Stochastic Effects in Nonaxisymmetric Charged Particle Beams

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Adiabatic beam theory provides the framework within which a drift kinetic treatment of charged particle beams is constructed. Transverse dynamics in an axisymmetric equilibrium constitute an integrable nonlinear Hamiltonian system. One is primarily interested, however, in nonaxisymmetric phenomena, for which theoretical understanding is as yet incomplete. We define nonaxisymmetric adiabatic charged particle beams to be those for which KAM tori exist in the transverse phase space. For such beams one is able to justify a drift kinetic description. This simplification of the dynamics has important consequences concerning numerical simulation of nonaxisymmetric beams. Our ultimate objective is to formulate a theory of the coupled Maxwell and drift kinetic equations for a weakly perturbed beam. We present this theory and justify it for situations in which the KAM structure is well enough preserved. We investigate stochastic effects of hose-like perturbations upon the transverse dynamics. Utilizing section maps we study the transverse orbit structure due to coupling resonances, between harmonics of circular drift and vortex gyration, driven by these weak hose-like perturbations.

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1 Introduction

When a charged particle beam undergoes a linear hose instability lateral deflections of the beam travel back from the head to the tail of the beam growing in amplitude as they do. In experiments with intense electron beams the result is more often in the nature of a catastrophic hose instability resulting in gross sideways deflection and destruction of the beam. In this situation the concept of a well behaved equilibrium undergoing weak perturbations of any sort is dubious, nevertheless, a great deal of theoretical work has been devoted to the linear theory of beam hosing [1],[2],[3],[4],[5],[6],[10].

Full nonlinear treatment of the mode has not been attempted to date nor will it be in this paper. Here, in order to have a well defined model to discuss, we are going to investigate linear hosing of a Bennett equilibrium [9]. The Bennett equilibrium is one of a class of beam equilibria which arise naturally as a result of statistically random perturbations of the beam particle distribution. An initially non-Bennett distribution has been shown both experimentally [7] and theoretically [8], as a result of an H-theorem, to evolve to the Bennett distribution. As the Bennett distribution is the natural quasi-equilibrium and possesses many remarkable features, not the least of which is analytical tractability, we have selected this beam distribution for our present work.

Anharmonic phase mixing effects are of great importance when attempting to analyze nonaxisymmetric phenomena such as hose deflections. The mixing arises due to radially dependent betatron frequencies of transverse particle orbits in the nonlinear pinch potential. The pinch potential is defined as $\psi = \beta_b A_z - \phi$ where A_z is the axial component of the vector potential, ϕ the scalar potential and $\beta_b = v_b/c$ the beam velocity in the paraxial approximation. The pinch potential completely determines the transverse particle dynamics. In terms of the pinch potential the radially dependent betatron frequency is

$$\Omega^2_{\beta_b}(r) = \frac{q}{\gamma_b m}\left(-\frac{1}{r}\frac{d\psi}{dr}\right) \tag{1}$$

In this equation $\gamma_b^2 = 1/(1-\beta_b^2)$, m is the beam particle rest mass and q its charge.

If we imagine the beam to be composed of a sequence of slices then as the hose disturbance passes a given slice, as the disturbance propagates tailward, the slice experiences a shaking back and forth. Individual particles in the slice respond to this nonaxisymmetric time dependent perturbation in a manner which is dependent upon betatron frequency. Since the particle betatron frequency is radially dependent the responses of classes of particles at different radii get out of phase with one another in time, that is, the response of the slice as a whole damps due to phase mixing amongst the classes of particles. If $\Omega = \omega - \beta_b ck$ is the hose frequency Doppler-shifted from the lab to the beam rest frame, where ω is the hose frequency and k the wave vector in the lab, then particles for which $\Omega \sim \Omega_\beta$ are maximally coupled to the perturbation and the beam is therefore maximally deformed. The earliest low frequency theories treated the beam as an internally rigid bending rod [1]-[2]. The essential fact, however, is that each slice responds as if it were a collection of oscillators, coupled to the perturbation, each of different fundamental frequency.

The first model, the "spread-mass" model, to attempt to accurately treat anharmonic effects exploited the known damping and convection due to real relativistic mass spread by introducing a fictitious mass spread amongst the particles in the beam [4]. Subsequent work involved dividing the beam into groups of particles with differing transverse energy [5]. This approach was improved upon with the development of the "multi-component" model in which the beam is divided into groups of particles according to azimuthal frequency [6].

The multicomponent model successfully duplicates the important analytical and resonance structure of an exact Vlasov theory [5], including radial localization of the resonance. The Vlasov theory does not lend itself to further analytical development yielding as it does a dispersion relation in the form of an intractable differential equation and its boundary conditions.

In a future publication we intend to present a new model of linear hose instability, the "multi-ring" model, which incorporates phase mixing effects without any ad hoc constructions. In the multi-ring model phase mix damping develops naturally from the basic particle dynamics.

In this paper we will describe an alternative approach to the linear hose instability, which is the most important nonaxisymmetric beam mode, based upon the "adiabatic" model [12]. In the next section we will describe the essentials of the full adiabatic theory. Then we will look at stochastic effects driven by the linear hose instability. Our conclusion is that for the amplitudes typically considered in the linear theory, $\delta y/a \sim 10^{-6} - 10^{-4}$ where y is the lateral displacement of the beam center of mass and a the Bennett radius, the invariants upon which the adiabatic theory is based hold up.

2 Adiabatic Theory

In the Lorentz gauge the Maxwell field equations describing a charged particle beam propagating in resistive plasma are

$$\beta_b\left(\nabla_\perp^2+\frac{1}{\gamma_b^2}\frac{\partial^2}{\partial\varsigma^2}\right)\frac{\partial\psi}{\partial\varsigma}-\frac{4\pi}{c}\nabla_\perp\cdot\sigma\nabla_\perp\phi=\frac{4\pi}{c}\frac{1}{\gamma_b^2}\frac{\partial}{\partial\varsigma}J_z \tag{2}$$

$$\left(\nabla_\perp^2+\frac{1}{\gamma_b^2}\frac{\partial^2}{\partial\varsigma^2}\right)\frac{\partial}{\partial\varsigma}(\psi+\phi)=-\beta_b\frac{4\pi}{c}\frac{\partial}{\partial\varsigma}J_z \tag{3}$$

where the current J_z is

$$J_z=J_b-\sigma\frac{\partial\psi}{\partial\varsigma} \tag{4}$$

The conductivity σ is usually modeled by a simple rate equation which accounts for beam driven impact ionization of the ambient gas

$$\frac{\partial\sigma}{\partial\varsigma}=\kappa J_b \tag{5}$$

where κ is a gas dependent constant.

In deriving these equations we have made the paraxial approximation that the transverse particle energy is negligible compared to the axial energy. We have also assumed the z dependence of the fields is weaker than the $\varsigma=\beta_b ct-z$ dependence. Physically this is because the fields are expected to look like fixed patterns moving in the z direction, evolving only slowly in the "time" variable z, whereas the dependence upon the "slice" variable ς may be strong if there is axial field structure in the beam rest frame. We have also decoupled the transverse vector potential $\vec{A}_\perp$ from the determination of the scalar potential and axial vector potential A_z by setting $\partial \vec{A}_\perp / \partial \varsigma = 0$ in the transverse induced current $J_\perp$. This gives a closed set of equations for the scalar and pinch potentials. Taking $\gamma_b \longrightarrow \infty$ in these equations yields the EMPULSE field equations of Lee [11].

The lab frame Hamiltonian of a particle in the beam is

$$H = \gamma mc^2 = \sqrt{(mc^2)^2 + (\vec{p}c)^2} + q\phi \tag{6}$$

This may be expanded in the paraxial approximation in the form

$$H = \gamma_b mc^2 + \frac{1}{2\gamma_b m}\left(p_r^2 + \frac{P_\theta^2}{r^2}\right) - q\psi \tag{7}$$

where the beam "fluid gamma" γ_b is defined by

$$(\gamma_b mc^2)^2 = 1 + \left(\frac{P_z}{mc}\right)^2 \tag{8}$$

and $\vec{p}$ denotes mechanical momentum whereas $\vec{P}$ denotes canonical momentum.

The adiabatic beam model is based upon the fact that there are transverse action invariants which remain sufficiently well defined despite the nonaxisymmetric potential and the coupling resonances driven by linear hose instability. The existence of action invariants enables the elimination of a fast variable, analogous to gyro-motion, here called vortex gyration, and the reduced description of the particle orbit as a circular orbit, here called circular drift. Particle orbits in a general charged particle beam resemble precessing ellipses in the transverse (r, θ) plane. The basic idea is to resolve the ellipse into a circle and an epicycle. Then the epicyclic "vortex gyration" is averaged out of the problem. This procedure is essentially a small gyro-radius expansion where the vortex radius plays the role of gyroradius.

Now let us introduce the radial drift $U(r)$, the circular drift frequency $\Omega(r)$, the vortex gyration frequency $\nu(r)$ and the vortex action J_r. We assume that the radial action J_r is of order ϵ^2 where $\epsilon \sim \delta r/r$ with δr the gyration radius of an orbit centered on r. The relations between U, Ω,

ν, and the mechanical radial momentum p_r and the canonical azimuthal momentum P_θ are

$$p_r = \gamma_b m U + \sqrt{2\gamma_b m \nu J_r} sin\xi \tag{9}$$

$$P_\theta = \gamma_b m r^2 \Omega + \frac{r\nu}{2\Omega}\sqrt{2\gamma_b m \nu J_r} cos\xi \tag{10}$$

where we have anticipated the fact that the action J_θ turns out to be related to the action J_r by $J_\theta = (r^2\nu^2/4\Omega^2)J_r$. If we insert these into the lab frame Hamiltonian and average over the gyration angle ξ we get the averaged Hamiltonian

$$< H >_\xi = \gamma_b mc^2 + \frac{1}{2}\gamma_b m\left(U^2 + \frac{(r^2\Omega)^2}{r^2}\right) + \frac{1}{2}\nu J_r\left(1 + \frac{\nu^2}{4\Omega^2}\right) - q\psi \tag{11}$$

We are only interested in the transverse part of this averaged Hamiltonian $\hat{H} \equiv < H >_\xi - \gamma_b mc^2$ and in what follows we will drop the "hat" from the averaged transverse Hamiltonian.

