{p}(t) = \frac{(f'M + g'N)\vec{R}_1 - (f'L + g'M)\vec{R}_2}{|(f'M + g'N)\vec{R}_1 + (f'L + g'M)\vec{R}_2|} \tag{6}$$

where $M = \hat{N} \cdot \vec{R}_{12}$, $N = \hat{N} \cdot \vec{R}_{22}$, and $L = \hat{N} \cdot \vec{R}_{11}$, and all quantities are to be evaluated along the curve $\vec{r}(t)$. The parallel surface derived from the tangent plane envelope surface has the form

$$\vec{R}_p^*(t,u) = \vec{R}^*(t,u) + h\hat{N}^*(t,u) \tag{7}$$

with $\vec{R}^*(t,u)$ being given by Eqs. (1) and (6) and h being the constant distance between the surfaces. However, since $\vec{R}^*$ is a developable surface, everywhere along a generator, the surface normal $\hat{N}^*$ is constant and equal to the value at the directrix. That is, $\hat{N}^*(t,u) = \hat{N}(t)$. We can therefore write

$$\vec{R}_p^*(t,u) = \vec{r}(t) + u\hat{p}(t) + h\hat{N}(t) \tag{8}$$

with $\hat{p}(t)$ being given by Eq. (6) and $\hat{N}(t)$ by Eq. (4). It should also be noted that if the surface in question is a coil winding surface, successive layers represent parallel surfaces and the last layer of the windings will be developable if the winding surface is (both sides of the winding pack can also be developable but the number of turns per layer may not be constant).

For a simple, but illustrative, example of the foregoing procedure, let the ideal surface be a sphere and the region to be approximated to be the surface between two lines of latitude. If the intermediate curve is chosen to be a line of latitude between the two bounding latitudes, the resultant developable surface is a cone tangent to the sphere along the intermediate line of latitude. The entire sphere can be approximated by such conical surfaces, with the optional addition of a cylindrical surface at the equator and circular plane surfaces at the poles. If, on the other hand, the surface is initially divided into equal regions by lines of longitude, the developable surfaces become cylinders and the lines of intersection ellipses; that is, the space curves of intersection of the developable surfaces deviate from the boundary curves by a radial distance which is greatest at the equator and zero at the poles.

Determination of the deviation vector of the curve of intersection of two approximating developable surfaces from the boundary curve on the ideal surface requires an explicit expression for their curve of intersection. The fact that the developable surfaces are ruled surfaces and can be expressed in the form of Eq. (1) greatly simplifies this task compared to the one of finding the intersection of two arbitrary surfaces. Denoting the directrix curves and unit ruling vectors for the two surfaces to be $\vec{r}_1$, $\hat{p}_1$ and $\vec{r}_2$, $\hat{p}_2$, and the curve of intersection to be $\vec{r}_I(t)$, we wish to find the distance u_1 for the intersection of a particular ruling $\vec{r}_1(t_1) + u\hat{p}_1(t_1)$ of the first surface with the second surface (see Fig. 2). The ruling of the first surface must intersect a ruling of the second surface (assuming the surfaces intersect); call the unit ruling vector of the second surface $\hat{p}_2(t)$, where t is the unknown to be found for solution of the problem. The intersecting rulings of

