Unfavorable Responses of Burning Plasma to a Perturbation of Fueling Rate

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ABSTRACT

It is found that a small variation of the fuel injection rate in a steady-state tokamak reactor leads to such a large variation of the plasma parameters that steady-state operation is no longer possible. Such an unfavorable response is the severest for a plasma governed by the empirical (Alcator) scaling law for particle and energy confinement.

1 Introduction

I have been developing a computer simulation and modeling methodology for fusion reactor plant availability analysis including degraded states of systems which has been applied to the STARFIRE tokamak fusion reactor power plant [1]. Such analyses require knowledge of the possible responses of the fusion power output to degraded and failed components in a fusion plant. In particular, the response of the burning plasma to the degradation of magnets, power injection systems, fuel injection systems, and impurity control systems must be studied.

I have performed some analyses to find possible derated steady-state operating parameters by solving the energy and particle balance equations for the point plasma. It is found that certain scaling laws of energy and particle confinement times lead to infinite responses of the steady-state plasma density and temperature to a perturbation of the fueling rate.

In the present report I will carry out detailed mathematical analyses to clarify the singular phenomenon by using a simple model of the plasma. In Section 2 I will describe the phenomenon by using the linear perturbation method. The analysis will be made for the steady-state energy and particle balance equations. In Section 3 I will discuss a relationship of the phenomenon with the so-called "thermal instability". In Section 4 some methods to avoid the singular responses will be discussed. It will be shown that control capabilities of plasma confinement are needed. The final section will conclude this report and some suggestions for the future will be made.

2 The Singular Responses

For the present analysis, I make the following assumptions:

- steady-state,
- point (0-dimensional) plasma model,
- equal electron and ion temperatures,
- 50% deuterium and 50% tritium,

- the impurity density is negligible,
- the radiation power loss and the injected power are negligible,
- the α-particle energy is transfered to the plasma before the particle escapes.

The energy and particle balance equations are given by

$$\frac{n^2}{2}F(T)kE_\alpha = \frac{3nkT}{\tau_E} \tag{1}$$

$$n^2F(T) + \frac{n}{\tau_p} = S \tag{2}$$

where T = plasma temperature in keV,
k = Boltzmann constant,
n = plasma particle density, $n_D = n_T = n/2$,
E_α = alpha particle energy,
S = fueling rate,
τ_E = energy confinement time,
τ_p = particle confinement time,
$F(T) =< \sigma v >_{DT} /2$ = D-T fusion reaction rate.

Suppose that there is a steady-state operating point representing a point (n_0, T_0, S_0) in the $n - T - S$ space. I will derive expressions for the perturbations of n and T with respect to a small variation of S in order to find a new steady-state operating point after the degradation of the fueling system occurs.

Assume that the energy and particle confinement times are not controllable; these depend on the plasma temperature and density. That is,

$$\tau_E = \tau_{E0}n^\alpha T^\beta \tag{3}$$

$$\tau_p = \tau_{p0}n^\alpha T^\beta \tag{4}$$

where τ_{E0}, τ_{p0}, α, and β are constants and are obtained from physics data bases.

Substituting Eqs. (3) and (4) into Eqs. (1) and (2), we have

$$E_\alpha F(T)/2 = 3n^{1-\alpha}T^{1-\beta}/\tau_{E0} \tag{5}$$

$$n^{1-\alpha}T^{-\beta}/\tau_{p0} + n^2 F(T) = S \tag{6}$$

The steady-state operating parameters n_0, T_0, and S_0 satisfy Eqs. (5) and (6).

For a perturbation of $S, S_0 + \delta S$, n and T can be represented as $n_0 + \delta n$ and $T_0 + \delta T$. Substitute these into Eqs. (5) and (6) and keep only terms with the first order perturbations. Using the following formulas:

$$F(T_0 + \delta T) = F(T_0) + F'(T_0)\delta T \tag{7}$$

$$(x_0 + \delta x)^m = x_0^m + m x_0^{m-1}\delta x, \tag{8}$$

we have

$$(1+\alpha)F(T_0)\frac{\delta n}{n_0} = ((1-\beta)F(T_0) - T_0 F'(T_0))\frac{\delta T}{T_0} \tag{9}$$

$$((1-\alpha)S_0 + (1+\alpha)n_0^2 F(T_0))\frac{\delta n}{n_0} + (n_0^2(T_0 F'(T_0) + \beta F(T_0)) - \beta S_0)\frac{\delta T}{T_0} = \delta S \tag{10}$$