To close the set of field, current, and conductivity equations we now have a drift kinetic equation of the form

$$\dot{f} \equiv \left(\frac{\partial}{\partial t} - L_H\right) f = 0 \tag{12}$$

where L_H is the angle averaged Lie derivative and f is the single particle phase space distribution function. Determination of the correct angle averaged Lie derivative will also yield U, Ω, and ν as nonlinear functionals of the pinch potential ψ. To do this we use the definitions (9) and (10) and $\dot{p}_r = -H_{,r}$ and $\dot{P}_\theta = -H_{,\theta}$ where H is here the exact Hamiltonian (7). The ordering scheme as relates to the various derivatives is

$$\epsilon \sim \frac{\partial}{\partial t} \sim \frac{\partial}{\partial \varsigma} \sim \frac{\partial}{\partial \theta} \sim U \tag{13}$$

Computing $\dot{\xi}$ results in terms of order ϵ^{-1}, ϵ^0, and ϵ^1. Since ν is taken to be order ϵ^0 and we want $\dot{\xi} = \nu(r) + O(\epsilon)$ we set the order ϵ^{-1} term to zero which yields the relation

$$\Omega^2 = -\frac{q}{\gamma_b m}\frac{1}{r}\frac{\partial\psi}{\partial r} \tag{14}$$

Setting an angle dependent order ϵ^0 term to zero yields

$$U = -\frac{2\Omega}{\nu^2}\left(\beta_b c \frac{\partial}{\partial z} r\Omega + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{2}r^2\Omega^2 - \frac{q}{\gamma_b m}\psi\right)\right) \tag{15}$$

This procedure yields $\dot{\xi} = \nu + O(\epsilon)$ with the definition

$$\nu^2 = \frac{1}{r^3}\frac{\partial}{\partial r}(r^2\Omega)^2 \tag{16}$$

If we hadn't taken $J_\theta = (r^2\nu^2/4\Omega^2)J_r$ then angle dependent terms would have arisen in the order ϵ^1 terms, setting these terms to zero would yield the given relation between J_θ and J_r. As an aside, note that the radial drift velocity has a simple interpretation as a balance equation, since it may be rewritten in the form

$$U\frac{\partial}{\partial r}\left(\gamma_b m r^2\Omega\right) + \beta_b c\frac{\partial}{\partial z}\left(\gamma_b m r^2\Omega\right) + \Omega\frac{\partial}{\partial \theta}\left(\gamma_b m r^2\Omega\right) = q\frac{\partial \psi}{\partial \theta} \tag{17}$$

which is an angular momentum balance, radial convection plus axial convection plus azimuthal convection equals torque on a wedge shaped element of the beam cross-section. Also notice that what we are calling the circular drift frequency Ω is in reality nothing but the betatron frequency (1).

The gyration averaged Lie derivative to lowest order is

$$L_H = -\beta_b c\frac{\partial}{\partial z} - U\frac{\partial}{\partial r} - \Omega\frac{\partial}{\partial \theta} + \frac{q}{\beta_b c}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial P_z} \tag{18}$$

so that, in terms of the beam variables (z, ς) the drift-kinetic equation assumes the form

$$\left(\beta_b c\frac{\partial}{\partial z} + U\frac{\partial}{\partial r} + \Omega\frac{\partial}{\partial \theta} + \frac{q}{\beta_b c}\frac{\partial \psi}{\partial \varsigma}\frac{\partial}{\partial P_z}\right) f = 0 \tag{19}$$

Equations (2), (3), (4), (5), (14), (15), (16), and (19) are a closed set of non-linear equations describing charged particle beam propagation in resistive plasma in the adiabatic theory. For low frequency modes the "displacement" terms on the RHS of the field equations may be dropped and for ultra-relativistic beams the EMPULSE field equations employed.

This theory is called the "adiabatic" theory because J_r is in fact an adiabatic action invariant of the system with drifts $U(r)$ and $\Omega(r)$ and an exact invariant of the system when U and Ω are constants. The question is whether the action remains adiabatically invariant in the nonaxisymmetric beam.

3 Bennett Equilibrium

The transverse Hamiltonian for a particle in a Bennett equilibrium written in the beam rest frame is

$$H_o = \frac{1}{2m}\left(p_\rho^2 + \frac{P_\theta}{a^2\rho^2}\right) + 2Tlog(1+\rho^2) + mc^2 - 2Tlog(\frac{a}{2}) \tag{20}$$

where $T = q\beta_b I_b(1 - 1/\beta_b^2/(1+f))/2c$ is the Bennett temperature, $\rho = r/a$ with a the Bennett radius and $f = I_p/I_b$ is the current neutralization fraction.

A nonaxisymmetry such as a linear hose instability results in the addition of another term δH to H_o. Before looking at δH we will transform H_o to action-angle variables. Expanding H_o about a reference circle orbit to lowest order, as in the original work on the adiabatic theory [12], yields a system of two harmonic oscillators. However, the system is inherently nonlinear so we expand to higher order to expose the lowest order nonlinearity. The ground state of the two oscillator system is nonlinear so that we may rely upon the KAM theorem [16] in its strongest and most useful form.

Expanding the Hamiltonian H_o to fourth order in the small parameter $\delta\rho/\rho_o \sim \epsilon$ we get

$$H_o = \Omega_o J_\theta + \epsilon^2\nu_o J + \epsilon^3\nu_1 J^{\frac{3}{2}} sin^2\xi + \epsilon^4\nu_2 J^2 sin^4\xi + \cdots \tag{21}$$

where the various parameters are defined

$$\Omega_o^2 = \frac{1}{a^2\rho_o}\frac{dV}{d\rho_o} \tag{22}$$

$$\nu_o^2 = 3\Omega_o^2 + \frac{1}{a^2}\frac{d^2V}{d\rho_o^2} \tag{23}$$

$$\nu_1 = \left(\frac{2}{\nu_o}\right)^{\frac{3}{2}} \left(\frac{1}{6a^3}\frac{d^3V}{d\rho_o^3} - 2\frac{1}{a}\frac{\Omega_o^2}{\rho_o}\right) \tag{24}$$

$$\nu_2 = \frac{1}{\nu_o^2}\left(\frac{1}{6a^4}\frac{d^4V}{d\rho_o^4} + 10\frac{1}{a^2}\frac{\Omega_o^2}{\rho_o^2}\right) \tag{25}$$

$$V = 2T\,log(1+\rho_o^2) \tag{26}$$

and the action J_θ is that of the circular drift, not to be confused with the usage of this symbol in the previous section where it referred to only the vortex component of the θ action and not the drift component. The radial action-angle variables are those of the order ϵ^2 terms of the radial contribution to the Hamiltonian

$$\delta\rho = \epsilon\sqrt{\frac{2}{\nu_o a^2}J}\,sin\xi \tag{27}$$

$$\delta p_\rho = \epsilon\sqrt{2\nu_o J}\,cos\xi \tag{28}$$

These are not the correct action-angle variables for the radial Hamiltonian as a whole. We employ Deprit's version [14] of Lie transform perturbation theory to get the action-angle variables and the Hamiltonian to the requisite order. This calculation is given in detail in [15]; the result is

$$H_o = \Omega_o J_\theta + \nu_o J_\rho + \epsilon^2\left(\frac{3}{8}\nu_2 - \frac{15}{32}\frac{\nu_1^2}{\nu_o}\right)J_\rho^2 \tag{29}$$

where the relation between J and J_ρ is given in [15].

This is the Hamiltonian of two oscillators, one linear and one nonlinear. A slight nonaxisymmetry will couple these oscillators together and generally the resulting Hamiltonian will be nonintegrable. The issue here is whether or not the invariants of the ground state will at least remain "adiabatic" invariants of the perturbed system. If the KAM structure in phase space holds up well enough then we will call the beam "adiabatic". For strong enough perturbations the KAM structure disintegrates and the beam trajectories become stochastic. When this happens the entire basis of linear theory, that the spectrum depends only upon the unperturbed system, breaks down and a quasilinear or fully nonlinear theory must be developed. This breakdown of linear theory vitiates the Vlasov treatment of the mode used in [5].

To nondimensionalize our system we will measure action in units of $a\sqrt{T}$, frequency in units of T/a^2, energy in units of transverse temperature T, and length in units of the Bennett radius a. For the Bennett beam our Hamiltonian now reads

$$H_o = \Omega_o J_\theta + \epsilon^2 \nu_o J_\rho + \epsilon^4 \alpha_o J_\rho^2 + \cdots \tag{30}$$

where the coefficients are

$$\Omega^2 = 4\frac{1}{1+\rho_o^2} \tag{31}$$

$$\nu_o^2 = 8\frac{2+\rho_o^2}{(1+\rho_o^2)^2} \tag{32}$$

$$\alpha_o = -\frac{1}{96}\nu_o^2 \frac{11\rho_o^6 + 66\rho_o^4 + 129\rho_o^2 + 72}{(2+\rho_o^2)^2} \tag{33}$$

Since α_o is negative the ground state system is of the "weak-spring" oscillator type, that is $\partial^2 H_o/\partial J_\rho^2 \leq 0$, so the twist mapping in the (J_ρ, ξ) plane is such that the larger actions revolve more slowly than the smaller actions. The fact that the ground state Hamiltonian depends linearly upon J_θ is of some consequence, resulting in primary resonances separated by an interval that is action independent.

In the ground state the actions and winding number of a given torus are given by

$$J_\theta(\rho_o) = \rho^2 \Omega_o \tag{34}$$

$$J_\rho(\rho_o, E) = -\frac{1}{2}\frac{\nu_o}{\alpha_o}\left(1 - \left(1 - 4\frac{\alpha_o}{\nu_o^2}(E - \rho_o^2\Omega_o^2\right)^{\frac{1}{2}}\right) \tag{35}$$

$$w(\rho_o, E) = \frac{\nu_o}{\Omega_o}\left(1 - 4\frac{\alpha_o}{\nu_o^2}\left(E - \rho_o^2\Omega_o^2\right)\right)^{\frac{1}{2}} \tag{36}$$

where, in deriving J_ρ we have selected the branch which goes to zero, as the radial energy goes to zero, $E \longrightarrow \rho_o^2\Omega_o^2$. Winding numbers are bounded above by a ρ_o dependent maximum ν_o/Ω_o where the upper bound itself is constrained to fall within the range $\sqrt{2} \leq \nu_o/\Omega_o \leq 2$. This restricts the primary resonances to $w = 1, 2$ which are equally separated from one another. At finite radius the only primary resonance of importance is the $w = 1$ resonance. The "sidebands" of secondary resonances ranging on either side of $w = 1$ are $w = 1 \pm m/n$ where $m \leq n$ so the sidebands form a dense set of resonances.

When we look at coupling resonances driven by the linear hose instability we shall be interested in whether or not the islands associated with the secondary resonances overlap one another. Technically, the "smoothness" condition of the KAM theorem requires good separation of these islands if invariant tori are to persist.

Inverting the winding number resonance condition yields the resonant energy spectrum

$$E_{pq} = \rho_o^2 \Omega_o^2 + \frac{1}{4} \frac{\Omega_o^2}{\alpha_o} \left(\frac{\nu_o^2}{\Omega_o^2} - \frac{p^2}{q^2} \right) \tag{37}$$

where p and q are integers and their ratio is of the form: unity plus or minus a fraction less than unity. We naturally speak of such an energy spectrum as there are many particles at any reference radius and these particles have a distribution in energy; the Bennett equilibrium is characterized by a transverse temperature. Changing the point of view, any particle with a given fixed energy has a nested tori family as orbit space and a torus in the family may be thought of as being parameterized by radius of the orbit.

Introducing the transverse energy spread described by T into our considerations we have a one parameter family of nested tori families, a so-called "VAK nest" [17]. The true dynamical system is the four-manifold and a nested tori family corresponding to an energy level submanifold is a "slice" of the four-manifold. For a description of the bewildering variety of bifurcational processes which may occur as one varies the energy parameter in such a system we refer the reader to [17].

If the radial action J_ρ ceases to be a good invariant due to any coupling resonance overlaps then the particle orbits will "diffuse" to other reference radii. This is because there is a one-to-one relationship between the reference radius and the azimuthal action J_θ. In fact, we could think of (ρ_o, J_ρ) as the invariants of the ground state system as well as (J_θ, J_ρ), so that diffusion in action is equivalent to radial diffusion.