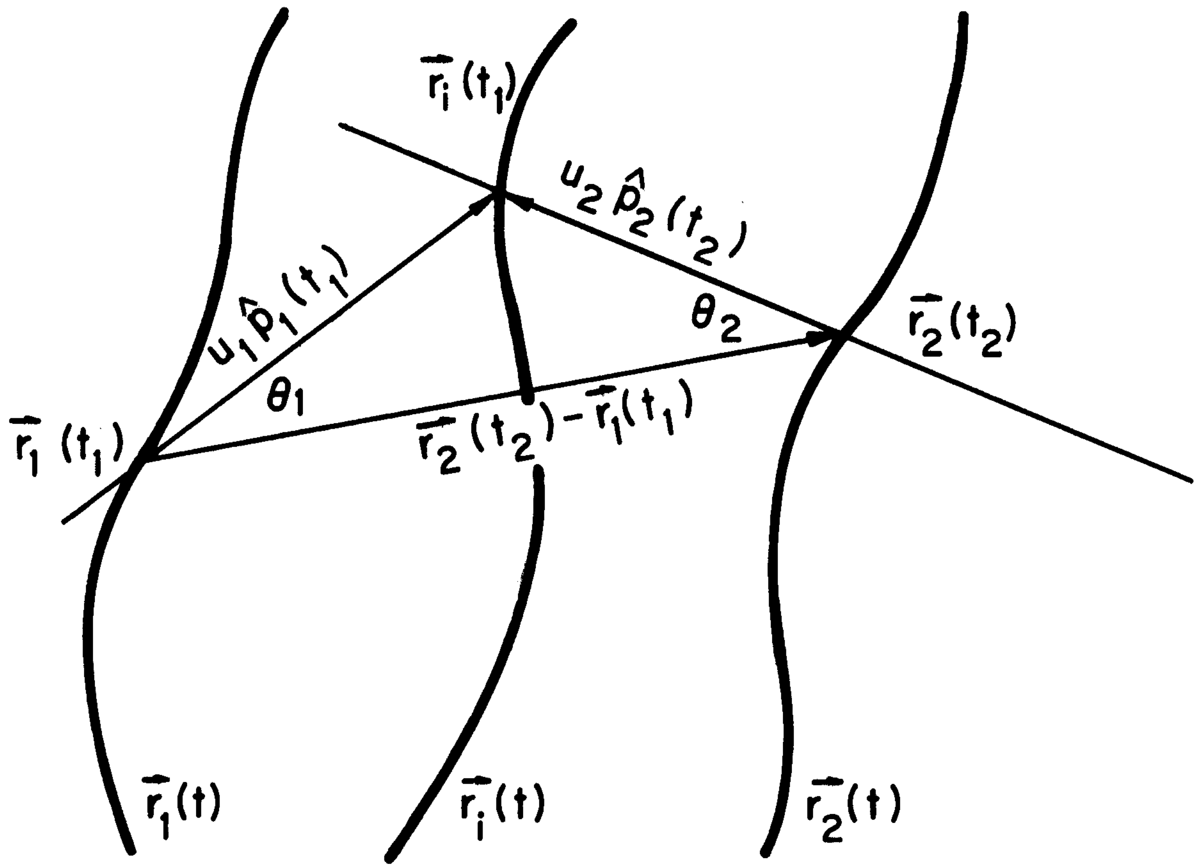

Fig. 2. Determination of the curve of intersection $\vec{r}_1$ of two developable surfaces. The directrix curves for the two surfaces are $\vec{r}_1$ and $\vec{r}_2$; the respective unit ruling vectors are $\hat{p}_1$ and $\hat{p}_2$.

the two surfaces and the difference vector $\vec{r}_2(t) - \vec{r}_1(t_1)$ between the corresponding points on the two directrix curves must be coplanar; that is

$$\hat{p}_1(t_1) \cdot [(\vec{r}_2(t) - \vec{r}_1(t)) \times \vec{p}_2(t)] = 0 \ . \tag{9}$$

Since all of the vector functions in it are known, Eq. (9) is of the form $\Psi(t) = 0$ and a point of intersection exists if and only if Eq. (9) has a solution for the range of t of interest. Assuming such a solution exists (call it t_2), then the problem is reduced to one of plane geometry. With the abbreviated notation $\vec{r}_2 = \vec{r}_2(t_2)$ and $\vec{r}_1 = \vec{r}_1(t_1)$, the angles θ_1 and θ_2 in Fig. 2 are given by

$$\cos\theta_1 = \frac{\hat{p}_1 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \tag{10}$$

and

$$\cos\theta_2 = \frac{- \hat{p}_2 \cdot (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|} \ . \tag{11}$$

The resultant expression for u_1 is then

$$u_1(t_1) = \frac{|\vec{r}_2 - \vec{r}_1|}{\cos\theta_1 + \left(\frac{1 - \cos^2\theta_1}{1 - \cos^2\theta_2}\right)^{1/2}\cos\theta_2} \tag{12}$$

The equation for the curve of intersection is

$$\vec{r}_1(t) = \vec{r}_1(t) + u_1(t)\ \hat{p}_1(t) \tag{13}$$

with $u_1(t)$ defined by Eq. (12).

The above curve of intersection is then to be compared to the corresponding curve of intersections for the ideal surfaces which the developable surfaces are intended to approximate. The degree of deviation of the lines of intersection of the developable surfaces from the boundary curves in the ideal surface is clearly dependent upon the choice of intermediate (directrix) curves and parallel displacement h. If desired, for example, a developable surface can be forced through any boundary curve by choosing the boundary curve itself as the directrix; however, the deviation at other points is then magnified.

The previously described prescription for constructing a developable surface which passes through a given space curve (Eqs. (1) and (6)) is certainly not unique; there is an infinite family of developable surfaces passing through $\vec{r}(t)$ which, instead of being tangent to the surface along the intermediate curve, cut it at some angle. The following section describes a method of finding such surfaces.

Frame Fields and Construction of a General Developable Surface from a Space Curve

The rulings of the envelope of tangent planes to a surface $\vec{R}(p,q)$ along a curve lie in the tangent planes to the surface at each point along the curve and intuitively can be thought to represent the lines along which paper can be folded to approximate the developable surface. The most general developable surface containing a space curve is formed from rulings lying in a family of planes which contain the tangent vector to the curve at each point in the curve and have an orientation which varies continuously and smoothly along the curve. These planes may, in general, cut a surface on which the curve is defined. This provides an extra degree of freedom in fitting a developable surface to a general bounded curved surface. One well-known example is the so-called rectifying developable of a space curve,(4) which offers certain advantages in coil winding. For this curve, the principal normal $\hat{n}$ to the curve (the curve being treated as a space curve) replaces the surface normal $\hat{N}$ in Eq. (3). The rulings then lie in the (rectifying) plane formed by the tangent and binormal vectors of the space curve.

The above ideas can be expressed mathematically by introducing the concept of a coordinate frame field, or translating and rotating trihedral of mutually orthogonal unit vectors, with origin at successive points along the curve and with one of the unit vectors always coinciding with the tangent vector of the curve. For the tangent plane envelope, this frame is the Darboux-Ribacour frame, defined to be $(\hat{T}, \hat{B}, \hat{N})$, where $\hat{T}$ is the tangent to the curve, $\hat{N}$ is the surface normal, and $\hat{B} = \hat{N} \times \hat{T}$. When referring curves and their associated developables to a surface the Darboux-Ribacour frame is the most natural to use; when reference is made to a space curve independently of a defining surface, the Frenet frame, composed of the tangent, principal normal, and binormal unit vectors, is the natural one to use.