Solve Eqs. (9) and (10) for $\delta n/n_0$ and $\delta T/T_0$. Making some manipulations, finally we have

$$\frac{\delta n/n_0}{\delta S/S_0} = \frac{1-\beta-\xi}{1-2\beta-\xi+x-\alpha(1-\xi-x)} \tag{11}$$

$$\frac{\delta T/T_0}{\delta S/S_0} = \frac{1+\alpha}{1-2\beta-\xi+x-\alpha(1-\xi-x)} \tag{12}$$

where

$$\xi = \frac{T_0 F'(T_0)}{F(T_0)} \tag{13}$$

$$x = \frac{n_0^2 F(T_0)}{S_0}. \tag{14}$$

The plasma beta is defined by

$$\beta_p = \frac{2nkT}{B^2/2\mu_0} \tag{15}$$

where B = magnetic field and μ_0 = permeability. The variation of β_p is given by

$$\frac{\delta\beta_p/\beta_{p0}}{\delta S/S_0} = \frac{2+\alpha-\beta-\xi}{1-\beta-\xi}\frac{\delta n/n_0}{\delta S/S_0} \tag{16}$$

Table 1: ξ VS. T

T in keV	5	10	15	20	25
ξ	3.72	2.42	2.03	1.78	1.61

The neutron power density, P_n, is given by

$$P_n = \frac{E_n}{2} n^2 F(T) \tag{17}$$

where E_n is the neutron energy. We have

$$\frac{\delta P_n/P_{n0}}{\delta S/S_0} = \frac{2-2\beta+(\alpha-1)\xi}{1-\beta-\xi} \frac{\delta n/n_0}{\delta S/S_0} \tag{18}$$

The denominator on the right-hand-side of Eq. (11) can be zero. This implies that a small variation of S leads to an infinite variation of the density n for a certain combination of α, β, ξ, and x unless $1-\beta-\xi=0$. As we see from Eqs. (16) and (18), the variations of the plasma beta and neutron power density behave like the particle density unless the numerators become zero. It is noted that the large change of the plasma beta is not desirable since the acceptable domain of the plasma beta is not wide, in particular, for the tokamak.

For $T \leq 25$ keV, $F(T)$ can be well approximated by

$$F(T) = \frac{1}{2} A_1 T^{-2/3} \exp(-A_2 T^{-1/3}) \tag{19}$$

where $A_1 = 3.68 \times 10^{-18}$ and $A_2 = 19.94$ [2]. Thus Eqs. (13) and (19) lead to

$$\xi = \frac{1}{3}(A_2 T^{-1/3} - 2) \tag{20}$$

The values of ξ are given for some temperatures in Table 1. For $10 \leq T \leq 25$ keV, $\xi = 2$ may be a good approximation.

Now let α_0 be defined by

$$\alpha_0 = \frac{1-2\beta-\xi+x}{1-\xi-x} \tag{21}$$

If $\alpha = \alpha_0$, the response given by Eqs. (11) and (12) as well as Eqs. (16) and (18) can become infinite. Using Eq. (2), x defined by Eq. (14) can be expressed as

$$x = (1 + 1/n_0\tau_p F(T_0))^{-1} \qquad (22)$$

For commercial reactors currently considered, $n_0\tau_p = 1 \times 10^{20}$ [1/m^{3}s], and $T_0 = 20$ keV (or $F(T_0) = 2.1 \times 10^{-22}$). Hence $x = 0.02$. This implies that the following is a good approximation for Eq. (21):

$$\alpha_0 = 1 + 2\beta \qquad (23)$$

To identify the location α and α_0, consider the scaling laws proposed for tokamak reactors[3]:

$$\tau = cn^{-1}T^{1/2} \quad \text{Neoclassical(NC)} \qquad (24)$$

$$\tau = cI_p^2 n^{-1}T^{1/2} \quad \text{Neoclassical/Pseudoclassical(NP)} \qquad (25)$$

$$\tau = cI_p^4 nT^{-7/2} \quad \text{Trapped Ion Mode(TIM)} \qquad (26)$$

$$\tau = cn \quad \text{Empirical} \qquad (27)$$

Here I_p is the toroidal current and c is a constant.