4 Linear Hose Perturbations

In the rest frame of a transverse slice of the beam the passage of a hose instability propagating from the head to the tail of the beam results in a nonaxisymmetric perturbation with sinusoidal time dependence. For the approximately autonomous case $\Omega \ll \Omega_o(\rho_o)$ the perturbation is of the form [15]

$$H = H_o + \sum_{lm} \delta H^c_{lm} cos(l\xi - m\theta) + \sum_{lm} \delta H^s_{lm} sin(l\xi - m\theta) \tag{38}$$

where the coefficients are given in detail in [15]. In this Hamiltonian the angle θ is that of circular drift and ξ is the vortex gyration angle. The linear hose instability couples the two oscillators nonlinearly.

At resonance the frequencies of the resonant term are related in such a fashion that the phase is stationary. All other terms have rapidly varying phases when the actions (J_θ, J_ρ) yield a stationary phase of a given term so they tend to average away as high frequency perturbations. The condition for coupling resonance is

$$\frac{d}{dt}(n\xi - s\theta) \approx 0 \tag{39}$$

Performing a transformation to the usual rotating (island) variables defined as

$$\hat{J}_r = \frac{1}{n} J_r \tag{40}$$

$$\hat{J}_\theta = \frac{s}{n} J_r + J_\theta \tag{41}$$

$$\hat{\xi} = s\theta - n\xi \tag{42}$$

$$\hat{\theta} = \theta \tag{43}$$

we may perform an average over the angle $\hat{\theta}$ to get the approximate second invariant which generically exists near an isolated coupling resonance of a two degree of freedom system. The only term which survives the averaging is that for which $ls - nm = 0$ so if we define $p = l/n$ we have $m = sp$ and $l = np$ and the angle averaged hose Hamiltonian in "island variables" is

$$\hat{H}_{ns} = \hat{H}_o + \sum_{p=1} \delta H^c_{np,sp} cosp\hat{\xi}_{ns} + \sum_{p=1} \delta H^s_{np,sp} sinp\hat{\xi}_{ns} \tag{44}$$

where $\hat{\xi}_{ns} = n\xi - s\theta$. With the Hamiltonian in this form we now see clearly the existence of an approximate second invariant

$$I = sJ_\rho + nJ_\theta \tag{45}$$

which demonstrates the integrability of the isolated resonance.

In order to justify the validity of our drift kinetic equation the stochastic effects of the perturbation must be considered. It turns out that only the one-one, the one-two, and the one-three islands need be examined. This is because only the θ fundamental and the fundamental and first two harmonics of ξ are of appreciable island width. If these islands overlap for lateral displacements of the magnitude considered in linear theory, $\delta y/a \sim 10^{-6} - 10^{-4}$, then linear theory is inappropriate. When the KAM structure disintegrates the system is no longer even approximately integrable and any theory based upon linearization about the unperturbed system fails, even though the perturbation may be small in magnitude. Our drift kinetic theory, which is nonlinear in the dependence upon the pinch potential, must be regarded as valid only when the KAM tori bound the gyration invariant. In this sense the theory is "linear" because clearly for large amplitude lateral deflections the invariants simply cannot survive.

In order to study the effects of the perturbation we use a Poincaré map induced by the flow on an energy level torus as it intersects a theta equals constant cross-section. The figures depict the numerical results of following orbits for various values of the lateral displacement $s = \delta h/a\rho_o$ and various reference orbits ρ_o. It is seen that for perturbations in the linear regime the tori do persist quite well. We view this as justification of the adiabatic theory in the linear regime.

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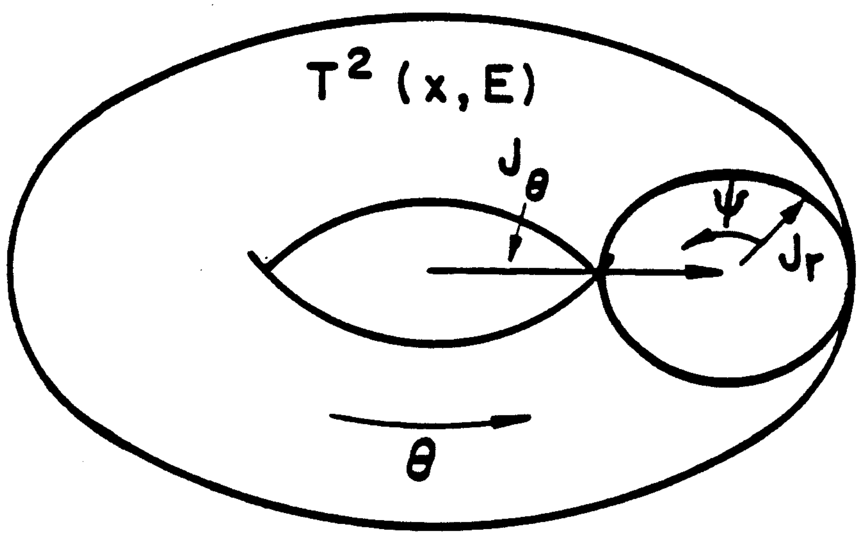

Figure 1 Ground State Tori

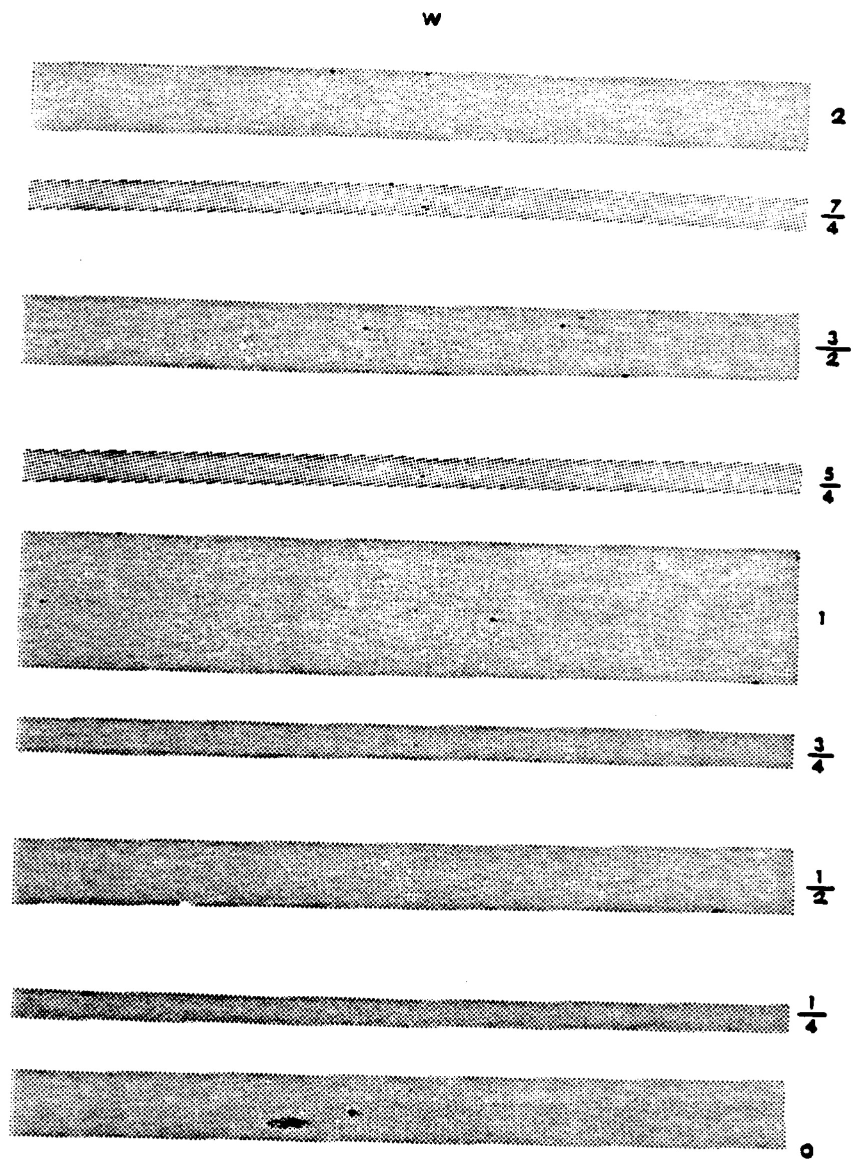

Figure 2 Secondary resonances arrayed on either side of the primary resonance at $w = 1$. There is a dense distribution of such secondary resonances, one at each rational w, however, the island amplitude decreases rapidly as the numerator and denominator of the fraction increase. The KAM theorem requires that the secondary resonances of appreciable amplitude be well separated in order that invariant tori may exist.

