Modification of Transverse Invariants

K.J. O'Brien and G.A. Moses

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Adiabatic Beam Theory: Resonant Modification of Transverse Invariants

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Contents

1 Introduction

When a particle beam undergoes a weak, if such a thing is possible, hose instability, hose waves travel back from the head to the tail of the beam, growing in amplitude as they do. In experiments on intense electron beam propagation the result is more often in the nature of a catastrophic hose instability and the concept of a well behaved equilibrium undergoing weak perturbations of any sort is dubious. Nevertheless, in order to have a well defined model to discuss, in this work we are going to postulate such benign hosing of a Bennett equilibrium. In any case for heavier particles any hosing is apt to be much less severe than it is for electron beams.

Anharmonic phase mixing effects are of great importance when attempting to analyze nonaxisymmetric phenomena such as hosing. The mixing arises due to the radially dependent betatron frequencies which occur when particles travel in a nonlinear potential such as the pinch potential. The pinch potential is defined as $\psi = \beta A_z - \phi$ and determines the transverse particle dynamics. In terms of the pinch potential the radially dependent betatron frequency is:

$$\omega_\beta^2(r) = \frac{q}{\gamma m}\left(-\frac{1}{r}\frac{d\psi}{dr}\right)$$

If we imagine the beam to be composed of a sequence of slices then as the hose disturbance passes a given slice the slice experiences a shaking back and forth. Individual particles in the slice respond in a manner which is dependent upon the betatron frequency. If $\Omega = \omega - \beta ck$ is the Doppler-shifted hose frequency then particles for which $\Omega \lesssim \omega_\beta$ are maximally affected and the beam is plastically deformed. It has been known since the original work in the 1950's [1] that radially dependent frequencies, and in the case of relativistic beams, relativistic mass spread, cause phase mix damping of the hose motion, that is, damping of the response of a slice, if one takes this point of view, and the convection of the hose backwards in the beam frame.

Lee [2] exploited the known damping and convection due to relativistic mass spread, in the first model of hosing beams to attempt to accurately model anharmonic effects, by introducing a fictitious mass spread, that is, a distribution of particle masses, chosen to recover known simulation growth rates. In his distributed mass model one imagines the beam to be composed of groups of particles the masses of which vary, from group to group, from a certain minimum mass to effectively infinite mass. Each slice is composed of a superposition of disks labeled by the mass of the particles in the disk, and the radial shape of each disk is that of the undisturbed beam profile. Each disk undergoes rigid harmonic oscillations in response to the linearized JxB force, averaged over the radial profile of the disk. Phase mixing results due to the different shaking frequencies of the disks. While this model is useful it has the feature that the radial dependence of the betatron frequency is not accounted for since each disk has the same shape.

Subsequent work by Uhm and Lampe [3] involved dividing the beam into groups of particles with differing transverse energy, each energy group having a partial density profile flat out to an energy dependent maximum radius beyond which it vanishes. The energy group model thus incorporates radial dependence of betatron frequencies. Each energy group is treated as a rigid wafer whose center of mass oscillates in the transverse plane in response to the disk averaged, linearized JxB force.

Sharp, Uhm, and Lampe [4] improved upon this approach by developing the multicomponent model, in which the beam is divided up into groups of particles each with a particular narrow range of azimuthal frequency. Each component is localized within a maximum radius which in this case is dependent upon azimuthal frequency. Each disk oscillates rigidly and harmonically in response to the disk-averaged linearized JxB force. As demonstrated by these authors, the multicomponent model has the powerful advantage of successfully duplicating the important analytical and resonance structure of the exact Vlasov theory, in particular, radial localization of the resonance.

The adiabatic beam theory of Mark et al.[5] opens the door to yet another multi-group approach to model nonaxisymmetric beam phenomena. In the context of Mark's model; it is possible to construct a multi-ring model; in place of disks one uses rings of particles, each of which consists of particles with a narrow distribution of azimuthal angular momentum, centered upon a reference circle orbit determined by the mean angular momentum of the particles in the ring. Each ring responds rigidly to the ring-averaged, linearized, JxB force.

A multi-ring approach has the distinct advantage of allowing one to build up hollow equilibria, something which cannot be done with multi-group models involving disks. As such hollowing beams have been theoretically predicted [6] and numerically observed [7] this is of some importance.

In this work we are going to consider the hypothetical but physically realistic (at least for heavier ion beams) situation of a Bennett beam undergoing weak and slow, in a sense to be made precise later, hose perturbations. The intent is to investigate the effect of such a nonaxisymmetric perturbation upon the dynamics of transverse particle motion. It is clear that in case of catastrophic destruction, due to large amplitude hosing, all bets are off in any save a fully nonlinear theory. If intense particle beams ever propagate for many betatron wavelengths they will certainly undergo benign hosing at worst, so we feel that the weak hose situation is a fruitful case to study.

We will not have time to adequately discuss the philosophy underlying Mark's adiabatic beam theory, its potential application to simulation, within a drift kinetic or even fully hydrodynamic picture, or its relation to more conventional particle simulation codes [24] except to say that one important reason for studying adiabatic beam theory is the hope that computer simulations based upon such fluid-kinetic hybrids might allow one to study beam propagation over many more betatron wavelengths than is possible using brute force particle simulation. There is also the fact that the particle codes exhibit certain numerical pathologies and predict certain nonphysical results [24] when simulating nonaxisymmetric beam phenomena. For more information on adiabatic beam theory one should consult the papers of Mark et al. [38].

The organization of this report is as follows: In section 2 we briefly develop the relativistically covariant formulation of Bennett equilibria in preparation for the investigation of weak hosing of such equilibria. Following this, in section 3 we proceed to our discussion of transverse beam dynamics in an equilibrium Bennett beam as a Hamiltonian system. In section 4 we employ canonical Lie transform perturbation theory to get the ground state Hamiltonian in action angle variables. In section 5 we analyze the families of nested tori of the equilibrium beam system. In section 6, within the slow hose context, we end up with an autonomous perturbed system involving nonlinear coupling resonances whose properties we examine in section 7. In section 8 we investigate resonance overlap using various tools available in the modern arsenal of Hamiltonian mechanics; our main concern will be with the extent to which the adiabatic assumptions underlying drift-kinetic beam models hold up for the benign, slow, weak, hose. In section 9 we present our conclusions. For completeness we include Appendices A and B dealing with Lie transforms.