When referred to the Darboux-Ribacour frame, the plane in which the rulings of the general developable lie is defined by a rotation of the $\hat{T}$ - $\hat{B}$ plane about the $\hat{T}$ axis. That is, the normal $\hat{S}$ to the new plane is given by

$$\hat{S} = \hat{N} \cos\theta + \hat{B} \sin\theta \tag{14}$$

where θ is a continuous, smoothly varying function of the curve parameter t. The surface will be developable if and only if

$$\hat{p} = \frac{\hat{S} \times \hat{S}'}{|\hat{S} \times \hat{S}'|} . \tag{15}$$

(The above condition plus the fact $\hat{S}$ is perpendicular to $\hat{T}$ is equivalent to that of Eq. (2).) In order to find an expression for $\hat{S}'$, we write

$$\hat{S}' = \hat{N}' \cos\theta - \hat{N} \sin\theta\theta' + \hat{B} \sin\theta + \hat{B} \cos\theta\theta' . \tag{16}$$

Again, differentiation refers to variation of quantities along the curve as the curve parameter t is varied. The derivatives of the unit vectors $\hat{N}$ and $\hat{B}$ are given in terms of the Darboux-Ribacour frame field itself by the equations

$$\begin{aligned} \hat{T}' &= (K_g\hat{B} + K_n\hat{N})\ s' \\ \hat{B}' &= (- K_g\hat{T} + \tau_g\hat{N})\ s' \\ \hat{N}' &= (- K_n\hat{T} - \tau_g\hat{B})\ s' \end{aligned} \tag{17}$$

where s is arc length along the curve, K_g the geodesic curvature, K_n the normal curvature, τ_g the geodesic torsion of the directrix curve, and s' denotes ds/dt. The geodesic curvature K_g is given by the expression

$$K_g = \frac{V}{W^3} [f'^3\Gamma_2^{11} - g'^3\Gamma_1^{22} + (2\Gamma_2^{12} - \Gamma_1^{11}) f'^2g' - (2\Gamma_1^{12} - \Gamma_2^{22}) f'g'^2$$

$$+ f'g'' - g'f''] \qquad (18)$$

where the subscripted Γ_s are Christoffel symbols and V and W are normalization factors defined as follows:

$$W = |\vec{R}_1 f' + \vec{R}_2 q'| \qquad (19)$$

$$V = |\vec{R}_1 \times \vec{R}_2| \ . \qquad (20)$$

The above Christoffel symbols are determined by the ideal surface independently of the choice of intermediate curve (but, of course, are to be evaluated along the intermediate curve). Introducing the vectors $\vec{F} = \vec{R}_2 \times \hat{N}$ and $\vec{G} = \vec{R}_1 \times \hat{N}$, we can write for the Christoffel symbols

$$\Gamma_1^{11} = (\vec{R}_{11} \cdot \vec{F})/V \qquad \Gamma_2^{11} = -(\vec{R}_{11} \cdot \vec{G})/V$$

$$\Gamma_1^{12} = (\vec{R}_{12} \cdot \vec{F})/V \qquad \Gamma_2^{12} = -(\vec{R}_{12} \cdot \vec{G})/V \qquad (21)$$

$$\Gamma_1^{22} = (\vec{R}_{22} \cdot \vec{F})/V \qquad \Gamma_2^{22} = -(\vec{R}_{22} \cdot \vec{G})/V \ .$$

The normal curvature K_n is given by

$$K_n = \frac{Lf'^2 + 2Mf'g' + Ng'^2}{W^2} . \tag{22}$$

Finally, the geodesic torsion τ_g is given by the formula

$$\tau_g = \frac{(FL - EM)f'^2 + (GL - EN)f'g' + (GM - FN)g'^2}{VW^2} . \tag{23}$$

Inspection of Eqs. (18), (22), and (23) shows that only K_g contains second derivatives of f and g; K_n and τ_g contain only first derivatives. This implies that all curves on $\vec{R}$ passing through $\vec{r}(t)$ for fixed t with the same tangent will have the same K_n and τ_g; however, only curves with both the same first and second derivatives will have the same K_g. These considerations can be given a geometrical interpretation (see Fig. 3). The principal normal $\hat{n}$ to the curve $\vec{r}(t)$ lies in the $\hat{B}$ - $\hat{N}$ plane as shown. The derivative of $\vec{T}$ with respect to arc length is $\kappa\hat{n}$ where κ is the curvature of $\vec{r}$ considered as a space curve. Denoting the angle between $\hat{N}$ and $\hat{n}$ by ω, we then have for K_n and K_g

$$K_n = \kappa\cos\omega$$

$$K_g = \kappa\sin\omega. \tag{24}$$

The geodesic curvature is thus the curvature of the projection of $\vec{r}$ onto the $\hat{T}$ - $\hat{B}$ plane. (In the case shown in Fig. 3, K_n is negative and K_g is positive.) The normal curvature is the curvature of the projection of $\vec{r}$ onto