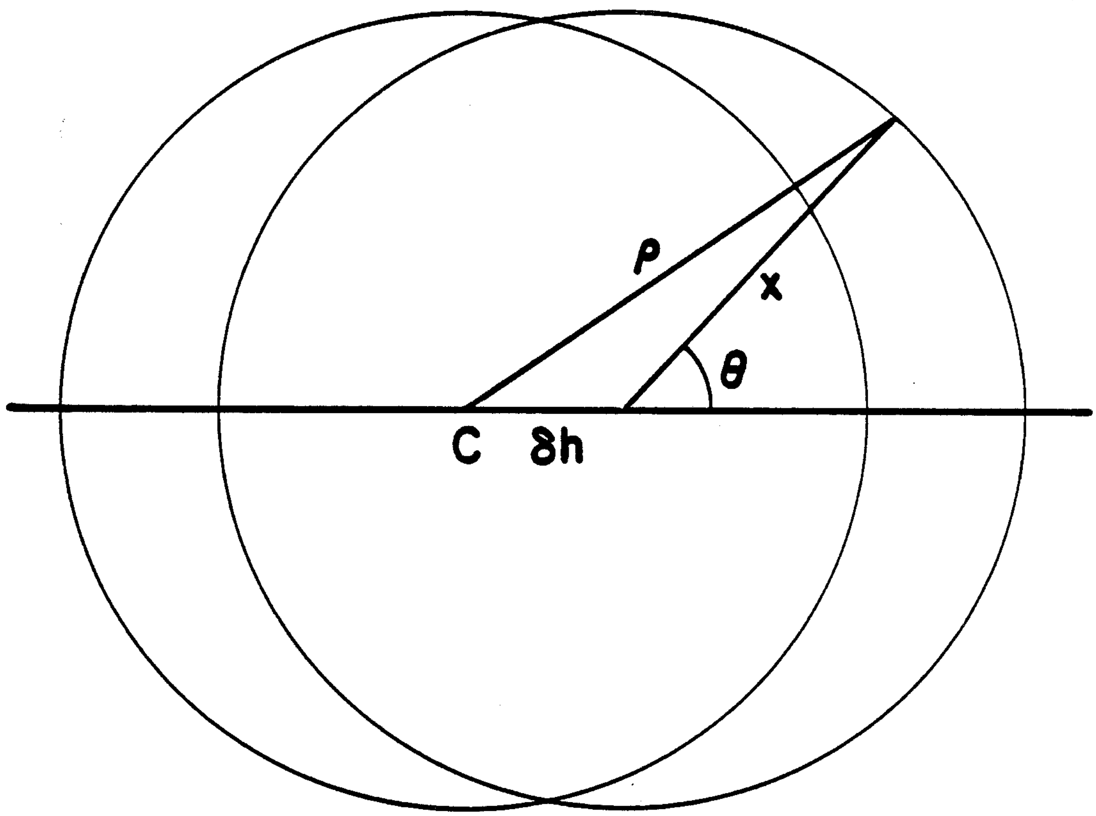

Figure 3 Lateral Hose Deflection of a Beam

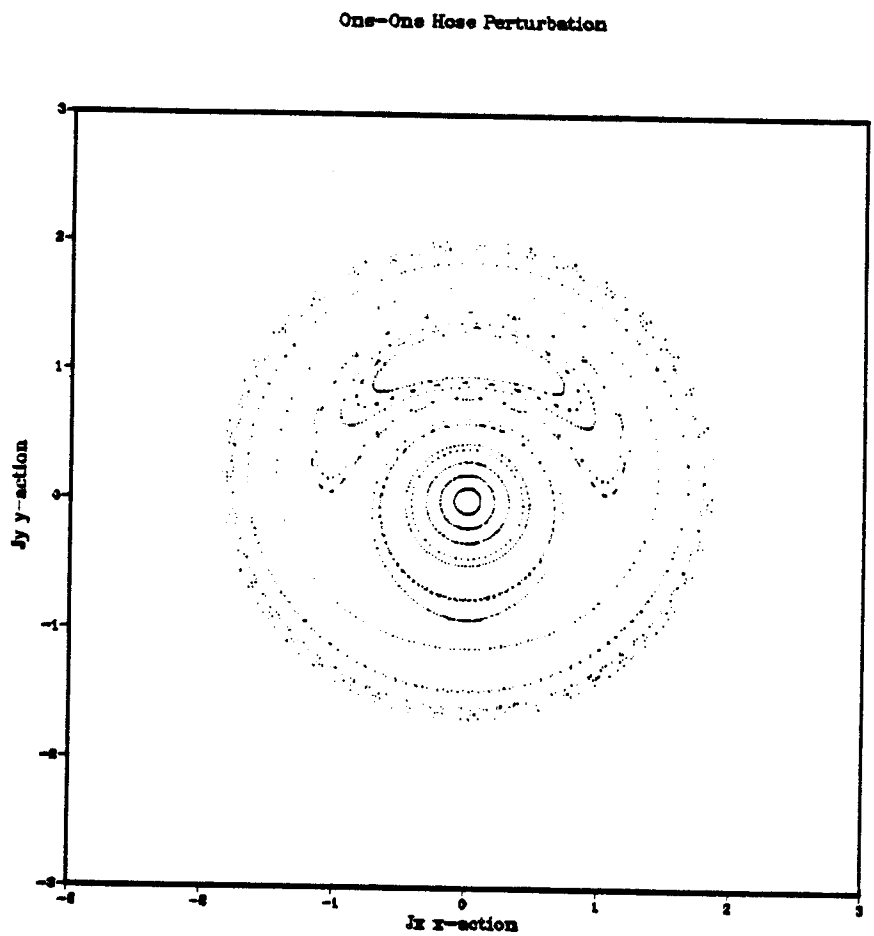

Figure 4 $x_o = 0.5$ and $s = 0.01$; depicting the growth of a primary island with elliptic point at $\psi = \pi/2$ and hyperbolic point at $\psi = -3\pi/2$. The separatrix orbit is not shown.

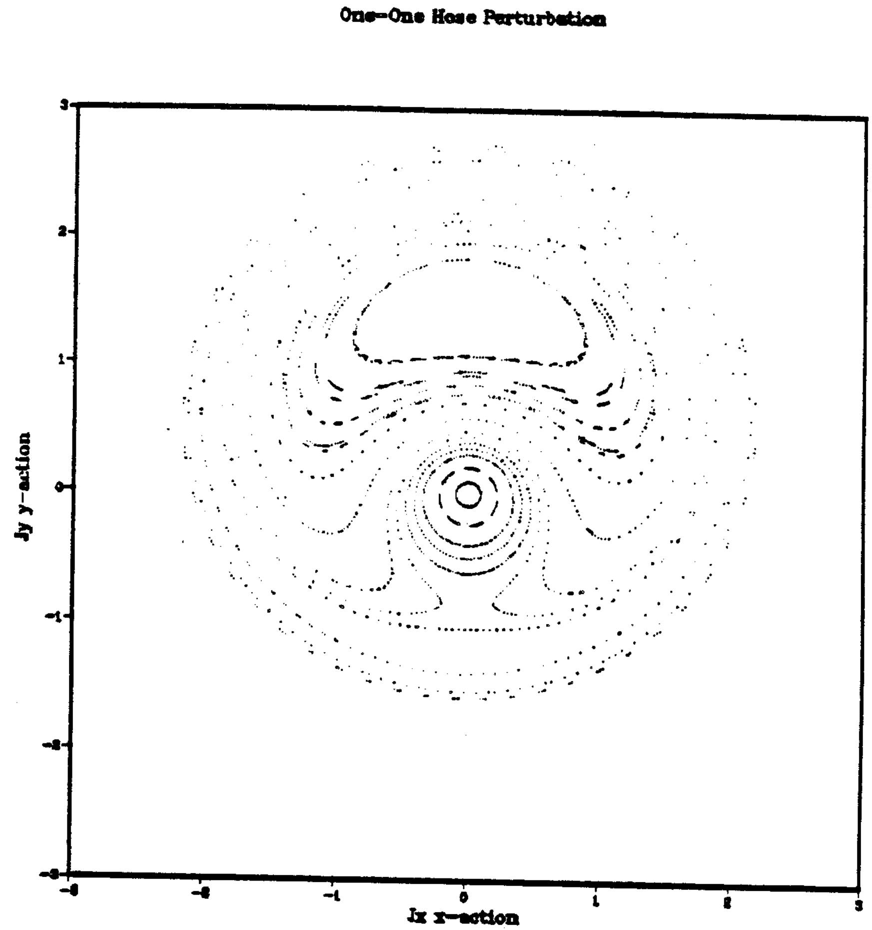

Figure 5 $x_o = 1.0$ and $s = 0.01$. At one Bennett radius the primary island has grown to an appreciable amplitude for this very small lateral displacement.

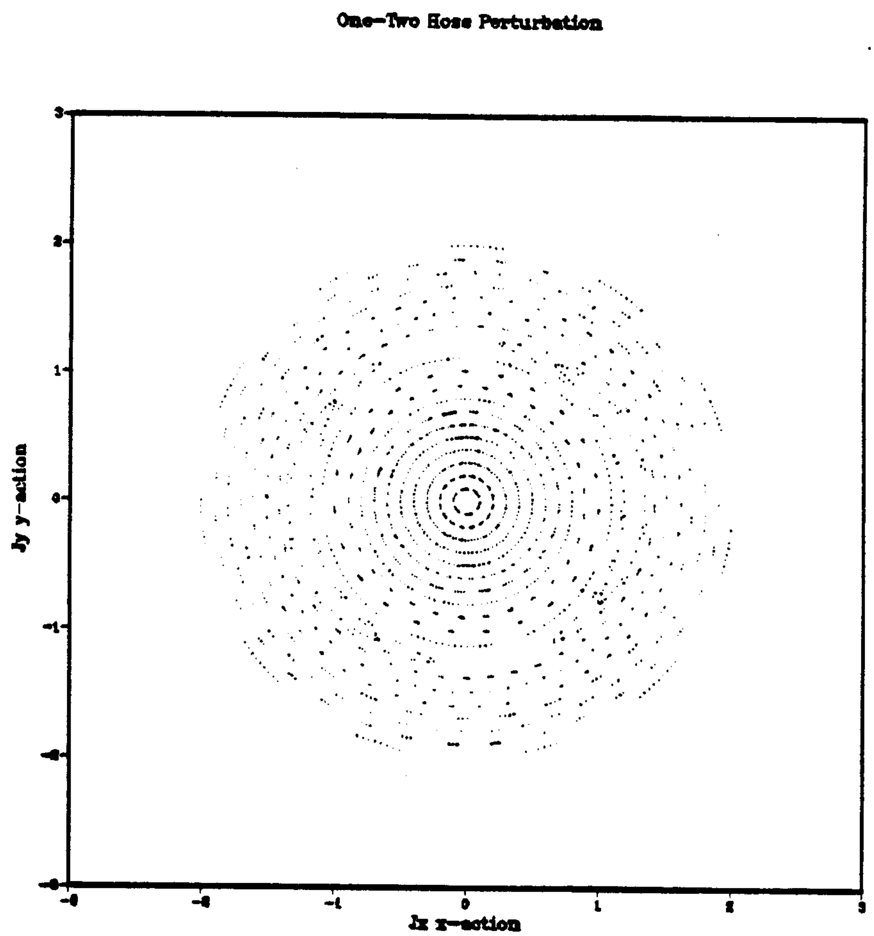

Figure 6 $x_o = 0.5$ **and** $s = 0.01$**. The two thin islands are visible.**

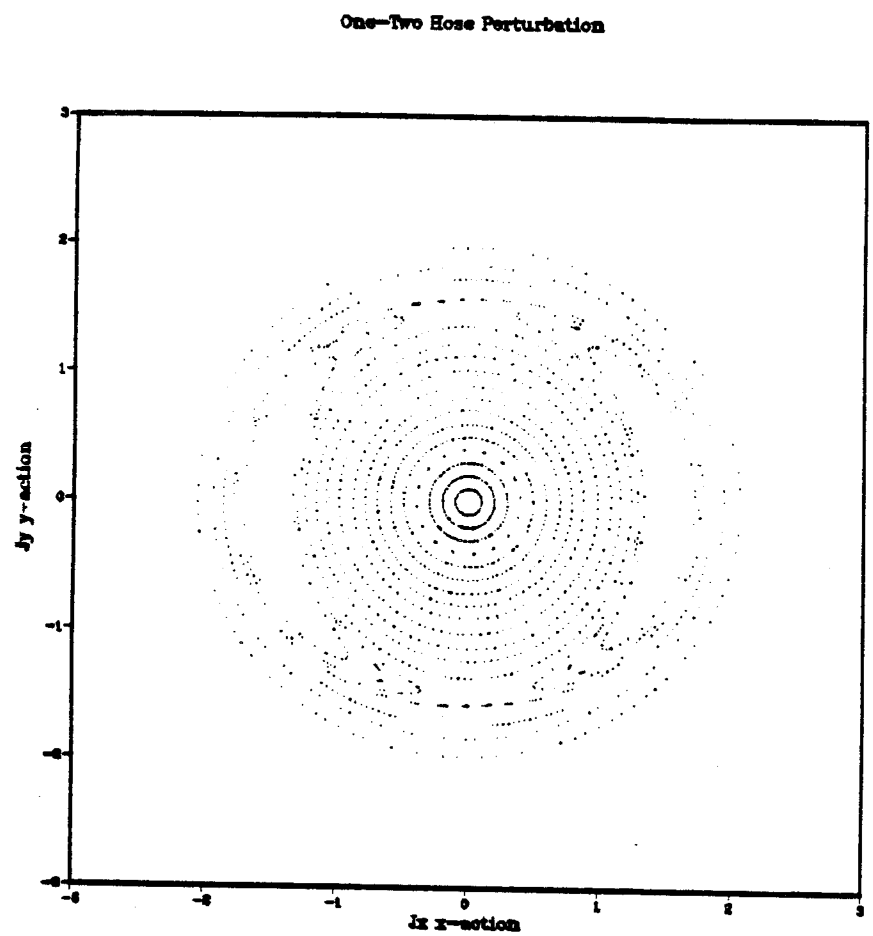

Figure 7 x_o = 1.0 and s = 0.01. The section shows again the increase in amplitude as compared to one-half the Bennett radius.