2 Relativistically Covariant Bennett Equilibria

Consider a superposition of charged particle streams each with fluid velocity directed along the z-axis of an otherwise arbitrary laboratory frame. In the original work [8] Bennett spoke of a superposition of two oppositely charged particle streams; in the formulation presented here we find a natural restriction to only two streams. The basic assumption is that each particle stream is distributed in phase space as a Maxwell-Boltzmann distribution in its own fluid rest frame. In any frame the Maxwell-Boltzmann distribution for species σ is defined as:

$$f_\sigma = N_\sigma exp(-a^\sigma_\mu P^\mu_\sigma)$$

which is a Lorentz scalar. In the lab frame the four-vectors a and P have contravariant components (we use the Feynman metric $g = diag(1, -1, -1, -1)$):

$$a^\mu_\sigma = (\frac{c}{T_\sigma}, 0, 0, \frac{v_\sigma}{T_\sigma})$$

$$P^\mu_\sigma = (\frac{H_\sigma}{c}, 0, 0, P^\sigma_z)$$

where T is the temperature, v the fluid velocity, H the Hamiltonian, and P the canonical momentum. The vector a is timelike in the lab frame:

$$a_\mu a^\mu = (\frac{c}{\gamma T})^2$$

and is thus timelike in any frame. If we contract a with the four-vector potential $A^\mu = (\phi, \vec{A})$ we arrive at the covariant pinch potential:

$$\psi_\sigma = \frac{q_\sigma}{c} A_\mu a^\mu$$

Note that this is $-q_\sigma/T_\sigma$ times the pinch potential which we mentioned in the introduction. In a frame for which $\phi = 0$ the particle stream is said to be a pure pinch. In the fluid rest frame of the sigma stream the vector a has no spacelike components and the covariant pinch potential is:

$$\psi_\sigma = \frac{q_\sigma}{T_\sigma} \phi$$

This is the pinch potential we shall be concerned with in the next section of the paper. The current density four vector for the sigma species is defined as:

$$j^\mu = (n_\sigma c q_\sigma, \vec{J}_\sigma)$$

$$n_\sigma = \int d^3 p f_\sigma$$

$$\vec{J}_\sigma = q_\sigma c \int d^3 p f_\sigma \vec{p} (p^2 + m_\sigma^2 c^2)^{-\frac{1}{2}}$$

Calculating j^μ explicitly yields the result:

$$j_\sigma^\mu = c q_\sigma N_\sigma Q_\sigma a_\sigma^\mu e^{-\psi_\sigma}$$

$$Q_\sigma = \frac{4\pi m_\sigma^2 c^2}{a_\mu^\sigma a_\sigma^\mu} K_2[m_\sigma c \sqrt{a_\mu^\sigma a_\sigma^\mu}]$$

where K_2 is the modified Bessel function of the second order. In this formula all terms are manifestly covariant and one therefore writes the total current in an arbitrary frame (with z axis along that of the lab and streams) as:

$$j^\mu = \sum_\sigma j_\sigma^\mu$$

The total current provides the source in the inhomogeneous Maxwell equation:

$$(\partial^\lambda \partial_\lambda g_\nu^\mu - \partial^\mu \partial_\nu) A^\nu = \frac{4\pi}{c} j^\mu$$

If we contract the inhomogeneous Maxwell equation (in the Lorentz gauge) with the four vector $\frac{q_l}{c} a_\mu^l$ we get a nonlinear equation for the pinch potential

$$\frac{1}{r} \frac{d}{dr} r \frac{d}{dr} \psi_l = -\frac{4\pi}{c} \sum_\sigma q_\sigma q_l N_\sigma Q_\sigma a_\mu^\sigma a_l^\mu e^{-\psi_\sigma}$$

where we assume an axisymmetric ground state and treat the hypothetical case of a long beam so that $\partial/\partial z = 0$. To get Bennett equilibria we invoke an ansatz which converts this equation into the Poisson-Boltzmann equation. To this end let us force the covariant pinch potentials of the streams to be related to one another by:

$$e^{-\psi_\sigma} = b_\sigma e^{-\psi_0}$$

where b_σ is to be determined by forcing self-consistency, that is, forcing ψ_0 to solve the Poisson-Boltzmann equation:

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\psi_0 - 2e^{-\psi_0} = 0$$

This equation results only if we have:

$$\frac{2\pi}{c}\sum_\sigma q_\sigma q_l N_\sigma Q_\sigma b_\sigma a^\sigma_\mu a^\mu_\sigma = -1$$

so that if we define the matrix m:

$$m_{l\sigma} = -\frac{2\pi}{c} q_\sigma q_l N_\sigma Q_\sigma a^\sigma_\mu a^\mu_l$$

we may solve for the unknown coefficients by inversion of the system of linear equations :

$$b_\sigma = (m^{-1})_{\sigma l} 1_l$$

The matrix m is square and S by S where S is the number of streams. It is easily proved that m is in fact singular for more than two streams, hence the theory only accommodates two streams of particles. The vector potential A_μ is gotten by inverting the system:

$$\psi_\sigma = \frac{q_\sigma}{c} a^\sigma_\mu A^\mu$$

If we explicitly refer to ions and electrons $\sigma = (i, e)$ we arrive at the following formulae for the scalar and vector potentials:

$$A_0 = \frac{c}{q_i q_e}\frac{q_i a^i_3 - q_e a^e_3}{a^e_0 a^i_3 - a^i_0 a^e_3}\psi_0 = \phi$$

$$A_3 = \frac{c}{q_i q_e}\frac{q_e a^e_0 - q_i a^i_0}{a^e_0 a^i_3 - a^i_0 a^e_3}\psi_0 = -A_z$$

From the relation for the scalar potential we can see under what conditions a pure pinch results in an arbitrary frame, with respect to which this formula is defined, namely:

$$\frac{v_e q_e}{T_e} = \frac{q_i v_i}{T_i}$$

in terms of physical variables. This is the same as the condition one arrives at in a nonrelativistic treatment. Notice that a pure pinch involves counter-streaming of the ion and electron beams. One can show that the pure pinch results in the same three-vector current density in all three frames of reference, lab, ion fluid, and electron fluid rest frames; it is of course this net current which provides the pinch force. We note that in the fluid rest frame of species σ the covariant potential ψ_σ is a pure scalar potential as mentioned earlier.

The Poisson-Boltzmann equation is a "classical" equation of mathematical physics [10] and admits a class of self-similar solutions [11] which are invariant under the group of transformations:

$$r' = e^{-\frac{1}{2}\alpha} r$$

$$\psi_0' = \psi_0 + \alpha$$

Here we find the mathematical basis for the self-similar expansion, that is, the Nordsieck expansion, of self-pinched particle beams [9]. The most general solution yielding a positive density $n = n_0 e^{-\psi_0}$ is:

$$\psi_0 = 2log[\frac{\Re^2\eta + \Im^2\eta + 1}{\sqrt{\frac{\partial}{\partial\vec{r}}\Re\eta \cdot \frac{\partial}{\partial\vec{r}}\Re\eta}}]$$

where η is an arbitrary harmonic function $\nabla^2\eta = 0$. Generalized Bennett solutions, gotten by taking the most general case of axisymmetry:

$$\Re\eta = (\frac{r}{a})^n cosn\theta$$

$$\Im\eta = (\frac{r}{a})^n sinn\theta$$

are given by:

$$\psi_0 = 2log\frac{a}{2n}[(\frac{a}{r})^{n-1} + (\frac{r}{a})^{n+1}]$$

where n and a are arbitrary positive real numbers. The self-similarity is manifest providing that the scale radius a scales as r, since r scales as $e^{-\frac{1}{2}\alpha}$ under the invariance group. The Nordsieck expansion of a with time preserves the group invariance. Taking $n = 1$ we have the classical Bennett solution:

$$\psi_0 = 2log(1 + \frac{r^2}{a^2}) - 2log\frac{a}{2}$$

taking n larger than 1 yields hollow equilibria, however, here we are concerned only with the classical Bennett. For more information on the family of generalized Bennett distributions one should consult the review [12].