For the NC and NP modes, $\beta = 1/2$. Hence $\alpha_0 = 2$, while $\alpha = -1$. Thus the singularity is avoided. However, the toroidal current drive scheme proposed for tokamak reactors to achieve a steady-state plasma leads to difficulty because the total toroidal current is proportional to the particle density[4]; Eq. (25) gives $\alpha = 1$. This is close to α_0. For the TIM mode, $\beta = -7/2$, hence $\alpha_0 = -6$. Thus this may not cause the difficulty. For the empirical law, $\beta = 0$, hence $\alpha_0 = 1$. Since $\alpha = 1$, this law leads to great difficulty. To visualize the singularity, the response of the particle density given by Eq. (11) is illustrated as a function of α for $\beta = 0$, $\xi = 2$, and $x = 0.02$ in Fig. 1.

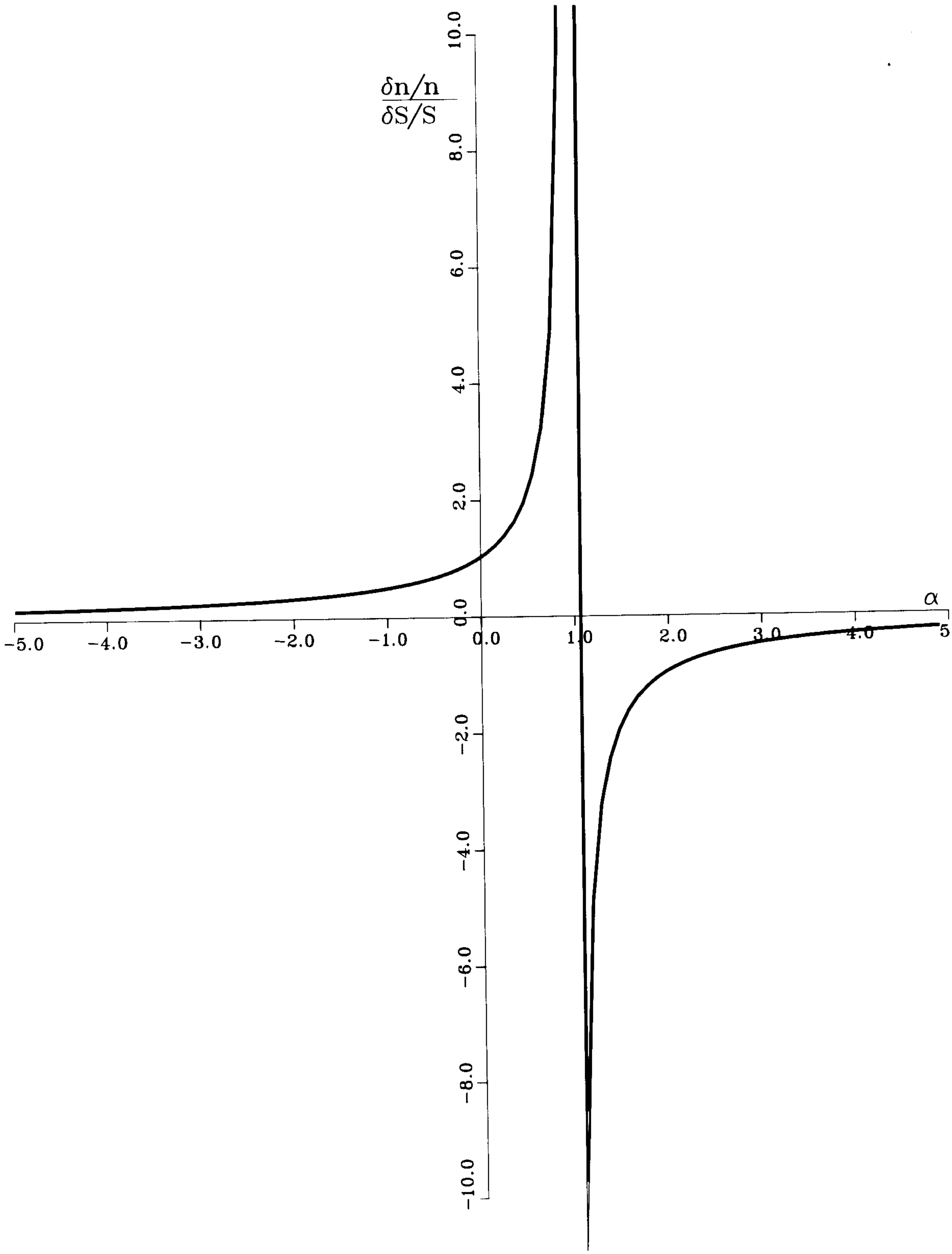


Figure 1: The Density Response VS. α

3 The Thermal Instability

Suppose that the particle density is kept constant and only the plasma temperature can vary. Furthermore, assume that the plasma temperature is one at which the fusion reaction rate increases as the plasma temperature increases and the energy confinement scaling is such that a higher temperature leads to better plasma confinement. Then a small rise in the plasma temperature will lead to a more active fusion reaction, higher temperature, and better confinement; consequently, the temperature rises significantly until the plasma disrupts or a stable point is reached. This is the most simplified picture of the thermal instability[5].