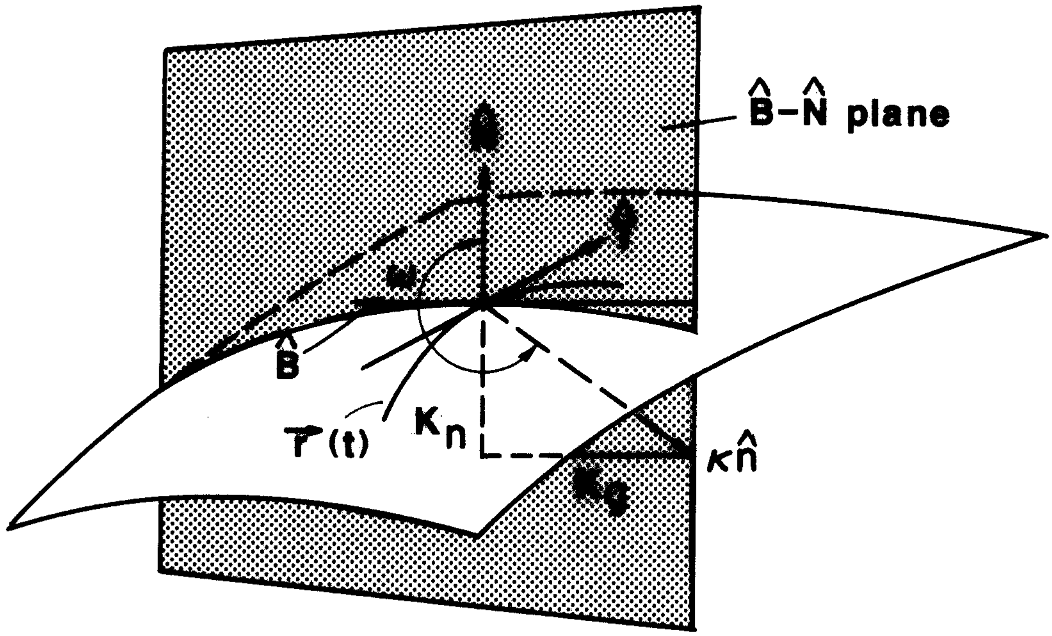

Fig. 3. Geometrical interpretation of normal and geodesic curvature.

the $\hat{T}$ - $\hat{N}$ plane. The geodesic torsion τ_g is the rate of change of $\hat{N}$ with respect to arc length projected onto the $\hat{B}$ - $\hat{N}$ plane. With these preliminaries out of the way, Eqs. (16) and (17) yield the following result for $\hat{p}$:

$$\hat{p} = \frac{(\tau_g s' - \theta')\,\hat{T} - \cos\theta\,(K_n\cos\theta + K_g\sin\theta)\,s'\hat{B} + \sin\theta\,(K_n\cos\theta + K_g\sin\theta)\,s'\hat{N}}{[(\tau_g s' - \theta')^2 + (K_n\cos\theta + K_g\sin\theta)^2 s'^2]^{1/2}} \cdot \tag{25}$$

Setting $\theta = 0$ yields a compact expression for the unit ruling vector for the envelope of tangent planes, derived previously in a more direct way to give Eq. (6):

$$\hat{p} = \frac{\tau_g\hat{T} - K_n\hat{B}}{[\tau_g^2 + K_n^2]^{1/2}} \cdot \tag{26}$$

It is easily shown that Eqs. (6) and (26) are equivalent.

In summary, then, the construction of a developable surface to fit an ideal surface bounded by two curves requires the following steps:

1. Selection of an intermediate curve for the directrix of the developable ruled surface.
2. Determination of a function $\theta(t)$ for input to Eq. (14) which minimizes the deviations of concern to the designer.
3. Iteration of steps 1 and 2 until a satisfactory surface is found.

It may, of course, be necessary to subdivide the original region into subregions to keep overall deviations within tolerable limits.

Flattening a Developable Surface - Continuous Case

Up to this point, developable surfaces have been discussed in their three-dimensional form. For a fabricator of the surfaces who wants to know what shapes his flat sheet metal pieces must be to start with and at what angles they should be bent to form the desired three-dimensional surfaces, this is not enough. Explicit expressions are therefore required for the shapes, curves, etc. after the isometric transformation which flattens the developable surface to a plane is performed on the three-dimensional shapes. This is done in this section in the continuum limit; discrete approximations to developable surfaces are discussed in a later section.

Angles between intersecting curves and arc lengths along curves are invariant under an isometry or isometric transformation. Moreover, rulings are straight lines both in the flattened and unflattened surfaces. Figure 4 shows two points lying an infinitesmal distance apart on the directrix of a developable surface which has been flattened out to a plane and referenced to an arbitrary coordinate system. The associated unit ruling vectors are denoted $\hat{p}$ and $\hat{p} + d\hat{p}$ and the angles which the ruling vectors make with the directrix curve are denoted ϕ and $\phi + d\phi$. The angles θ and $\theta + d\theta$ are the angles which $\hat{p}$ and $\hat{p} + d\hat{p}$, respectively, make with the x axis. Finally, the infinitesmal angle $d\psi$ is the angle between $\hat{p}$ and $\hat{p} + d\hat{p}$ in the flattened surface. (Rulings do not, in general, intersect in the unflattened surface; they are tangent to the edge of regression.) However, the angle $d\psi$ can be defined for the unflattened surface by transporting the generator at t by the displacement vector $\hat{t}ds$. This is allowable since the surface is developable, and therefore the component of $d\hat{p}$ out of the $\hat{T} - \hat{p}$ plane is of second order. The angle $d\psi$ is then the angle in the unflattened surface between the