3 Transverse Hamiltonian in a Bennett Equilibrium

In this section we are going to derive the Hamiltonian describing transverse dynamics of a beam ion. In the interest of careful presentation we prefer to develop the theory, as in the previous section, in a covariant fashion. The relativistically covariant formula which implicitly determines the Hamiltonian is:

$$(mc)^2 = p_\mu p^\mu$$

Minimal coupling between a charged particle and an electromagnetic field is described by the replacement:

$$p^\mu \longrightarrow P^\mu - \frac{q}{c}A^\mu$$

where the canonical momentum four vector is:

$$P^\mu = (\frac{H}{c}, \vec{P})$$

For particles in electromagnetic fields the relativistic and gauge covariant formula which implicitly determines the Hamiltonian is $(mc)^2 = p_\mu p^\mu$ subjected to the minimal coupling replacement. We may explicitly solve for the ground state Hamiltonian:

$$H_0 = \gamma_0 mc^2 + q\phi_0 = [(mc^2)^2 + (\vec{p}_0 c)^2]^{\frac{1}{2}} + q\phi_0$$

where the mechanical momentum with respect to our arbitrary frame is:

$$\vec{p_0} = \vec{P_0} - \frac{q}{c}\vec{A_0}$$

Small perturbations of the ground state manifest themselves in the addition of small, nonaxisymmetric, time-dependent terms to the potential, and corresponding terms to the canonical momentum:

$$A^\mu \longrightarrow A^\mu + \delta A^\mu$$

$$P^\mu \longrightarrow P^\mu + \delta P^\mu$$

or, in terms of the explicit time and space parts:

$$\phi = \phi_0 + \phi_1$$

$$\vec{A_0} = \vec{A_0} + \vec{A_1}$$

$$H = H_0 + H_1$$

$$\vec{P} = \vec{P_0} + \vec{P_1}$$

We may get an expression for the perturbation H_1 of the Hamiltonian by variation of the relation $(mc)^2 = p_\mu p^\mu$:

$$p_\mu \delta p^\mu = 0$$

Solving this for H_1 yields:

$$H_1 = q(\phi_1 - \frac{\vec{p_0} \cdot \vec{A_1}}{\gamma_0 mc}) + \frac{\vec{p_0} \cdot \vec{P_1}}{\gamma_0 m}$$

The last term may be eliminated by a redefinition of the canonical three-momentum but not in a gauge covariant fashion. This redefinition amounts to setting the perturbation $\vec{P_1}$ to zero, which one is free to do but such a choice destroys the gauge covariance of the theory. This lack of gauge covariance, in itself, is not necessarily a problem so long as the final results of the calculation are gauge covariant. A similar situation has arisen in a recent paper [13] wherein the authors opt to set the perturbation to zero but demonstrate the gauge covariance of their final results. Failure to obtain a gauge covariant result would place a calculation in direct opposition to the modern concept of gauge covariance as a guiding principle in physics, equal in status to Lorentz covariance and general coordinate covariance. We prefer to avoid tampering with gauge covariance, instead transforming to the frame in which p_{0z} is zero when we write down our Hamiltonian. We have discussed the derivation of H from the point of view of an arbitrary frame of reference in order to motivate the naturalness of the final choice of reference frame, which is neither the lab frame, nor, strictly speaking, the fluid rest frame. The perpendicular terms in $\vec{p_0}$ are not zero in this frame but we argue that the perpendicular components of $\vec{P_1}$ are entirely negligible. Utilizing the fundamental relation $(mc)^2 = p_\mu p^\mu$ we arrive at the Hamiltonian in this frame by the same procedure as above:

$$H_0 = mc^2 + \frac{1}{2m}(p_r^2 + \frac{P_\theta^2}{r^2}) + q\phi_0$$

$$H_1 = q\phi_1$$

Now we argue that the dispersion of the z velocityh in the fluid rest frame is of negligible magnitude. This implies that the special frame we are working in has p_{0z} equals zero for essentially all the ions in the beam; stronger yet, this means that the frame we are in is essentially the fluid rest frame. In the fluid rest frame we recall that the covariant pinch potential takes the form of the last equation in section 1 so we arrive at our final result for the Hamiltonian in the fluid rest frame:

$$H_0 = \frac{1}{2m}(p_r^2 + \frac{P_\theta^2}{r^2}) + 2T\log(1 + \frac{r^2}{a^2}) + mc^2 - 2T\log(\frac{a}{2})$$

$$H_1 = q\phi_1$$

where, of course, the last two terms in H_0 are of no consequence to dynamics. There still remains the calculation of ϕ_1, the perturbed potential due to the hose wave passing through the fluid frame. In the next section we process the ground state Hamiltonian, by means of Lie transforms, into a more useful form in preparation for the application of KAM theory. We derive in section 6 the hose perturbation rendered also in action-angle variables.

4 Action-Angle Variables for the Near-Circle Hamiltonian

Expanding the Hamiltonian H_0 about a reference circle orbit, to lowest order, one gets a system of two linear oscillators for the radial and azimuthal degrees of freedom. If one uses this lowest order Hamiltonian and perturbs it with a weak hose one gets a nonlinear coupling of the two degrees of freedom which is delicate to study since the ground state system fails to satisfy the sufficient nonlinearity condition of the KAM theorem. The linearity of the ground state is not inherent in the physical system but only in the expansion to lowest order in noncircularity. In this work we are investigating the breakdown of adiabaticity and find it necessary to expand the ground state to higher order to expose the inherent nonlinearity. This alleviates the problem with the weak nonlinearity, i.e., the linear ground state, and results in fulfillment of the conditions of the KAM theorem in its strongest form. For more information on KAM theory, in the context in which we are using it, we find Chirikov's report [14] and the recently published book by Lichtenberg and Lieberman [15] to be particularly valuable. Expanding to fourth order in the small parameter $\epsilon \sim \delta r/r_0$ we get the transverse Hamiltonian:

$$H_0 = \Omega_0 J_\theta + \epsilon^2 \nu_0 J_r + \epsilon^3 \nu_1 J_r^{\frac{3}{2}} \sin^3 \psi + \epsilon^4 \nu_2 J_r^2 sin^4 \psi$$

where the various parameters are defined by:

$$\Omega_0^2 = \frac{1}{r_0}\frac{dV}{dr_0}$$

$$\nu_0^2 = 3\Omega_0^2 + \frac{d^2V}{dr_0^2}$$

$$\nu_1 = \left(\frac{2}{\nu_0}\right)^{\frac{3}{2}} \left(\frac{1}{6}\frac{d^3V}{dr_0^3} - 2\frac{\Omega_0^2}{r_0}\right)$$

$$\nu_2 = \frac{1}{\nu_0^2}\left(\frac{1}{6}\frac{d^4V}{dr_0^4} + 10\frac{\Omega_0^2}{r_0^2}\right)$$

$$V = 2T log(1 + \frac{r^2}{a^2})$$

and the radial action-angles variables are those of the order ϵ^2 terms of the radial contribution to the Hamiltonian:

$$\delta r = \epsilon(\frac{2}{\nu_0})^{\frac{1}{2}} J_r^{\frac{1}{2}} sin\psi$$

$$\delta p_r = \epsilon(2\nu_0)^{\frac{1}{2}} J_r^{\frac{1}{2}} cos\psi$$

These are not the correct action-angle variables for the radial Hamiltonian as a whole. We employ Deprit's version [16] of Lie transform perturbation theory to get the action-angle variables and the Hamiltonian to the requisite order. Lie transforms operate directly on the functions defined on the phase space manifold and thereby avoid the well-known problem of the classical Poincaré-Von Ziepel canonical perturbation theory, cf. [16], that is, mixing together of the new and old variables. Our problem is easily formulated in canonical variables so that we have no need of the more general noncanonical perturbation methods pioneered in plasma physics by Littlejohn [17].

The azimuthal contribution is already in its exact action-angle variables. In what follows we are concerned only with the radial contribution:

$$H_0 = \nu_0 J_r + \epsilon\nu_1 J_r^{\frac{3}{2}} sin^3\psi + \epsilon^2\nu_2 J_r^2 sin^4\psi$$

where we have divided out ϵ^2 to save writing. We will refer to the various terms of H according to the power of ϵ in this equation in what follows. The missing powers of ϵ will be replaced at the end of the calculation.

The terminology we use is largely that of Abraham and Marsden whose magnum opus [18] is eminently worth studying. We mention now that in the physical literature the operator that one calls a Lie transform is geometrically a "pullback by a flow". Unfortunately we simply will not have time to fill in much of the beautiful differential geometric interpretation of the formalism we are applying; for the whole story one must consult [19]. For other treatments in the plasma physics literature, devoted strictly or more completely to the methods, one could read Littlejohn [20] or Cary [21], or, for a wide ranging overview of this and many related topics, Lichtenberg and Lieberman [15]. To carry out the calculation one needs the H_0 flow, the pullback, and the pushout by the H_0 flow. The pullback by the H_0 flow is used to invert the inhomogeneous Liouville operator, which occurs at each order of the perturbation calculation, on the orbits of the H_0 flow (method of characteristics) . One selects at each order a function whose vector field generates a flow the pullback by which yields the new variables at that order. This function is called the Lie generating function. The pushout by this flow yields the new Hamiltonian at each order.

A general operator formalism was introduced by Dewar [22], cf. Appendix A, but we follow Fermi's dictum: "when in doubt... expand!" , to get Deprit's version of the formalism, which in fact preceded Dewar's more general treatment. To this end we expand everything in sight in powers of epsilon, cf. Appendix

B. For our system the H_0 flow , i.e., the "time development mapping", is quite trivial:

$$F_t(J,\psi) = (J, \psi + \nu_0 t)$$

The pullback by the H_0 flow, i.e., the "time development operator" is:

$$F_t^*(J,\psi) = f \circ F_t(J,\psi) = f(J, \psi + \nu_0 t)$$

The first order term of our generating function is gotten from the equation:

$$(\frac{\partial}{\partial t} - \{H_0, \})w_1 = \bar{H}_1 - H_1$$

Inverting the inhomogeneous Liouville operator on the H_0 flow we get:

$$w_1 = \int^t dt' [\bar{H}_1(J, \psi + \nu_0(t' - t)) - H_1(J, \psi + \nu_0(t' - t))]$$

To rid w_1 of secular terms we define $\bar{H}_1$ to be the time average of H_1 on an H_0 orbit:

$$\bar{H}_1 = \frac{\nu_0}{2\pi} \int_0^{2\pi/\nu_0} dt H_1(J, \psi + \nu_0 t) = 0$$

hence, to this order we have:

$$\bar{H}_1 = 0$$

$$w_1 = \frac{1}{12} \frac{\nu_1}{\nu_0} \bar{J}^{\frac{3}{2}} (9cos\bar{\psi} - cos\bar{\psi})$$

The old variables in terms of the new are gotten from the formulae:

$$J = \bar{J} + \epsilon\{w_1, \bar{J}\}$$

$$\psi = \bar{\psi} - \epsilon\{w_1, \bar{\psi}\}$$

We also compute the updated version of δr in terms of the new action and angle; leaving out the rather lengthy computations we tabulate the results of the order ϵ perturbation:

$$\bar{H} = \nu_0 \bar{J}$$

$$w = 1 + \epsilon \frac{1}{12} \frac{\nu_1}{\nu_0} \bar{J}^{\frac{3}{2}} (9cos\bar{\psi} - cos3\bar{\psi})$$

$$J = \bar{J} + \epsilon \frac{1}{4} \frac{\nu_1}{\nu_0} \bar{J}^{\frac{3}{2}} (sin3\bar{\psi} - 3sin\bar{\psi})$$

$$\psi = \bar{\psi} + \epsilon \frac{1}{8} \frac{\nu_1}{\nu_0} \bar{J}^{\frac{1}{2}} (cos3\bar{\psi} - 9cos\bar{\psi})$$

$$\delta r = (\frac{2\bar{J}}{\nu_0})^{\frac{1}{2}} (sin\bar{\psi} - \epsilon \frac{1}{4} \frac{\nu_1}{\nu_0} \bar{J}^{\frac{1}{2}} (3 + cos2\bar{\psi}))$$

where functions with overbars are "new", those without, "old". What is significant is that the Hamiltonian has not changed; evidently we need to go to higher order yet in order to expose the nonlinearity of the ground state. The second order term of the generating function is the solution of:

$$(\frac{\partial}{\partial t} - \{H_0, \ \})w_2 = 2\bar{H}_2 + \frac{1}{16}J^2[-(4\nu_2 + 3\frac{\nu_1^2}{\nu_0})cos4\psi$$

$$+ (16\nu_2 - 12\frac{\nu_1^2}{\nu_0})cos2\psi - (12\nu_2 - 15\frac{\nu_1^2}{\nu_0})]$$