To demonstrate a relationship between the thermal instability and the singular responses described in the previous section, I will carry out a linear perturbation analysis for the plasma model used in that section; in this section a time-dependent case is dealt with. The equations are as follows:

$$\frac{d(3nT)}{dt} = \frac{n^2}{2}F(T)E_\alpha - \frac{3nT}{\tau_E} \tag{28}$$

$$\frac{dn}{dt} = S - n^2F(T) - \frac{n}{\tau_p} \tag{29}$$

Note that the energy balance equation was divided by the Boltzmann constant.

I have studied the response of the particle density and the plasma temperature to a small variation of the fueling rate. Suppose that a steady-state plasma exists before time 0 and parameters n_0, T_0, and S_0 satisfy Eqs. (1) and (2). At time 0, the fueling rate is varied by δS and the fueling rate $S_0 + \delta S$ is maintained after that. The particle density and the temperature after the disturbance can be represented by

$$n(t) = n_0 + \delta n(t) \tag{30}$$

$$T(t) = T_0 + \delta T(t) \tag{31}$$

To find the induced variations $\delta n(t)$ and $\delta T(t)$, first let new variables X and Y be defined by $X = \delta n/n_0$ and $Y = \delta T/T_0$. Now substitute Equations (30) and (31) into Eqs. (28) and (29) and use Eqs. (1) and (2) with n_0, T_0, and S_0 inserted. Keeping the first order terms of the perturbations, we have

$$\frac{dX}{dt} + \frac{dY}{dt} = \frac{E_\alpha n_0}{6T_0}\{F_0(1+\alpha)X + [T_0F_0' - (1-\beta)F_0]Y\} \tag{32}$$

$$\frac{dX}{dt} = \frac{\delta S}{n_0} - [2n_0F_0 + \frac{1-\alpha}{\tau_p}]X - [n_0T_0F_0' - \frac{\beta}{\tau_p}]Y \tag{33}$$

where $F_0 = F(T_0)$ and $F_0' = F'(T_0)$.

Equations (32) and (33) can be transformed to the following forms:

$$\frac{dX}{dt} = a_1X + b_1Y + c \tag{34}$$

$$\frac{dY}{dt} = a_2X + b_2Y - c \tag{35}$$

where

$$a_1 = -2n_0F_0 - \frac{1-\alpha}{\tau_p} \tag{36}$$

$$b_1 = \frac{\beta}{\tau_p} - n_0T_0F_0' \tag{37}$$

$$a_2 = -a_1 + \frac{E_\alpha n_0}{6T_0}F_0(1+\alpha) \tag{38}$$

$$b_2 = -b_1 + \frac{E_\alpha n_0}{6T_0}(T_0F_0' - (1-\beta)F_0) \tag{39}$$

$$c = \frac{\delta S}{n_0} \tag{40}$$

The analytical solutions of Eqs. (34) and (35) can be obtained for four separate cases: [I] $D \neq 0,\ g_2 \neq 0$, [II] $D \neq 0,\ g_2 = 0$, [III] $D = 0,\ g_2 \neq 0$, and [IV] $D = g_2 = 0$.

[I]

$$X(t) = \frac{(c+\lambda_2y_0)exp(\lambda_1t) - (c+\lambda_1y_0)exp(\lambda_2t)}{\lambda_1 - \lambda_2} + y_0 \tag{41}$$

$$Y(t) = \frac{-(c-\lambda_2z_0)exp(\lambda_1t) + (c-\lambda_1z_0)exp(\lambda_2t)}{\lambda_1 - \lambda_2} + z_0 \tag{42}$$

where $y_0 = -(b_1 + b_2)c/g_2$ and $z_0 = (a_1 + a_2)c/g_2$.

[II]

$$X(t) = \frac{(c-p)(exp(\lambda_1 t) - exp(\lambda_2 t))}{\lambda_1 - \lambda_2} + pt \tag{43}$$

$$Y(t) = -\frac{(c+q)[exp(\lambda_1 t) - exp(\lambda_2 t)]}{\lambda_1 - \lambda_2} + qt \tag{44}$$

where $p = (b_1 + b_2)c/g_1$ and $q = -(a_1 + a_2)/g_1$.