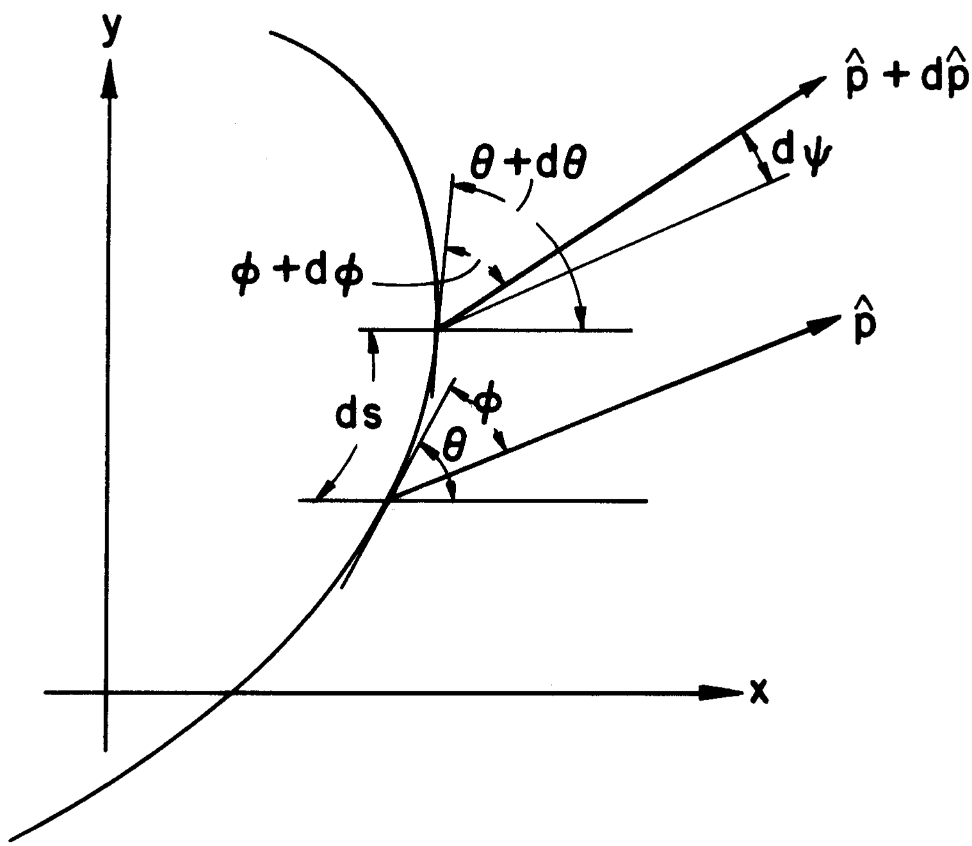

Fig. 4. Determination of the position and orientation of the directrix curve and rulings in the developed surface.

generator at $\vec{r}(t + dt)$ and the transported generator from $\vec{r}(t)$. This angle is just $d\psi = (\hat{p}' \cdot \hat{q})\, ds = \pm |\hat{p}'|\, ds$ where $\hat{q}$ is a unit vector perpendicular to $\hat{p}$ in the $\hat{T} - \hat{p}$ plane. The second equality follows because the surface is developable (Eq. (2)). Denoting the quantity $\hat{p}' \cdot \hat{q}$ by a, one can write

$$\theta = \theta_0 + \phi - \phi_0 + \int_0^s a(s')\, ds' \tag{27}$$

where θ_0 and ϕ_0 are the values for the respective angles at an arbitrary starting point of integration and s the arc length therefrom. We also have for differentials of distance along the coordinate axes in the flattened system

$$dx = \cos\theta\, ds$$

$$dy = \sin\theta\, ds\ .$$

The angle ϕ is determined by the relation $\cos\phi = \hat{p} \cdot \hat{t}$. We can therefore express the coordinates of the directrix curve in the flattened system in terms of integrals as follows:

$$y = y_0 + \int_0^s \sin\left[\theta_0 + \phi - \phi_0 + \int_0^{s'} a(\sigma)\, d\sigma\right] ds'$$

$$\tag{28}$$

$$x = x_0 + \int_0^s \cos\left[\theta_0 + \phi - \phi_0 + \int_0^{s'} a(\sigma)\, d\sigma\right] ds'\ .$$

The angle ϕ between the rulings and the directrix curve tangents in the flattened system is the same as in the unflattened system. Equations (27) and (28), together with the parametric form for the developable surface (Eq. 2) and its first derivative, provide all of the mathematical expressions needed to map a point or curve, including boundaries, from the unflattened surface onto the flattened surface.

The task remaining, from a manufacturing standpoint, is to describe how to bend the flat sheets inscribed with the directrix curves into the desired three-dimensional surfaces. In the continuum limit, this requires finding the normal curvature everywhere on the surface and bending the surface continuously to give this normal curvature.

In a developable surface, the direction of largest normal curvature is in the direction perpendicular to the rulings, or in the $\hat{q}$ direction. (In the following, the directrix curve will be parameterized by arc length s for simplicity.) In the $\hat{q}$ direction, the differentials in s and u, the two surface parameters, are related as follows:

$$du = - \cos\phi \; ds \; . \tag{29}$$

Substituting Eqs. (2) and (29) in Eq. (22), one obtains for the curvature K_n in the $\hat{q}$ direction

$$K_1 = \frac{\kappa(\hat{n} \cdot \hat{N}) + u(\hat{p}'' \cdot \hat{N})}{\sin^2\phi + 2ua\sin\phi + u^2a^2} \tag{30}$$

where $\hat{n}$ is the principal normal of the directrix curve. Equation (30) can be further simplified by reducing the second term to a quantity proportional to $\hat{n} \cdot \hat{N}$. The result is

$$K_1 = \frac{\kappa(\hat{n} \cdot \hat{N})}{\sin\phi(\sin\phi + ua)} . \tag{31}$$

Equation (31) for u = 0 can be compared with the first of Eqs. (24). The normal curvature in the $\hat{q}$ direction is one of the principal curvatures; the other principal curvature is zero because the surface is developable. Thus, Eqs. (24) and (31) for u = 0 are consistent with the Gaussian formula for normal curvature at an angle ϕ with respect to the second principal direction of curvature and 90° - ϕ with respect to the first direction of curvature, since the two directions are orthogonal. That is,

$$K_n = \sin^2\phi K_1 + \cos^2\phi K_2 = \sin^2\phi K_1 \tag{32}$$

with

$$K_1 = \frac{K(\hat{n} \cdot \hat{N})}{\sin^2\phi} = \frac{\kappa\cos\omega}{\sin^2\phi} . \tag{33}$$

Appearance of the quantity ua in the denominator in Eq. (31) for u not equal to zero expresses the fact that away from the directrix, the infinitesmal distance between nearby rulings varies with u because the nearby rulings are inclined at an infinitesmal angle from each other.