We choose $\bar{H}_2$ to eliminate secularity and invert the Liouville operator on the H_0 flow to arrive at:

$$\bar{H}_2 = (\frac{3}{8}\nu_2 - \frac{15}{32}\frac{\nu_1^2}{\nu_0})\bar{J}^2$$

$$w_2 = \frac{1}{16}\bar{J}^2[-(\frac{\nu_2}{\nu_0} + \frac{3}{4}\frac{\nu_1^2}{\nu_0})\sin 4\bar{\psi} + (8\frac{\nu_2}{\nu_0} - 6\frac{\nu_1^2}{\nu_0})\sin 2\bar{\psi}]$$

The second order contributions to the old action and angle in terms of the new are computed from the formulae:

$$J_1 = \frac{1}{2}\{w_1, \frac{\partial w_1}{\partial \bar{\psi}}\} + \frac{1}{2}\frac{\partial w_2}{\partial \bar{\psi}}$$

$$\psi_2 = -\frac{1}{2}\{w_1, \frac{\partial w_1}{\partial \bar{J}}\} - \frac{1}{2}\frac{\partial w_2}{\partial \bar{J}}$$

After a lengthy calculation one arrives at the results:

$$J_2 = \frac{1}{32}\bar{J}^2[-(4\frac{\nu_2}{\nu_0} + 6\frac{\nu_1^2}{\nu_0^2})\cos 4\bar{\psi} + (16\frac{\nu_2}{\nu_0} - 24\frac{\nu_1^2}{\nu_0^2})\cos 2\bar{\psi} + 15\frac{\nu_1^2}{\nu_0^2}]$$

$$\psi_2 = \frac{1}{128}\bar{J}[-\frac{\nu_1^2}{\nu_0^2}\sin 6\bar{\psi} + (8\frac{\nu_2}{\nu_0} + 18\frac{\nu_1^2}{\nu_0^2})\sin 4\bar{\psi} + (-64\frac{\nu_2}{\nu_0} + 27\frac{\nu_1^2}{\nu_0^2})\sin 2\bar{\psi}]$$

Inserting the results for J and ψ into δr and expanding to order ϵ^2 we get the result for δr in terms of the second order correct action and angle:

$$\delta r = (\frac{2}{\nu_0})^{\frac{1}{2}}\bar{J}^{\frac{1}{2}}\sin\bar{\psi} - \epsilon\frac{1}{4}\frac{\nu_1}{\nu_0}(\frac{2}{\nu_0})^{\frac{1}{2}}\bar{J}(\cos 2\bar{\psi} + 3)$$

$$+\epsilon^2\frac{1}{256}(\frac{2}{\nu_0})^{\frac{1}{2}}\bar{J}^{\frac{3}{2}}[\frac{\nu_1^2}{\nu_0^2}\sin 7\bar{\psi} - 19\frac{\nu_1^2}{\nu_0^2}\sin 5\bar{\psi} + (-16\frac{\nu_2}{\nu_0}$$

$$+69\frac{\nu_1^2}{\nu_0^2})\sin 3\bar{\psi} + (-96\frac{\nu_2}{\nu_0} + 189\frac{\nu_1^2}{\nu_0^2})\sin\bar{\psi}]$$

The second order correct Hamiltonian which now contains the nonlinearity is:

$$\bar{H} = \nu_0\bar{J} + \epsilon^2(\frac{3}{8}\nu_2 - \frac{15}{32}\frac{\nu_1^2}{\nu_0})\bar{J}^2$$

In the next section we will analyze this Hamiltonian, our intention being to set up the family of ground state tori which describes the energy level manifold of the transverse dynamics in a classical Bennett equilibrium.

5 Ground State Tori of the Classical Bennett Equilibrium

Time evolution of two degree of freedom dynamical systems may be described as a toral flow on a two-torus whose major and minor radii are actions and whose toroidal and poloidal angles are the corresponding angles; see Figure 1. When one considers the collection of tori parameterized by the ranges of actions consistent with the total energy of the system one arrives at a family of nested two-tori, an idea which apparently can be traced back to Lagrange in 1762, cf. [18]. It is interesting that the behavior of the phase space orbits in this family of tori is perfectly analogous to the behavior of magnetic field lines in a tokamak. This is because one may in fact arrive at a description of field line behavior as a two degree of freedom Hamiltonian system.

Toral flows have been investigated since the time of Jacobi who, in 1835, proved that orbits with irrational winding numbers densely cover the torus, cf. [23]. Poincaré [25] initiated the modern qualitative period of dynamics in the late 1800's the development of which continued with the work of Birkhoff [26] and others, including Kolmogorov whose fundamental paper of 1954 [27], followed by work of Arnold [28] and Moser [29] in the early sixties, resulted in KAM theory and the resolution of the problem of small denominators which had plagued celestial mechanics for well nigh a hundred years. Chirikov and others began introducing these ideas into plasma physics in the late 1950's. In 1966 a paper applying these ideas directly to the physics of tokamaks appeared [30]. These ideas must have seemed all the more compelling for tokamak physicists, in light of the direct analogy mentioned above. In fact, one might look at this aspect of tokamak physics as a particularly beautiful physical manifestation of an otherwise abstract concept, that is, Hamiltonian systems as toral flows!

With this brief bit of lore behind us we move on to describe the tori of the classical Bennett ground state. We may easily non-dimensionalize the problem and by so doing scale away any mention of the transverse temperature T and the Bennett scale radius a. To this end we will measure action in units of $a\sqrt{T}$, frequency in units of T/a^2, energy in units of transverse temperature T, and length in units of the Bennett radius a. If we wish to scale to oof ther beams we simply use the following interbeam scaling laws:

$$\tilde{\Omega} = \frac{\tilde{T}}{T}\left(\frac{a}{\tilde{a}}\right)^2 \Omega$$

$$\tilde{E} = \frac{\tilde{T}}{T} E$$

$$\tilde{J} = \frac{\tilde{a}}{a}\left(\frac{\tilde{T}}{T}\right)^{\frac{1}{2}} J$$

We therefore have the near-circle Hamiltonian:

$$H = \Omega_0 J_\theta + \epsilon^2 \nu_0 J_r + \epsilon^4 \alpha_0 J_r^2$$

where the various quantities have the same meaning as before but are now pure numbers and we have dropped the overbars on the new Hamiltonian and variables. As explicit functions of the dimensionless radius the various frequencies are:

$$\Omega_0^2 = 4\frac{1}{1+x_0^2}$$

$$\nu_0^2 = 8\frac{2+x_0^2}{(1+x_0^2)^2}$$

$$\alpha_0 = -\frac{1}{96}\nu_0^2\frac{11x_0^6 + 66x_0^4 + 129x_0^2 + 72}{(2+x_0^2)^2}$$