[III]

$$X(t) = [(y_0 + c)t - y_0]exp(\lambda t) + y_0 \tag{45}$$

$$Y(t) = [(z_0 - c)t - z_0]exp(\lambda t) + z_0 \tag{46}$$

where $y_0 = (b_1 + b_2)ct/g_1$ and $z_0 = -(a_1 + a_2)ct/g_1$.

[IV]

$$X(t) = c\,t\;exp(\lambda t) - 0.5(b_1 + b_2)ct^2 \tag{47}$$

$$Y(t) = -c\,t\;exp(\lambda t) + 0.5(a_1 + a_2)ct^2 \tag{48}$$

Here $g_1 = a_1 + b_2$, $g_2 = a_1 b_2 - a_2 b_1$, and $D = g_1^2 - 4\,g_2$. λ_1 and λ_2 (λ if $\lambda_1 = \lambda_2$) are the roots of a quadratic equation

$$\lambda^2 - g_1 \lambda + g_2 = 0 \tag{49}$$

The real part of the roots λ must be negative so that $X(t)$ and $Y(t)$ do not grow infinitely. The necessary conditions for this are

$$g_1 \leq 0 \tag{50}$$

$$g_2 \geq 0 \tag{51}$$

g_1 and g_2 can be expressed in terms of x, ξ, α, and β by using Eqs. (13), (14), (36), (37), (38), and (39). A cumbersome calculation lead to

$$g_1 = \frac{S_0}{n_0}[(1-x)\alpha + ((1+r)x - 1)\beta - 1 + x(r+1)(\xi - 1)] \tag{52}$$

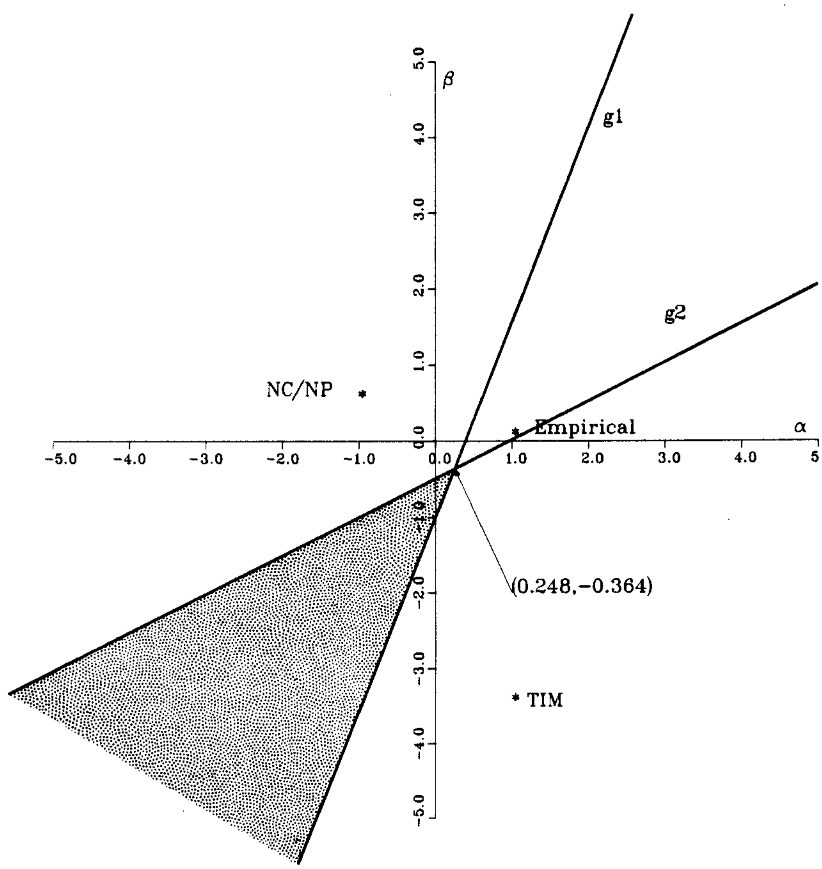
$$g_2 = rF_0 S_0[x - \xi + 1 - 2\beta + (x + \xi - 1)\alpha] \tag{53}$$

where $r = E_\alpha/6T_0$.