Discrete Approximations to a Developable Surface

The previously described continuum description of forming developable surfaces from flat stock is not entirely satisfactory from a manufacturer's point of view; producing a continuously varying radius of curvature orthogonal to varying ruling directions is difficult to achieve in practice. Instead, various discrete approaches to approximating developable surfaces for which analytic expressions are known (or, at least, for which first and second derivatives are piecewise continuous) can be used. The simplest is to segment the surface into plane regions with angle 2θ between planes. In fabricating such a surface, the manufacturer simply produces fixed angle bends along a series of lines scribed on the flat stock. These lines approximately, but not exactly, coincide with rulings of the corresponding continuous developable surface. The procedure for finding the bend lines is as follows.

An arbitrary staring point $\vec{r}_1$ on the directrix curve of the surface is chosen. The normal $\hat{N}_1$ to the surface at $\vec{r}_1 = \vec{r}(t_1)$ is determined. Next the function $F(t) = \hat{N}_1 \cdot \hat{N}(t)$ is evaluated along the curve. Let t_2 be the value of t for which $F(t) = \cos 2\theta$ and the corresponding directrix curve coordinates and unit ruling vector be denoted $\vec{r}_2, \hat{p}_2$, respectively. The two surface normals $\hat{N}$ and $\hat{N}_2$ and the corresponding directrix curve coordinates $\vec{r}_1$ and $\vec{r}_2$ define planes. A point in these planes can be conveniently expressed parametrically in terms of the unit ruling vectors $\hat{p}_1$ and $\hat{p}_2$ and the in-plane orthogonal unit vectors $\hat{q}_1$ and $\hat{q}_2$. For the line of intersection, one has

$$\vec{r} = \vec{r}_1 + u\hat{p}_1 + v\hat{q}_1 = \vec{r}_2 + w\hat{p}_2 + x\hat{q}_2 \ . \tag{34}$$

Equation (34) represents three equations (one for each component of $\vec{r}$) in the four unknowns u, v, w, x and therefore can yield a relation of any one variable in terms of another. Choosing first v(u) as the desired solution, one takes the scalar product of $\hat{N}_2$ with Eq. (34). The scalar products of $\hat{N}_2$ with $\hat{p}_2$ and $\hat{q}_2$ vanish and the result is

$$(\vec{r}_2 - \vec{r}_1) \cdot \hat{N}_2 - u(\hat{p}_1 \cdot \hat{N}_2) - v(\hat{q}_1 \cdot \hat{N}_2) = 0 \qquad (35)$$

or

$$v = m_1 u + b_1 \qquad (36)$$

with $m_1 = -(\hat{p}_1 \cdot \hat{N}_2)/(\hat{q}_1 \cdot \hat{N}_2)$ and $b_1 = (\vec{r}_2 - \vec{r}_1) \cdot \hat{N}_2 / (\hat{q}_1 \cdot \hat{N}_2)$. The analogous solution in terms of the variables w and x of the second plane is obtained by taking the scalar product of $\hat{N}_1$ with Eq. (34); the result is

$$x = m_2 w + b_2 \qquad (37)$$

where $m_2 = -(\hat{N}_1 \cdot \hat{p}_2)/(\hat{N}_1 \cdot \hat{q}_2)$ and $b_2 = -(\vec{r}_2 - \vec{r}_1) \cdot \hat{N}_1/(\hat{N}_1 \cdot \hat{q}_2)$.

The process can be continued around the closed curve (if it is closed) until the starting point is reached. If there is a discrepancy in position (as will almost always be the case), a small adjustment can be made in θ to make the final and initial fold lines coincide.

The straight boundary lines on the sides of the pieces are determined by the tangent vectors to the boundary curves at their intersections with the rulings at $\vec{r}_1$ and $\vec{r}_2$. A continuous boundary curve $\vec{r}_b(t)$ can be written as

$$\vec{r}_b(t) = \vec{r} + f(t)\hat{p}(t) \ . \qquad (38)$$

Then the unit tangent vector at $\hat{r}_b(t_1)$ is

$$\hat{t}_b(t_1) = \frac{\vec{r}'(t_1) + f'(t_1)\, \hat{p}_1 + f(t_1)\, \hat{p}'(t_1)}{|\vec{r}'(t_1) + f'(t_1)\, \hat{p}_1 + f(t_1)\, \hat{p}'(t_1)|} \; . \tag{39}$$

In the frame $\hat{p}_1$, $\hat{q}_1$, $\hat{t}_b(t_1)$ can be written

$$\hat{t}_b(t_1) = \frac{(\cos\phi_1\; s'(t_1) + f'(t_1))\, \hat{p}_1 + (\sin\phi_1\; s'(t_1) + f(t_1))\, \hat{q}_1}{[(\cos\phi_1\; s'(t_1) + f'(t_1))^2 + (\sin\phi\; s'(t_1) + f(t_1))^2]^{1/2}} \; . \tag{40}$$

The corner of the piece is just the intersection of the line $\hat{r}_b(t_1) + r\, \hat{t}_b(t_1)$ where r is an arbitrary parameter, with the folding line v(u) (Eq. (36)). The remaining corners are found in an analogous way.