Since α_0 is manifestly negative the ground state system is of the "weak-spring" oscillator type, that is $\partial^2 H_0/\partial J_r^2 \leq 0$ so that the twist mapping in the (J_r, ψ) plane is such that the larger actions revolve more slowly than the smaller actions. The fact that the ground state Hamiltonian depends linearly upon J_θ is of some consequence, resulting in the primary resonances being separated by a frequency interval that is independent of action and therefore the same for all adjacent pairs of primary resonances. Of course one only considers the primary resonances that actually exist and in general there are only finitely many primary resonances consistent with allowable winding numbers. In the ground state the actions and winding number of a given torus are explicitly given by:

$$J_\theta(x_0) = x_0^2 \Omega_0$$

$$J_r(x_0, E) = -\frac{1}{2}\frac{\nu_0}{\alpha_0}[1 - (1 - 4\frac{\alpha_0}{\nu_0^2}(E - x_0^2\Omega_0^2))^{\frac{1}{2}}]$$

$$w(x_0, E) = \frac{\nu_0}{\Omega_0}[1 - 4\frac{\alpha_0}{\nu_0^2}(E - x_0^2\Omega_0^2)]^{\frac{1}{2}}$$

where, in deriving J_r we have selected the branch that goes to zero as the radial energy goes to zero, $E \longrightarrow \Omega_0^2 x_0^2$. Winding numbers are bounded above by an x_0 dependent maximum ν_0/Ω_0 where the upper bound itself is constrained to fall within the range $\sqrt{2} \leq \nu_0/\Omega_0 \leq 2$.This restricts the primary resonances to $w = 1, 2$ which are equally separated from one another, as depicted in Figure 2. At any finite radius the only primary resonance of importance is $w = 1$. The secondary resonances are $w = 1 + \frac{m}{n}$ and $w = 1 - \frac{m}{n}$ where $m \leq n$ so there is a dense distribution of secondary resonances on either side of the primary resonance; see Figure 2. Later when we investigate the coupling resonances we will be interested in whether the islands of the secondary resonances overlap one another or whether they remain isolated. Technically, the smoothness condition of the KAM theorem requires the latter condition if invariant tori are to exist.

We may invert the winding number condition for resonance to get the resonant energy spectrum:

$$E_{pq} = x_0^2\Omega_0^2 + \frac{1}{4}\frac{\Omega_0^2}{\alpha_0}\left(\frac{\nu_0^2}{\Omega_0^2} - \frac{p^2}{q^2}\right)$$

where p and q are integers and their ratio is of the form one plus or minus a fraction less than one. We naturally speak of such an energy spectrum as there are many particles at any reference circle and these particles have a distribution in energy. If we change our point of view and fix a given energy then the nested tori for any particle with that energy are parameterized by the radius of the reference orbit.

If we introduce the energy spread into our considerations we have a one parameter family of nested tori families, that parameter being the energy. The true dynamical system is the four-manifold and a nested tori family corresponding to an energy level manifold is a "slice" of this manifold. For a description of the bewildering variety of bifurcational processes which may occur as one varies the energy parameter in such a system we refer the reader again to [18].

If the radial action ceases to be a good invariant due to any resonance overlaps then the particle orbit will diffuse to other reference radii. This is because there is a one-to-one relationship between the reference radius and the azimuthal action J_θ . In fact, we could think of (x_0, J_r) as the invariants of the ground state system as well as (J_θ, J_r), so that diffusion in action is equivalent to radial diffusion.

For our system, as with all of these systems, resonant tori are densely distributed in the energetically accessible region of action space. We have a dense distribution in radius, at fixed energy, of resonant tori, or a dense distribution in energy at a fixed radius. Both situations actually exist in the beam which consists after all of many particles with a distribution in transverse energy.

Resonance structure in action space tends to enhance the diffusive effects of weak extrinsic random noise. We are not considering in this paper the effects of extrinsic diffusion, such as elastic interparticle collisions, which tend to cause radial diffusion. Collisions can remove a phase point from a nonresonant torus and place it on a resonant torus or island. Such resonant diffusion, taking place in the absence of resonant interaction, amounts to an enhancement over the nonresonant extrinsic diffusion, as the islands provide "stepping stones" for a phase point to random walk about action space propelled from island to island by the collisions. We note that even KAM tori cannot prevent such diffusion as the particle is able to "hop over" the tori. Here we are investigating only the intrinsic diffusion driven by the interaction of neighboring islands.

The KAM theorem establishes the fact that "most" of the tori bearing incommensurate frequencies survive with only a small distortion, a small nonintegrable perturbation. By a small distortion we mean one without any ripping or topological changes, i.e., we end up with a deformed manifold which is topologically a torus. On the other hand, tori bearing periodic orbits, or nearly periodic orbits, i.e. incommensurate but with winding number approximated extremely well by n/m (where n and m are relatively small integers), are grossly, that is to say topologically, deformed.

Ultimately the distinction between rational and irrational winding numbers is meaningful only in a time asymptotic sense and has no real physical significance for finite times. Mathematically one distinguishes between the concepts of an orbit and a trajectory. An orbit is a geometrical concept, independent of time, that is, the set of points which may be gotten from one another by action of the flow map or its inverse. A trajectory is the act of evolution along an orbit. One might think of the orbit as a wire and the trajectory as the motion of a signal down the wire. Periodic orbits whose winding numbers are the ratios of very large relatively prime numbers look very much like irrational orbits, that is, tend to cover the torus well and close upon themselves only after a very great, but finite, path length. It is important to realize that finite segments of all orbits, corresponding to flow for a finite time along the orbits, which do not happen to close during that time, look, and are, topologically identical. This point is nicely discussed by Greene [31].

6 Hose Perturbations

In the rest frame of a transverse slice of the beam the passage of a hose wave, propagating from the head to the tail of the beam, results in nonaxisymmetric perturbations upon the potential with a sinusoidal time dependence. For an overall review of hose theory we find the reports of Lee [2] to be particularly enlightening; here we are only going to derive the perturbation due to a hose wave whose Doppler shifted frequency is slow compared to the time scales of transverse particle motion. The phase of the wave in the lab frame is $kz - \omega t$ so, in terms of the Doppler shifted frequency $\Omega = \omega - c\beta k$, one may formulate the problem in two different but fundamentally equivalent ways, depending upon whether one eliminates ω or k between the two equations. If one eliminates ω one arrives at the phase $kZ - \Omega t$ where $Z = z - \beta ct$ is the fixed label of the slice, whereas if one eliminates ω one arrives at the phase $-\Omega z/\beta c - \omega\varsigma/\beta c$ where $\varsigma = \beta ct - z$ is the fixed label of the slice. In the hose literature both formulations have been used by various authors. Which choice one adopts is determined primarily by whether one wants the frequency or the wave vector as the independent variable, which choice in turn depends upon the experimental situation one has in mind. If one perturbs the beam with a fixed, known frequency in the lab, at a fixed position, then one wants the frequency to be the independent variable, i.e., one wants the dispersion relation to yield $k(\omega)$, in which case subsequent development of the mode is governed by $k(\omega)$ and $\Omega(\omega)$. This was the situation in early work of Weinberg [32] which was carried out in the context of the Astron experiment. In this work, however, we find it convenient to use the opposite formulation, that is, k is the independent variable, so that the phase of the hose wave in the slice rest frame is $kZ - \Omega t$ where Z is fixed. The hose wave therefore manifests itself in the appearance of factors such as:

$$\delta f(r) exp[in\theta] exp[i(kZ - \Omega t)]$$

in the pinch potential. We hope, in subsequent work, to investigate the effects of the time dependence, which destroys the autonomous nature of the system, making it a 5/2 dimensional problem and characteristically a more intricate system as well, although a version of the KAM theorem on invariant tori holds in this case also. For now we restrict attention to the autonomous case wherein frequencies of transverse motion are much faster than Ω. This removes the time dependent factor $exp[i(kZ - \Omega t)]$ from the pinch potential.