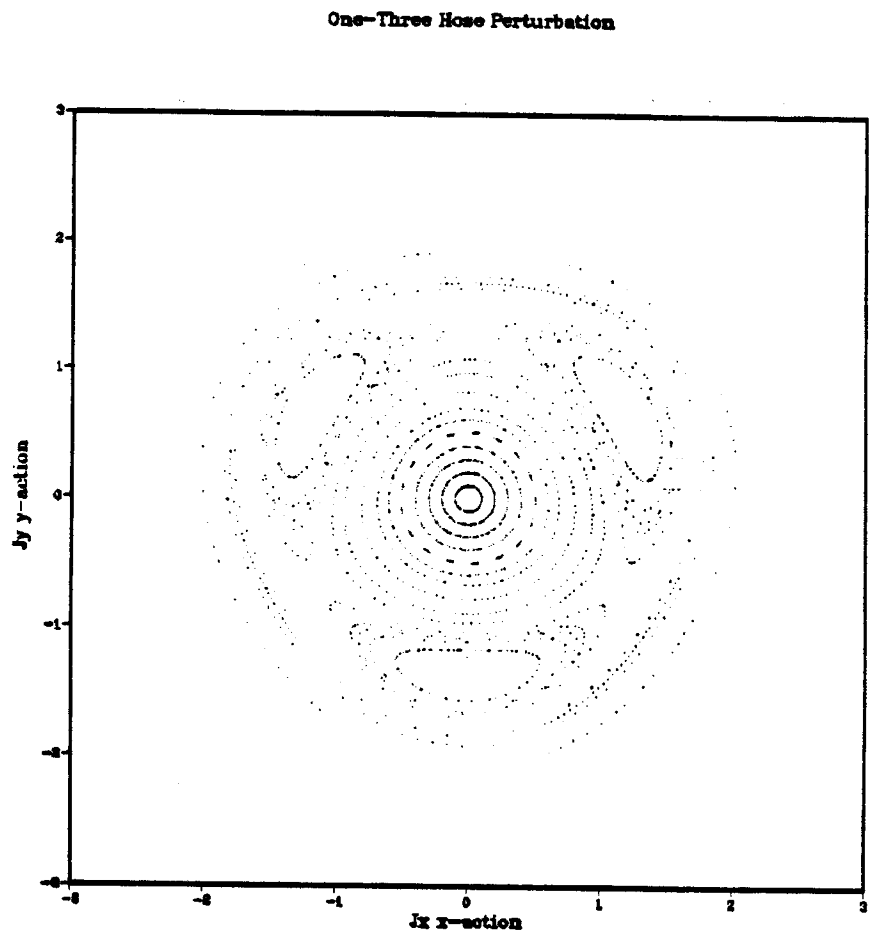

Figure 8 One–Three Isolated Resonance: $x_o = 0.5$ and $s = 0.01$.

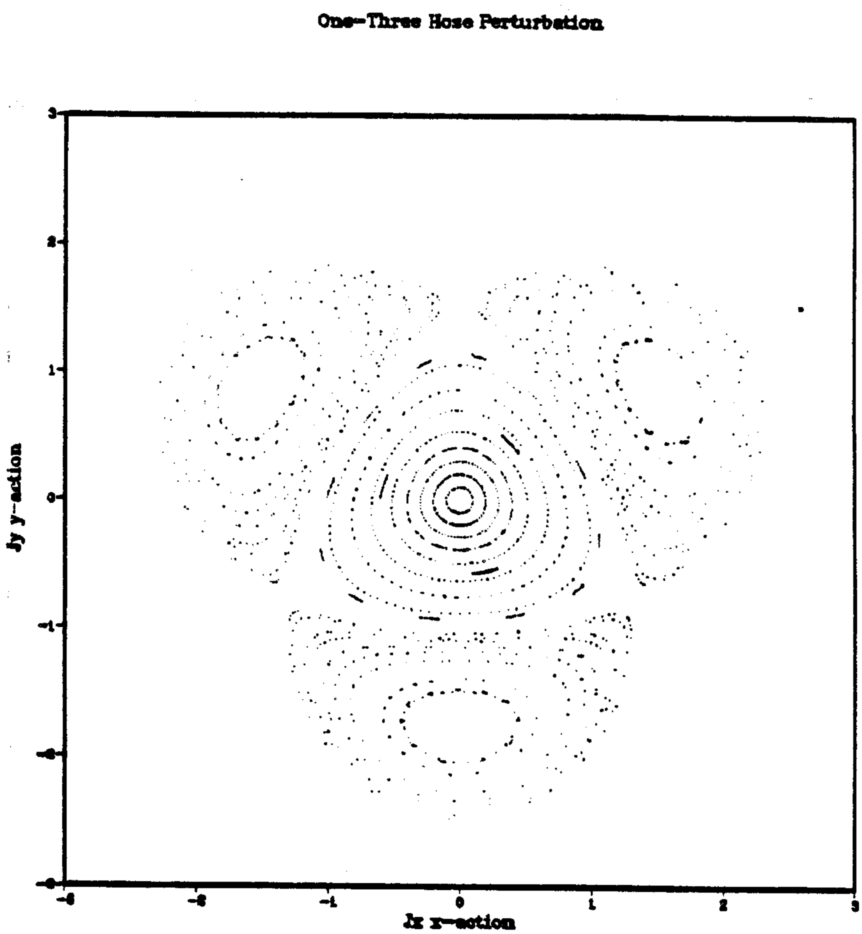

Figure 9 One–Three Isolated Resonance: $x_o = 1.0$ and $s = 0.01$.

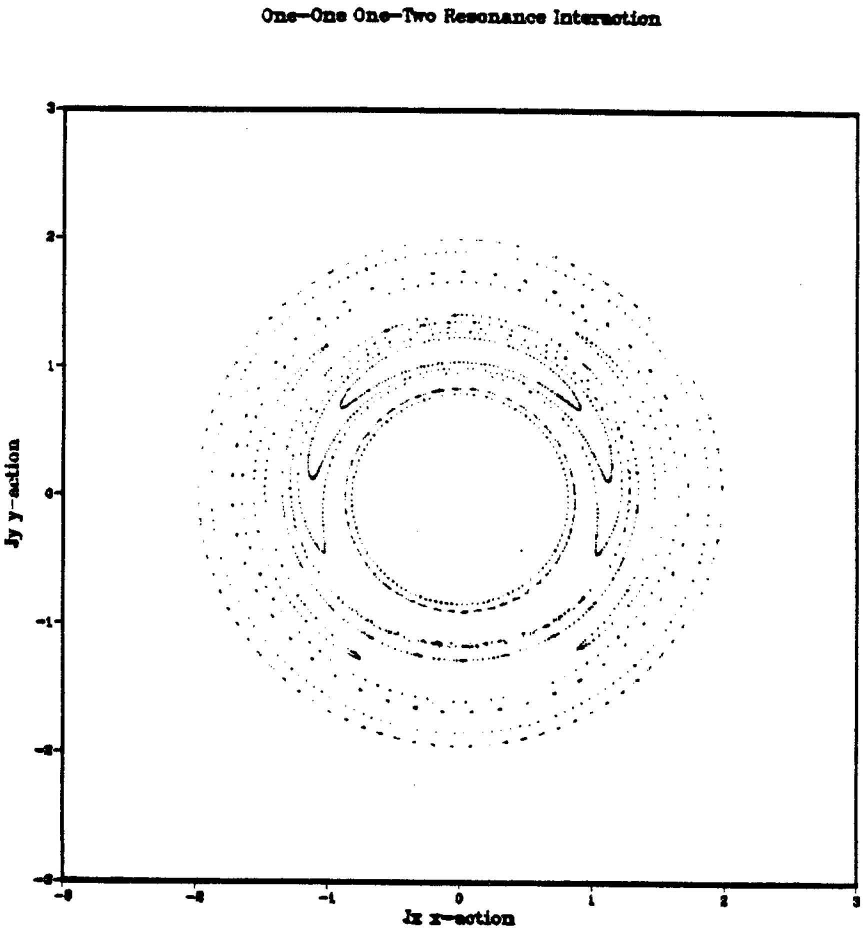

Figure 10 $x_o = 1.0$ **and** $s = 0.0005$**. A thin stochastic layer around the separatrix near the unstable hyperbolic point is emerging.**

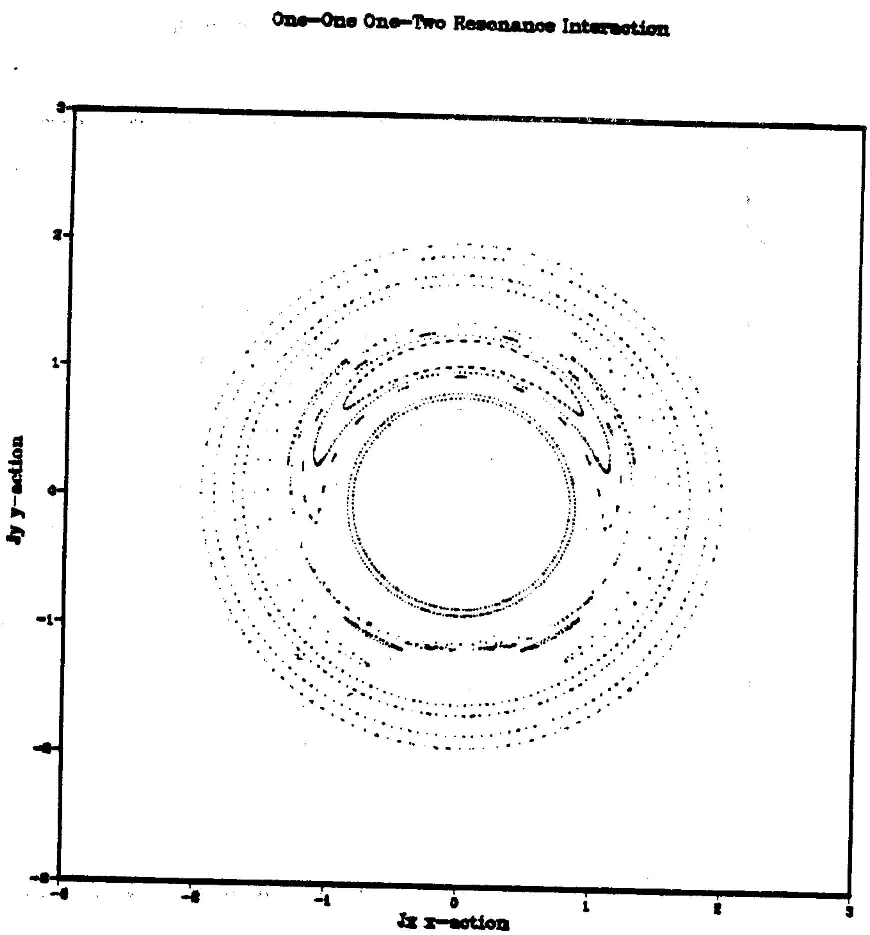

Figure 11 $x_o = 1.0$ and $s = 0.0006$. Island interaction has generated visible satellite islands and the stochastic layer width has increased.