The previously described method of approximating a developable surface by plane segments may not be acceptable, for example, to coil winders because of the finite angle bends. A better approximation to the continuous developable surface can be made by use of a combination of portions of circular conical surfaces and planes. The resultant surface has continuous tangents. A combination of planes and cones is required because in a general developable surface, nearby rulings do not intersect (as would be the case for a general conoid). This means that it is impossible, in general, to approximate the surface with cone segments alone without having discontinuities in the surface tangents. In general, between successive cone segments, trapezoidal plane pieces are required to bring the surfaces into tangency. A description of the specific method of finding these cone segments and flat pieces follows: To start, a pair of points $\vec{r}_1$ and $\vec{r}_2$ on the directrix with $\hat{N}_1 \cdot \hat{N}_2 = \cos 2\theta$ are found and the two planes with normals $\hat{N}_1$ and $\hat{N}_2$ are constructed as before. The first step is to find the parameters of a cone segment of arc angle 2θ

which is tangent to the first plane along the generator through $\vec{r}_1$ and which is also tangent to the second plane. The line of tangency of the cone with the second plane will not, in general, coincide with the ruling at $\vec{r}_2$. The angle between the generator at $\vec{r}_1$ and the fold line is $\alpha_1 = \tan^{-1} m_1$ (see Fig. 5). Denoting the apex angle of the cone by β (the apex angle being defined as the angle between any ruling of the cone and its axis), one has the relation

$$\sin\beta = \frac{m_1 \cos\theta}{(\sin^2\theta + m_1^2)^{1/2}} \cdot \tag{41}$$

The cone radius a at $\vec{r}_1$ is given by the formula

$$a = \frac{b_1}{m_1} \sin\beta \; . \tag{42}$$

The line of tangency of the cone with the second plane has the equation

$$x = b_1 \hat{q}_1 \cdot \hat{q}_2 - (\vec{r}_2 - \vec{r}_1) \cdot \hat{q}_2 + b_1 \cos(\alpha_1 - \alpha_2)$$

$$+ \tan(\alpha_1 - \alpha_2)[w - b_1 \hat{q}_1 \cdot \hat{p}_2 + (\vec{r}_2 - \vec{r}_1) \cdot \hat{p}_2 + b_1 \sin(\alpha_1 - \alpha_2)] \; . \tag{43}$$

The plane trapezoidal piece required to fill in between cones is thus the region bounded between the line of Eq. (43) and the line $x = 0$.

Determination of boundary curves is more complicated than in the plane case and requires determination of the intersections of cone segments approximating adjoining regions of the ideal surface. In practice, the fit with the cone segments to the continuous developable surface should be so

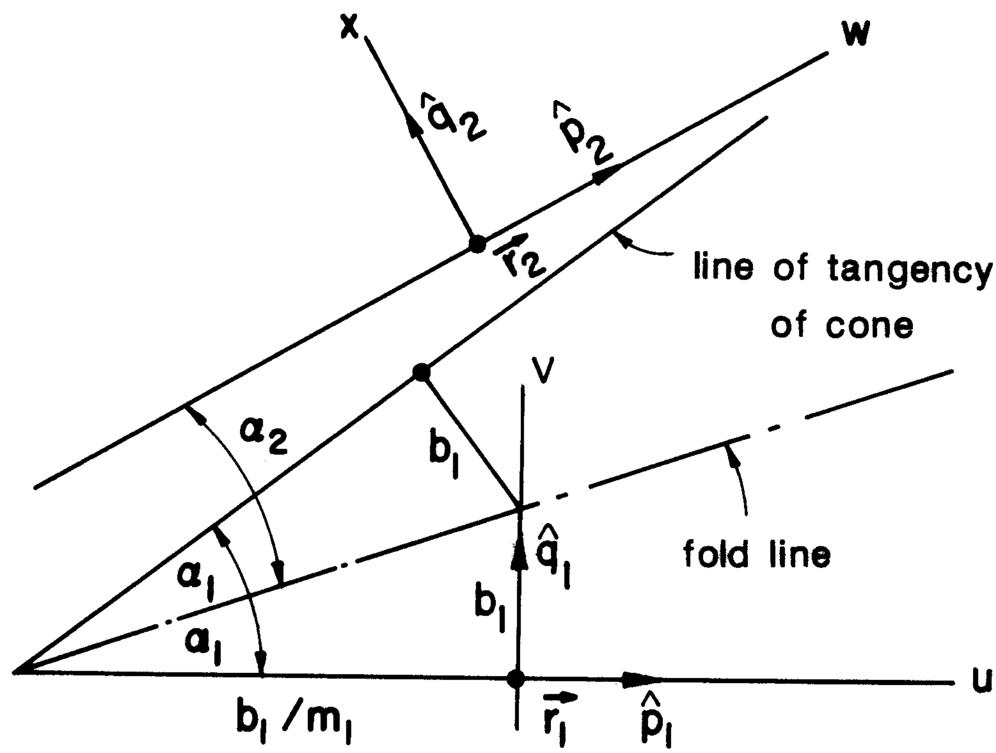

Fig. 5. Determination of the line of tangency of cone segments on successive tangent planes to a developable surface.

close that orthogonal projection of the boundary line in the continuous surface onto the cone segment surface will be sufficiently accurate for most fabrication purposes.