A lateral hose wave results in a lateral deflection of a slice and the appearance of angular factors of the following form:

$$\delta\rho(\delta h, \delta x, \theta) = \sum_{k=0}^{\infty} \delta\rho_k(\delta x, \delta h) cos k\theta$$

where $\delta\rho = \rho - x_0$. We have exploited the absence of θ dependence in δx to write this Fourier expansion of $\delta\rho$ in harmonics of θ . We are going to keep only the lowest few harmonics in both θ and ψ. In Figure 3 C refers to the center of the unperturbed beam, δh is the lateral displacement which is assumed small compared to the beam radius (i.e., roughly twice the Bennett radius), and x is the radial position of the particle. It is the lateral displacement which effectively determines the barrier transition to global stochasticity.

Next we expand $\delta\rho$ and $\delta\rho^2$ on the theta harmonics with the result being:

$$\delta\rho = \frac{\delta h^2}{4x_0^3}[\delta h \delta x cos\theta + (x_0\delta x - \delta x^2) cos 2\theta - \delta h \delta x cos 3\theta + \cdots]$$

$$\delta\rho^2 = \frac{\delta h \delta x}{2x_0^2}[(4x_0^2 - \delta h^2) cos\theta + (\delta h \delta x - \delta h x_0) cos 2\theta + \delta h^2 cos 3\theta + \cdots]$$

where ellipses in this case denote terms with no ψ dependence. The hose perturbation of the Hamiltonian is expanded also on the harmonics:

$$\delta H = \sum_{k}^{\infty} \delta H_k(\delta x) cos k\theta$$

which, in terms of $\delta\rho$ and $\delta\rho^2$ is explicitly given by:

$$\delta H = x_0^2\Omega_0^2\delta\rho + \frac{1}{2}(\nu_0^2 - 3\Omega_0^2)\delta\rho^2 + \cdots$$

Finally, the double Fourier expansion in harmonics of θ and ψ yields, after quite a considerable amount of algebra, the hose perturbations:

$$\delta H_{11} = 4h_1 J^{\frac{1}{2}} \sin(\theta - \psi) + \epsilon^2 h_1 (3\frac{\nu_1^2}{\nu_0^2} - \frac{3}{2}\frac{\nu_2}{\nu_0}) J^{\frac{3}{2}} \sin(\theta - \psi)$$

$$\delta H_{12} = -\epsilon h_1 \frac{\nu_1}{\nu_0} J \cos(\theta - 2\psi) + \frac{1}{4}\epsilon^2 h_1 \frac{\nu_2^2}{\nu_0^2} J^{\frac{3}{2}} \sin(\theta - 2\psi)$$

$$\delta H_{13} = \epsilon^2 h_1 \frac{\nu_1^2}{\nu_0^2} J^{\frac{3}{2}} \sin(\theta - 3\psi)$$

$$\delta H_{22} = (-h_2 \frac{4}{\nu_0} + \epsilon h_2 x_0 \frac{\nu_1}{\nu_0}(\frac{2}{\nu_0})^{\frac{1}{2}}) J \cos(2\theta - 2\psi) - \epsilon^2 \frac{3}{4} h_2 \frac{\nu_1}{\nu_0^2} J^{\frac{3}{2}} \cos(2\theta - 2\psi)$$
$$+\epsilon^2 3h_2 \frac{\nu_2}{\nu_0^2} J^2 \cos(2\theta - 2\psi) - \epsilon^2 \frac{1}{4} h_2 x_0 \frac{\nu_2}{\nu_0}(\frac{2}{\nu_0})^{\frac{1}{2}} J^{\frac{3}{2}} \sin(2\theta - 2\psi)$$

$$\delta H_{23} = (\epsilon 4h_2 \frac{\nu_1}{\nu_2} + \epsilon^2 h_2 x_0 \frac{\nu_1^2}{\nu_0^2}(\frac{2}{\nu_0})^{\frac{1}{2}}) J^{\frac{3}{2}} \sin(2\theta - 3\psi) + \epsilon^2 \frac{1}{2} h_2 \frac{\nu_2}{\nu_0^2} J^2 \cos(2\theta - 3\psi)$$

$$\delta H_{32} = -\epsilon h_3 \frac{\nu_1}{\nu_0}(\frac{2}{\nu_0})^{\frac{1}{2}} J \cos(3\theta - 2\psi) + \epsilon^2 \frac{1}{4} h_3 \frac{\nu_2}{\nu_0}(\frac{2}{\nu_0})^{\frac{1}{2}} J \sin(3\theta - 2\psi)$$

where $s = \delta h/a x_0$ is the "hose strength", and h_1, h_2, and h_3 are defined as:

$$h_1 = \frac{x_0}{8}(\nu_0^2 - 4\Omega_0^2)(\frac{2}{\nu_0})^{\frac{1}{2}}(s - \frac{1}{4}s^3)$$

$$h_2 = \frac{1}{32}s^2(\nu_0^2 - 4\Omega_0^2)$$

$$h_3 = x_0 s h_2$$

In the next section we analyze the isolated coupling resonances driven by these perturbations. If one considers a region of action space in which there is a resonance between the two degrees of freedom with no other resonances nearby then all terms save the resonant term have rapidly varying phases and thus tend to average away on the time scale of the resonant term. Such an isolated resonance is in fact integrable, that is, an invariant besides the energy exists enabling one to easily study the properties of the given isolated coupling resonance between the two degrees of freedom.

If one examines the perturbations we have derived one immediately realizes that there will be many pairs of interacting, that is, overlapping, coupling resonances, when the hose strength parameter exceeds a certain value[1]. One cannot hope that the system will retain good invariants in this situation. It is to the study of these resonances that we now turn our attention.

[1] One also realises that the perturbations vanish identically for a linear beam, that is, a beam with a Gaussian radial profile. We must confess that we do not understand as yet just why this should be.

7 Coupling Resonance Between Harmonics of Circular Drift and Vortex Gyration

Poincaré and Birkhoff realized that very few two degree of freedom Hamiltonian systems possess two independent invariants, that is, classical Hamilton-Jacobi theory fails for most systems, thus nonintegrable systems are properly thought of as generic or "garden variety" systems. This means that one may write down any Hamiltonian involving only actions, and perturb it with any perturbation involving angles, at random so to speak, and be reasonably certain that the combined system is nonintegrable. In fact, one is rather astounded if it turns out to be integrable, as in the case of the Toda lattice [33] which surprised Ford et al. [34] when they numerically found evidence of its integrability. This was analytically verified by Henon [36] who, inspired by Ford's work, sought and found n independent invariants for the n particle lattice showing that the Toda lattice is indeed "a jewel in physics" [34].