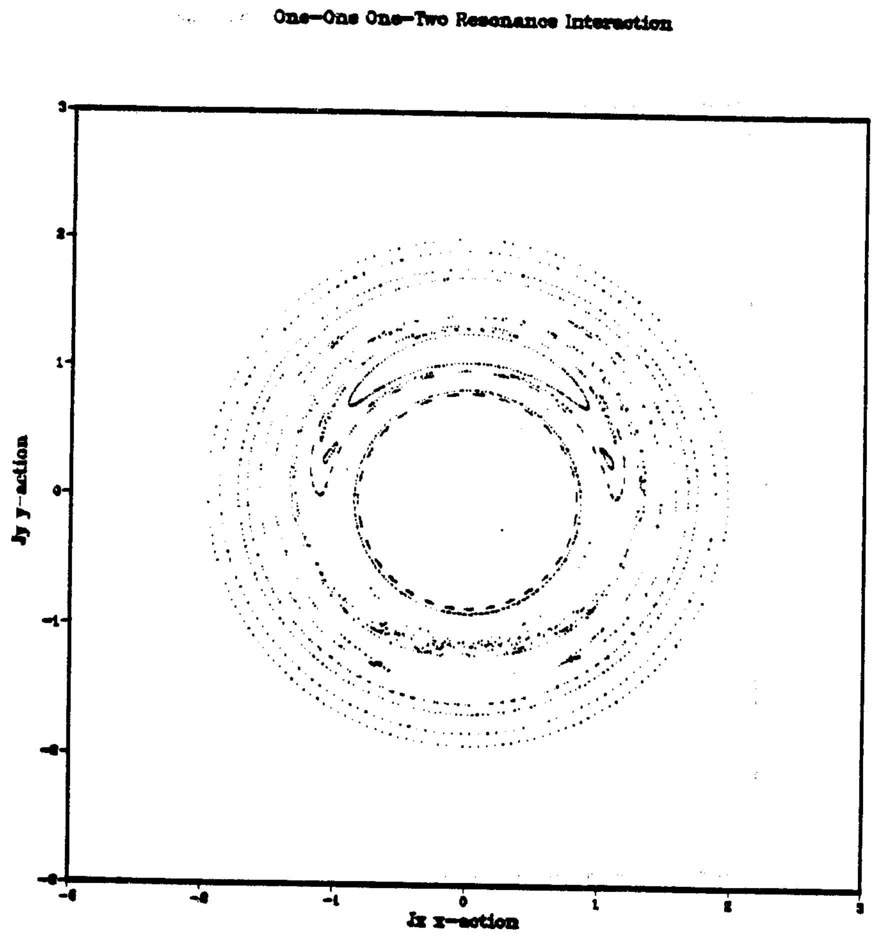

Figure 12 $x_o = 1.0$ **and** $s = 0.0007$. **Further widening of the stochastic layer and more satellite island structure is visible.**

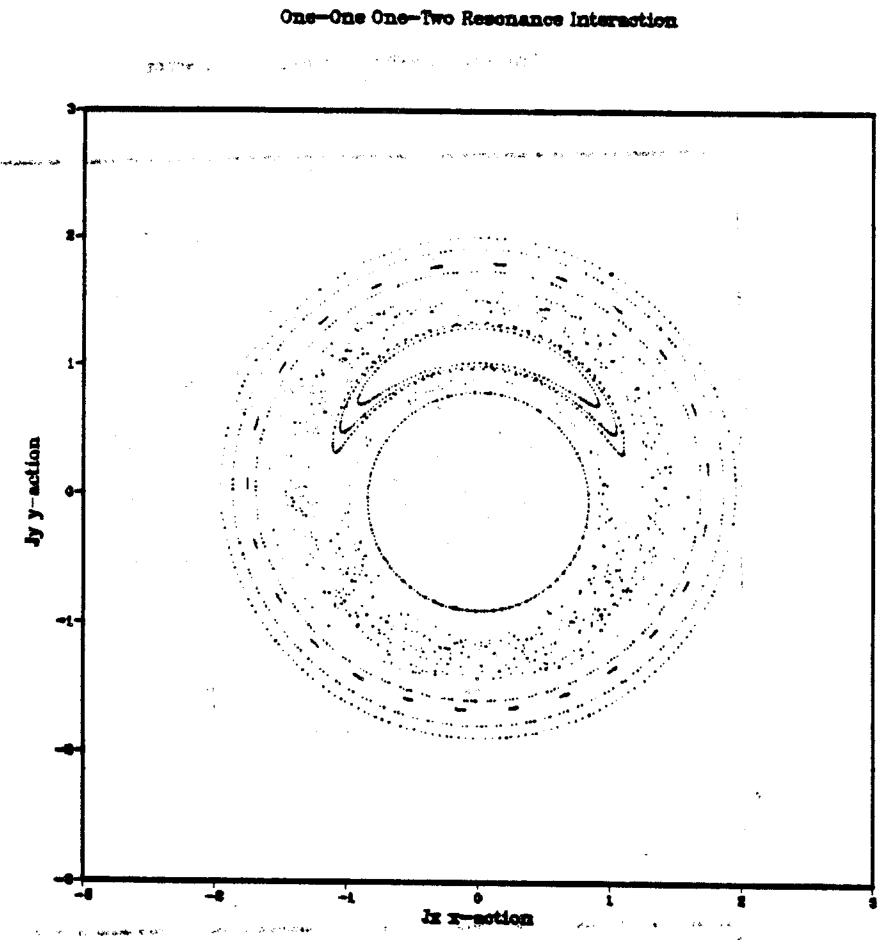

Figure 13 $x_o = 1.0$ and $s = 0.001$. The one–two islands have shrunken out of visibility; a wide stochastic layer surrounds the one–one island but is still well bounded by KAM tori.

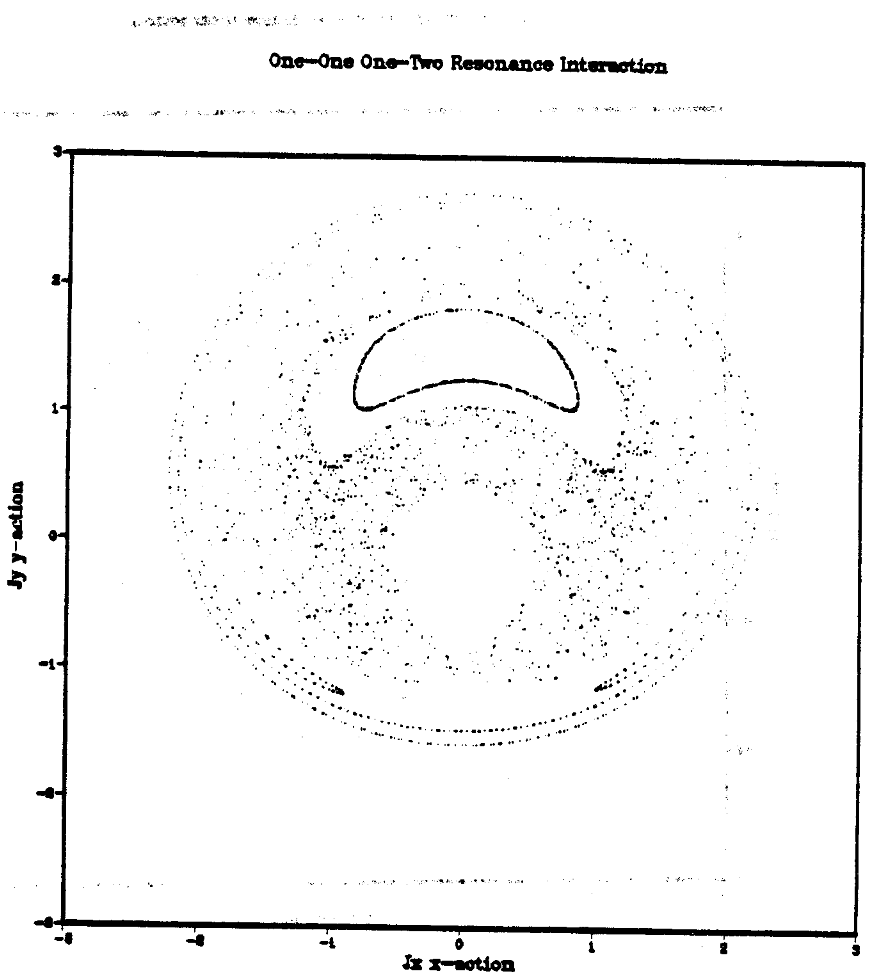

Figure 14 $x_o = 1.0$ and $s = 0.01$. The inner tori are shrinking as the stochastic region encompasses more and more of the section. One can see very thin remnants of the one–two islands.

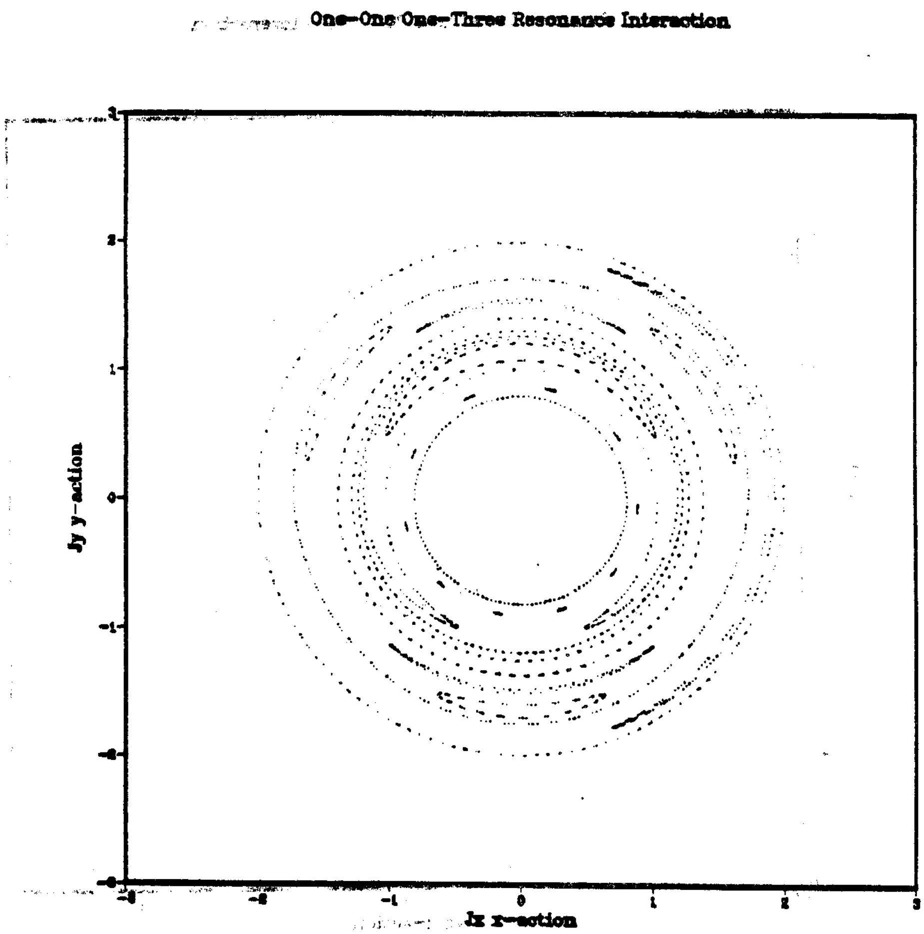

Figure 15 $x_o = 1.0$ and $s = 0.001$. Interaction between the one–one and one–three islands has generated satellites but there is no apparent instability yet.