Approximation of General Curved Surfaces with Planes

Various approaches to approximating curved surfaces with planes are well known (e.g. so-called geodesic domes). A general surface can always be approximated to any degree of accuracy by use of triangular plane elements. One question of importance from a fabrication standpoint is how many seams have to be made by welding, etc.: bending along a line is cheaper than cutting and welding. This is related to the question of the curvature of the surface in a region. In terms of the principal radii of curvature K_1 and K_2, a surface is locally categorized as:

1. Elliptic if $K_1K_2 > 0$.
2. Hyperbolic if $K_1K_2 < 0$.
3. Parabolic if $K_1 = 0$ or $K_2 = 0$.
4. Planar if $K_1 = K_2 = 0$.

A developable surface is everywhere parabolic or parabolic and/or planar. A sphere is elliptic. A ruled surface which is not developable is hyperbolic (but not all hyperbolic surfaces are ruled surfaces). The non-developable ruled surface will be the last surface discussed in this section; the question of the optimum representation of a surface by plane triangles with respect to the length of seams per unit area is left open; the fact that ruled surfaces are naturally hyperbolic suggests that they may be best suited to economically approximate general hyperbolic surfaces.

A general ruled surface has the parametric form of Eq. (1), but does not necessarily satisfy Eq. (2). This means that a plane segment approximation of such a surface requires intersecting fold lines as shown in Fig. 6. In this case, folds are made at the rulings through $\vec{r}_1$, $\vec{r}_2$, etc., but also along diagonals connecting the boundary points. In practice, a ruled surface of this type is most easily defined in terms of its boundary curves rather than Eq. (1).

Assuming the two boundary curves $r_l(t)$ and $\vec{r}_r(t)$ are parameterized with the same parameter t, a ruled surface can be defined by the form

$$\vec{R}(t,u) = \vec{r}_l(t) + u \frac{\vec{r}_r(t) - \vec{r}_l(t)}{|\vec{r}_r(t) - \vec{r}_l(t)|} \ . \tag{44}$$

Actually, Eq. (44) represents an infinite number of different ruled surfaces passing through $\vec{r}_l$ and $\vec{r}_r$, depending on the relative parameterizations of the two curves; the choice of the "best fit" surface is thus a choice of a parameterization which minimizes deviations. Therefore, Eq. (44) can be rewritten

$$\vec{R}(t,u) = \vec{r}_l(t) + u \frac{\vec{r}_r[\phi(t)] - \vec{r}_l(t)}{|\vec{r}_r[\phi(t)] - \vec{r}_l(t)|} \tag{45}$$

with $\phi(t)$ a function of t which reparameterizes the right-hand curve.

The fold angles $\cos\theta_{ij}$ are then determined by the scalar product of the normals $\hat{N}_i$ and $\hat{N}_j$ to adjoining triangular plane regions:

$$\cos\theta_{ij} = \hat{N}_i \cdot \hat{N}_j \ .$$

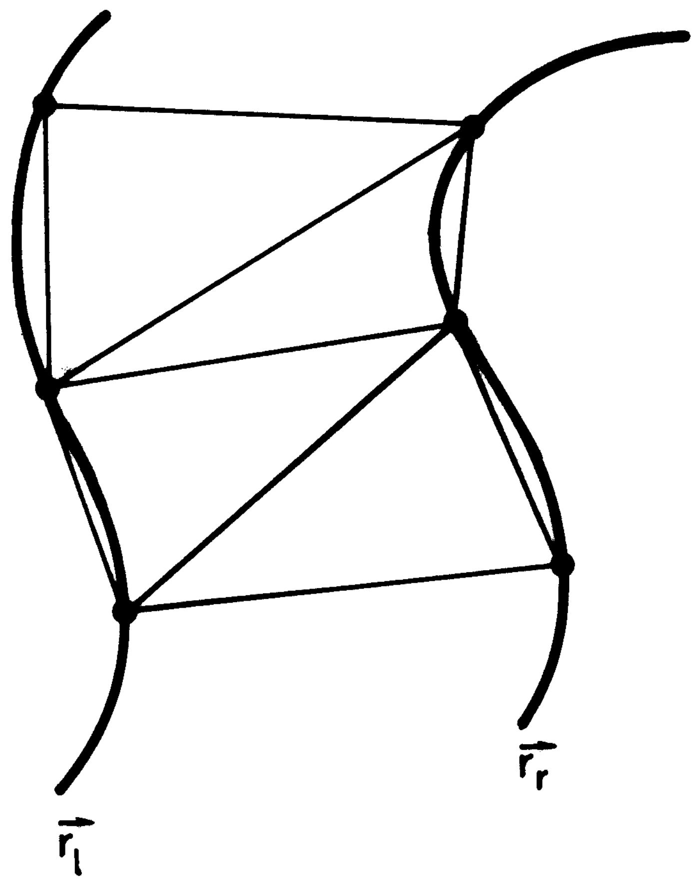

Fig. 6. Approximation of a general ruled surface by triangular elements.

The normals are readily determined from the boundary position vectors which define successive planes. Determination of the flattened shape of the plane-approximated surface is similarly straightforward.

Conclusions

Explicit methods for approximating general curved surfaces with developable surfaces have been described. Developable surfaces can be themselves approximated by a collection of plane surfaces connected by discrete-angle bends, or to greater accuracy, by a collection of segments of circular cones tangent to trapezoidal plane sections. These approximations provide practical methods of fabricating such surfaces. The mathematical formalism necessary to lay out patterns on the flat stock and form the finished three-dimensional pieces is described. Approximation of non-developable surfaces including elliptic and general ruled surfaces by triangular plane elements is also discussed.

The detailed implementation of the approximation methods previously described is problem-dependent and requires extensive development of numerical codes coupled with graphics software programs of the sort used with large finite element stress analysis codes.

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