For $x = 0.02$, $\xi = 2$, and $r = 30$, the domain given by the inequalities (50) and (51) is illustrated in Fig. 2 on the α-β plane. The plasma is thermally stable for the scaling laws with α and β in the shaded region in the figure. The points representing the scaling laws given by Eqs. (24) through (27) are also indicated in the figure. None of these are inside the stable region.

Comparing Eqs. (11) and (53), we discover that the singular response described in Section 2 occurs for the scaling laws whose α and β are on the $g_2 = 0$ line. There is a singularity near the $g_2 = 0$ line even if the scaling law is inside the stability region.

Figure 2: Stability region in α - β space.



4 Controlling the Plasma Confinement

The discussions in Sections 2 and 3 assumed that energy and particle confinements are determined by nature and no control of the plasma confinement is imposed. For real machines, there must be certain control schemes accommodated. To obtain a thermally stable plasma, some feedback control methods have been proposed[5]:

- varying the fraction of D and T particles,
- injection of high energy ions,
- enhancement of the radiation power loss,
- enhancement of the α particle loss.

The thermal instability can be mitigated by imposing a self-control mechanism on the plasma confinement. The singular response to the degradation of the fueling rate can be prevented by controlling the confinement time so that the variation of the particle density and the temperature can be made as small as possible. In the remainder of the present section I will analyze this approach by using the plasma model described in Section 2.

The energy and particle confinement times are forced to vary such that

$$\tau_E = \tau_{E0} + \delta\tau_E \tag{54}$$

$$\tau_p = \tau_{p0} + \delta\tau_p \tag{55}$$

The expressions for $\delta\tau_E$ and $\delta\tau_p$ are obtained so that the variations of the particle density and the temperature become negligible. By using Eqs. (54) and (55), the first order perturbation equations are derived from Eqs. (1) and (2). A lengthy manipulation leads to

$$\frac{\delta n}{n_0} = \frac{-x\xi\delta\tau_E/\tau_{E0} - (\xi - 1)(\delta S/S_0 + (1-x)\delta\tau_p/\tau_{p0})}{x - \xi + 1} \tag{56}$$

$$\frac{\delta T}{T_0} = \frac{\delta S/S_0 + (1-x)\delta\tau_p/\tau_{p0} + (1+x)\delta\tau_E/\tau_{E0}}{x - \xi + 1} \tag{57}$$

Let $\delta n/n_0 = 0$ and $\delta T/T_0 = 0$. Solving Eqs. (56) and (57) leads to

$$\frac{\delta\tau_p}{\tau_{p0}} = -\frac{1}{1-x}\frac{\delta S}{S_0} \tag{58}$$

$$\frac{\delta\tau_E}{\tau_{E0}} = 0 \tag{59}$$

Equations (56) and (57) give the necessary variations of the particle and energy confinement times so that the density and the temperature responses to the perturbation of the fueling rate are minimized. The equations imply that the particle confinement must be improved for the degradation of the fueling rate ($\delta S < 0$) and the energy confinement must be kept constant. One possible approach to this control is to operate the reactor with somewhat deteriorated particle confinement and remove the deterioration whenever the degradation of the fueling rate occurs. A difficulty of this control method is that the energy confinement must be kept constant when the particle confinement is improved.

5 Conclusions

A singular response of plasma parameters (the particle density, temperature, plasma beta, and neutron power output) to a perturbation of the fueling rate has been presented and analyzed by using a linear perturbation method. The phenomenon is related to the thermal instability. The unfavorable response can be mitigated by controlling the particle and energy confinement.

The analysis has been made by linearizing the non-linear particle and energy balance equations. Hence, the results are true only for the cases in which the amount of perturbation (= the ratio of the perturbation of a variable to its original value) is much smaller than 1. If the non-linearity is taken into account, the singular behavior discussed in this report may appear somewhat different. The truth can be revealed by solving the original time-dependent equations numerically.

When a control scheme is applied, the plasma parameters can be kept constant for a certain range of the perturbation of the fueling rate. Outside that range, however, the plasma parameters must also be perturbed. The range of controllability should be studied.

As mentioned in the Introduction, the motivation of the present analysis comes from the difficulty to find a derated steady-state operating point for a degraded fueling rate. The calculation was done by using the particle and energy confinement time scaling laws that depend on both the particle density and temperature. Since the confinement time can be controlled to obtain a thermally stable plasma, the calculation can be performed in a different way. That is, we search for a derated steady-state operating point by adjusting the confinement times. In this way the difficulty stemming from the singularity can be overcome.

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