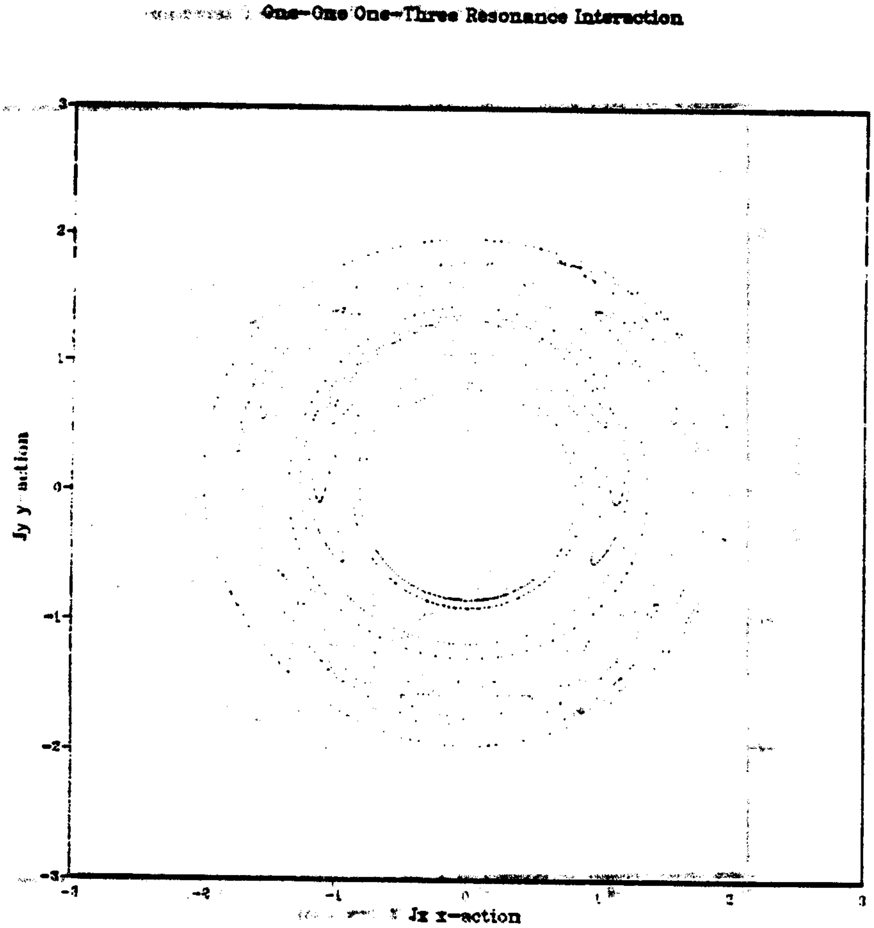

Figure 16 $x_o = 1.0$ **and** $s = 0.0003$. **One can see some instability around the separatrices of the one–three island chain bounded by as yet unaffected tori.**

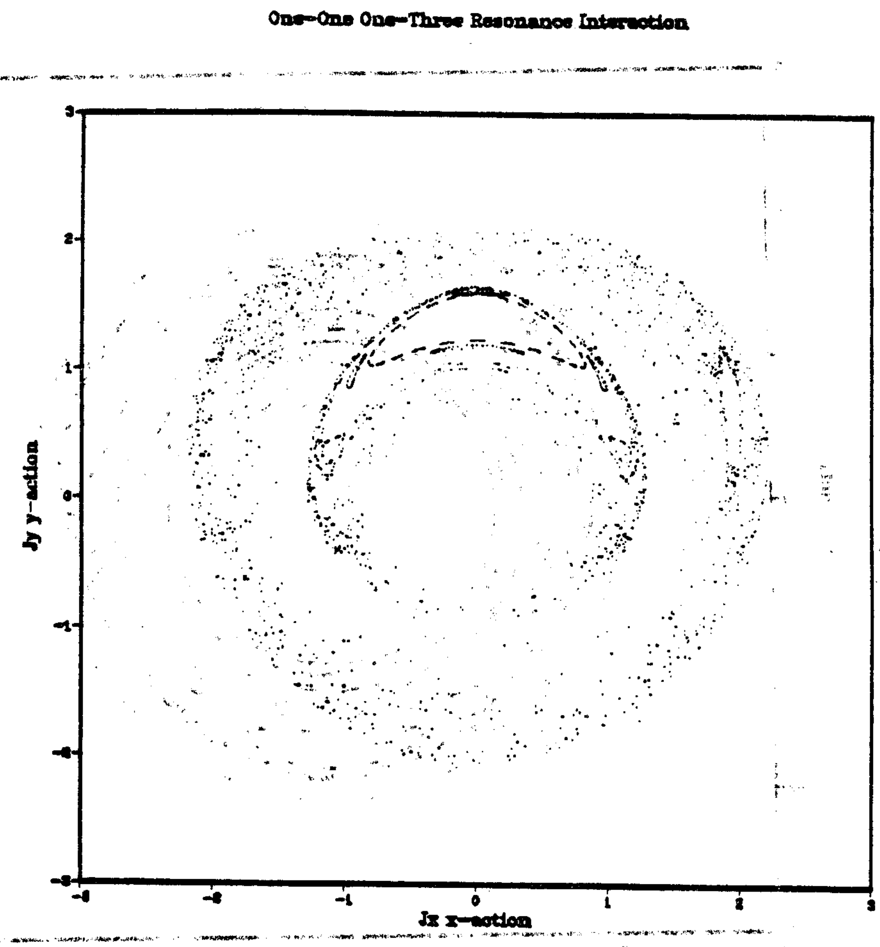

Figure 17 $x_o = 1.0$ and $s = 0.01$. Here the situation has changed dramatically, though good tori are still bounding the action quite effectively.

One–Two One–Three Resonance Interaction

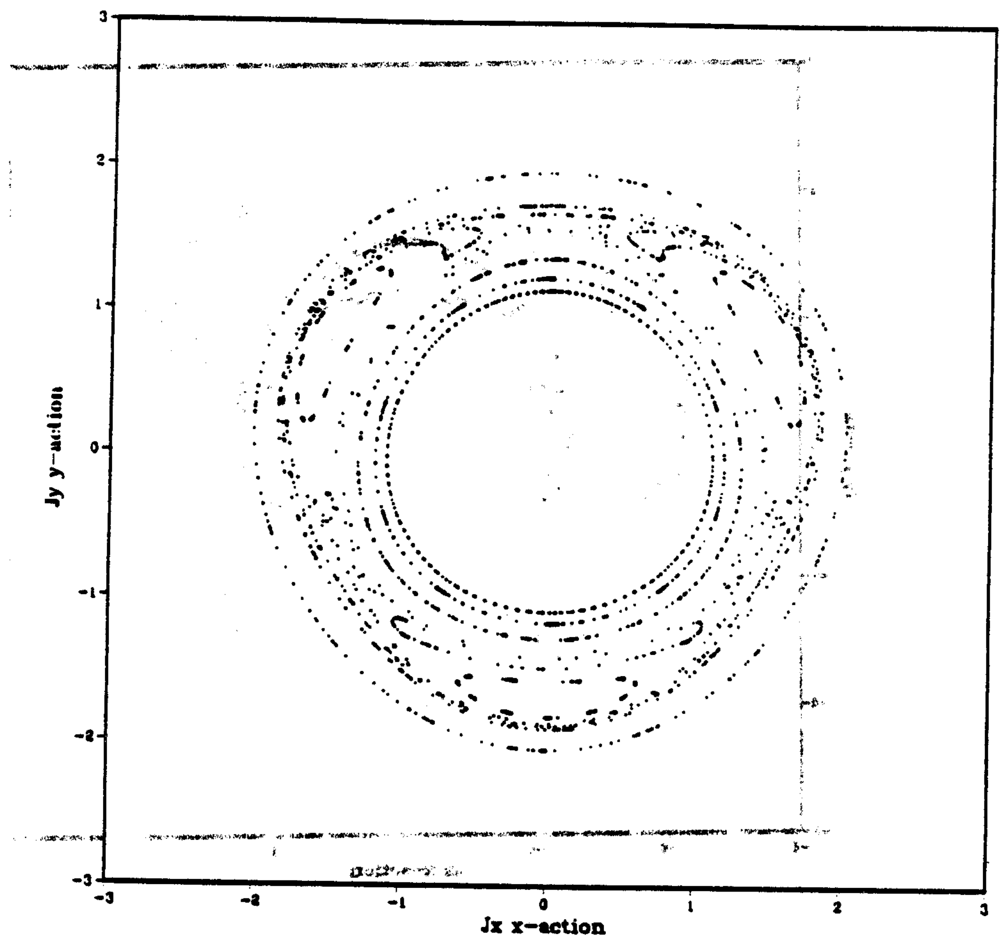

Figure 18 $x_o = 1.0$ **and** $s = 0.007$.

One–Two One–Three Resonance Interaction

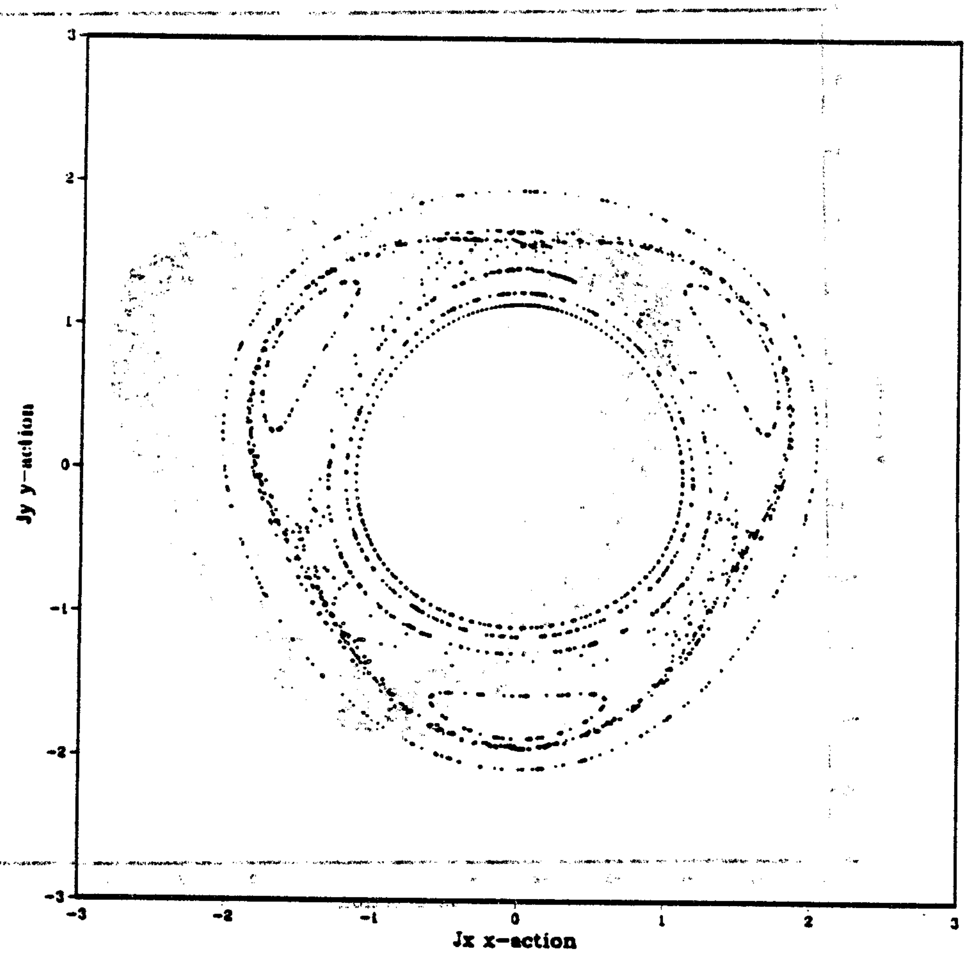

Figure 19. $x_o = 1.0$ and $s = 0.01$. The one–two islands are not even visible; there is some instability near the separatrices of the one–three islands.

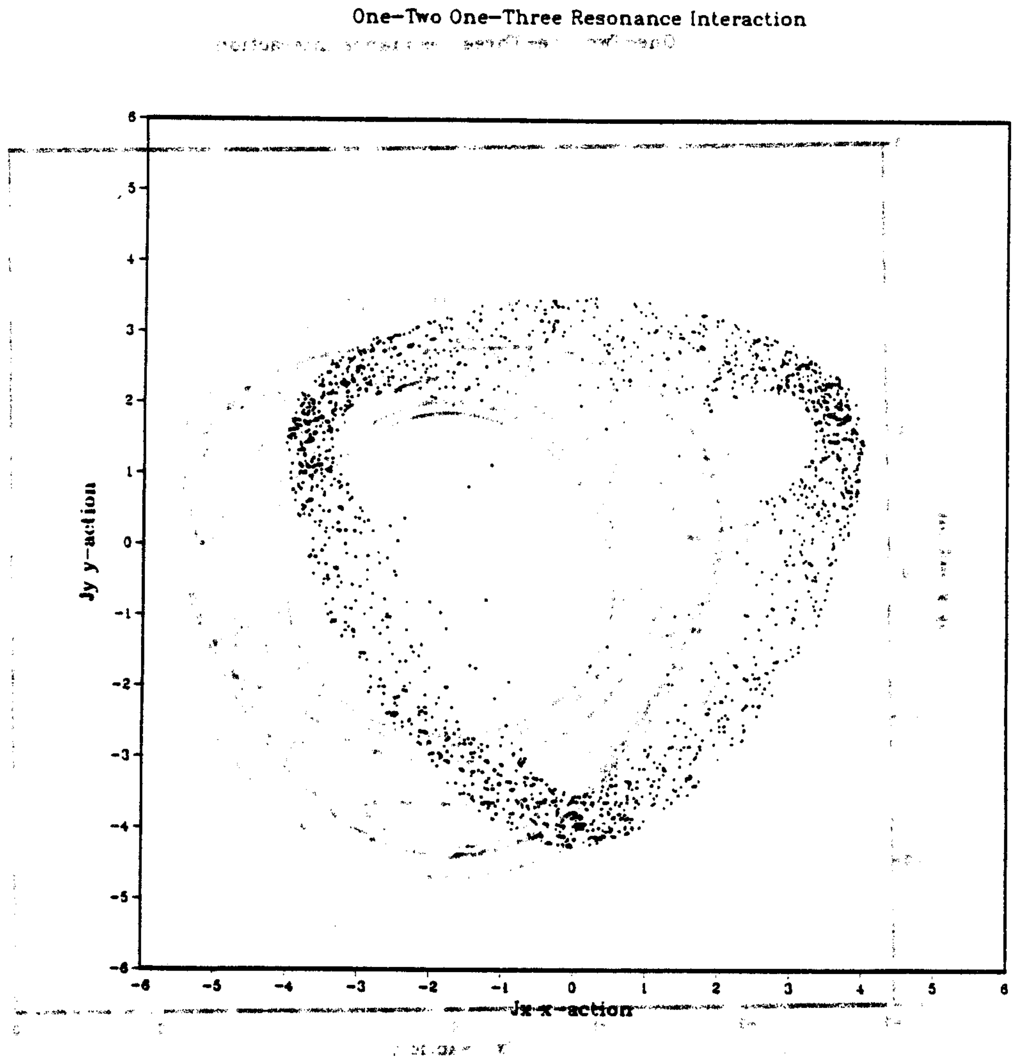

Figure 20 $x_o = 1.0$ **and** $s = 0.1$. **The nine orbits are very unstable but, and this is the significant point, are still bounded within a region of action space.**

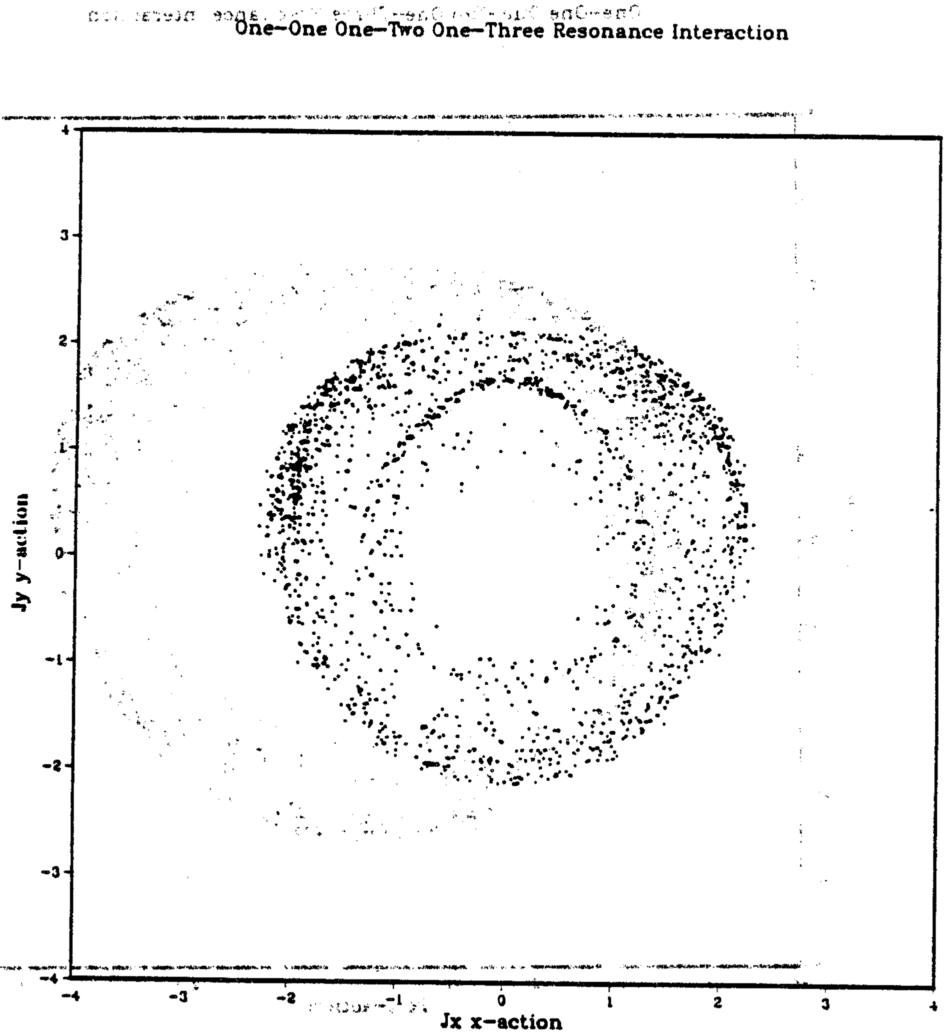

Figure 21 $x_o = 1.0$ **and** $s = 0.01$. **Orbits are locally unstable but well bounded by tori.**

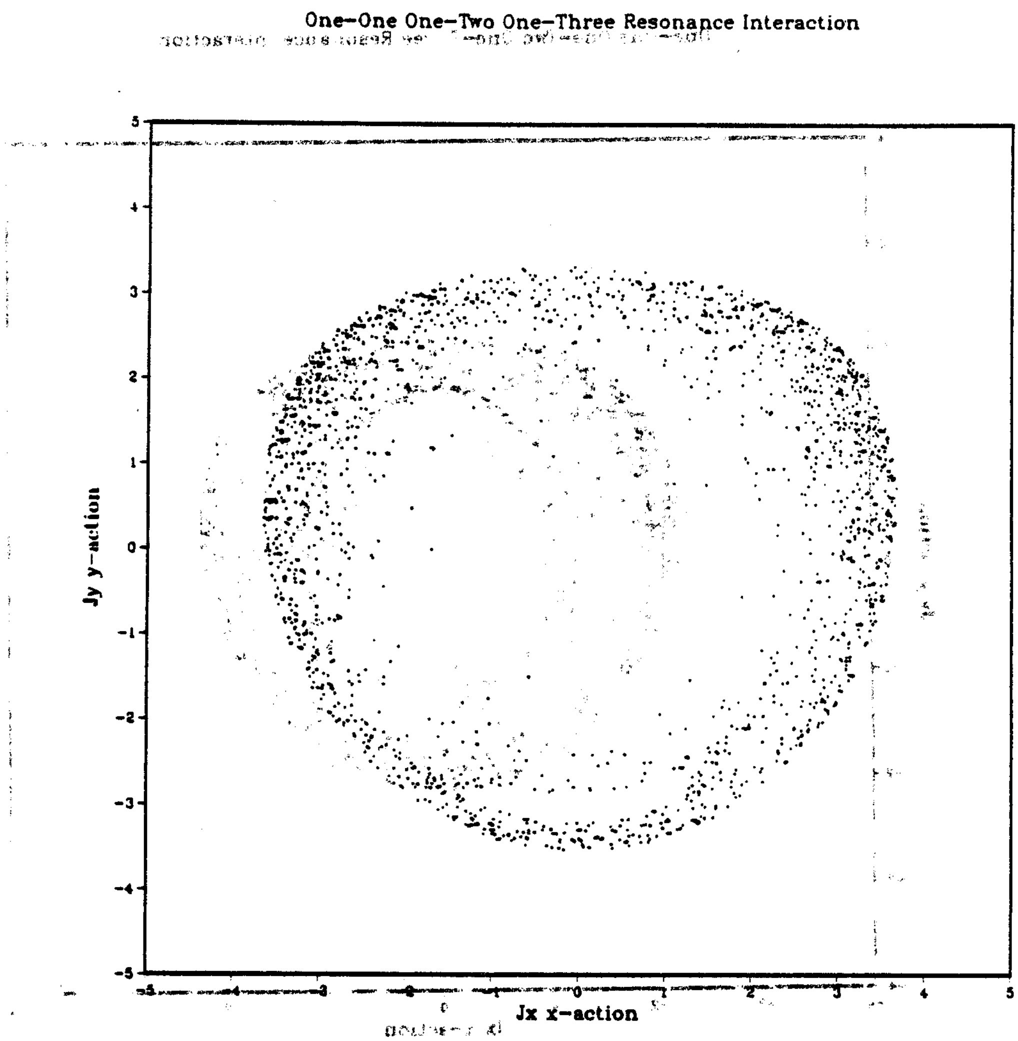

Figure 22 $x_o = 1.0$ and $s = 0.03$. The outer tori are expanding, indicating increasing transfer of energy from the azimuthal to the radial motion. We have found that this continues as the lateral displacement increases further, with no qualitative change in the orbit behavior within the torus.