On the one hand, for garden variety perturbations of integrable systems, one has the celebrated theorems of Kolmogorov, Arnold, and Moser to fall back upon. The existence of invariant tori enables one to effectively study a wide class of otherwise intractable systems which may be thought of as those systems for which perturbation theory succeeds, if done correctly. On the other hand, as one moves a system further and further from an integrable system the KAM tori begin to disintegrate, that is, perturbation theory fails outright. If one pushes a system far enough in this direction one may fall back upon essentially statistical methods.

Of particular interest is the behavior of a system undergoing transition between near-integrability and non-integrability. For two DOF systems such as the one we are studying the computation of the threshold of global stochasticity involves determination of the critical perturbation strength at which all of the KAM tori have disintegrated. The breakup and disappearance of the last KAM torus triggers global stochasticity. Even after the destruction of the KAM tori there are structures that provide resistance to global diffusion: invariant Cantor sets, the Cantori, stubborn remnants of KAM tori which impede global diffusion of actions and result in a divided or "clumpy" phase space. These partial barriers to diffusion are of considerable current interest, cf. [35] for example.

In the late 1950's Chirikov [37] began formulating a practical method of discerning and quantifying the point at which the last KAM torus between two fundamental resonances is destroyed, the overlap criterion. This was the criterion applied to tokamak magnetic field structures by Rosenbluth et al. in 1966 [30]. Although the overlap criterion is neither necessary nor sufficient for destruction of the torus [15] it is easy to apply and has great intuitive appeal [14].

Before investigating the resonance overlap and resultant diffusion of the radial action in the Bennett beam we wish in this section to present some graphical results concerning the Poincaré sections of the isolated resonances in the weakly hosing Bennett beam.

In the previous section we arrived at the hose Hamiltonian which turned out to be of the form:

$$H = H_0 + \sum_{lm} \delta H^c_{lm} cos(l\psi - m\theta) + \sum_{lm} \delta H^s_{lm} sin(l\psi - m\theta)$$

At resonance the frequencies of circular drift and vortex gyration are related in such a fashion that the phase of the resonant term is stationary:

$$\frac{d}{dt}(n\psi - s\theta) \approx 0$$

Performing a transformation to the usual rotating variables defined as:

$$\hat{J}_r = \frac{1}{n} J_r$$

$$\hat{J}_\theta = \frac{s}{n} J_r + J_\theta$$

$$\hat{\psi} = s\theta - n\psi$$

$$\hat{\theta} = \theta$$

we may perform an average over the angle $\hat{\theta}$ to get the approximate second invariant which we mentioned earlier. The only surviving term is that for which $ls - nm = 0$ so if we define $p = l/n$ we have $m = sp$ and $l = np$ and the angle averaged hose Hamiltonian in island variables is:

$$\hat{H}_{ns} = \hat{H}_0 + \sum_{p=1} \delta H^c_{np,sp} cosp\hat{\psi}_{ns} + \sum_{p=1} \delta H^s_{np,sp} sinp\hat{\psi}_{ns}$$

where $\hat{\psi}_{ns} = n\psi - s\theta$. Refering to the results of the last section we see that the resonance Hamiltonians are just H_0 plus the various perturbations previously derived taken one at a time, the only difference being that the ground state is now evaluated at

$$(\hat{J}_r, \hat{J}_\theta)$$

which amounts only to an inconsequential shift in the energy. We view this as justification for examining each isolated resonance individually prior to investigating the overlaps. With the Hamiltonian in this form we now see clearly the existence of an approximate second invariant:

$$I = sJ_r + nJ_\theta$$

which demonstrates the integrability of the isolated resonance. In action space, in close proximity to a given isolated resonance, the island Hamiltonian of that resonance may be cast into an alternate form with great intuitive significance, expanding around the fixed points of $\hat{H}$:

$$\hat{J}_r = \hat{J}_0 + \delta \hat{J}$$

$$\hat{\psi} = \hat{\psi}_0 + \delta \hat{\psi}$$

To second order in the small $\delta \hat{J}$ we arrive at the isolated hose resonance Hamiltonian in the perspicuous form which Chirikov has called the "standard form", i.e., a simple pendulum:

$$\hat{H}_{ns} = \frac{1}{2}\frac{\partial^2 \hat{H}_0}{\partial \hat{J}_0^2}\delta \hat{J}^2 + [\delta \hat{H}_{ns}^{c2} + \delta \hat{H}_{ns}^{s2}]^{\frac{1}{2}} cos(\hat{\psi} + \chi)$$

where everything is evaluated at the fixed point $\hat{J}_0$. We have used the fact that only $p = 1$ occurs in the isolated resonances except the one-one resonance, which includes the two-two as well. We neglect the two-two vis-à-vis the one-one as it is of much smaller amplitude. The action at the fixed point is determined by the condition that the unperturbed frequency vanish:

$$\hat{J}_0 = \frac{1}{2}\frac{s\Omega_0 - n\nu_0}{n^2\alpha_0}$$

and the angle at the fixed point is given by:

$$tan\hat{\psi}_0 = \frac{\delta \hat{H}_{ns}^{s}}{\delta \hat{H}_{ns}^{c}}$$

With this form of the island Hamiltonian we easily read off the island width which is twice the maximum $\Delta \hat{J}_{ns}$ on the separatrix orbit:

$$\Delta \hat{J}_{ns} = (-\frac{2}{\alpha_0})^{\frac{1}{2}}(\delta H_{ns}^{c2} + \delta H_{ns}^{s2})^{\frac{1}{4}} \sim h_n^{\frac{1}{2}}$$

The Poincaré map of interest is that induced by the flow on an energy level torus as it intersects a theta equals constant cross-section, for a given point on the section the Poincaré map is the first return map which takes the point of intersection at one time to the point of intersection one unit of time later. If the winding number on the torus is irrational the images of a single point will densely cover the circle of intersection of the torus with the section as one iterates the map infinitely many times. For intersections with rational tori the images of a single point will repeat themselves eventually. Here one might recall the discussion of the previous section concerning the distinction between rational and irrational tori.

In this section we want to study the islands formed by the perturbation harmonics taken one at a time. The sections were chosen at $\theta = 0$ for simplicity. A different choice would only rotate the pictures, corresponding to the winding of the island structure itself around the torus. One would expect, and this expectation is confirmed, that the amplitude of the Fourier harmonics decreases rapidly with increasing mode number. It turns out that of the perturbations we have computed only the one-one, one-two, and the one-three islands are important, that is, only the θ fundamental and the fundamental and first two harmonics of ψ. A series of sections are depicted below (Figs. 4–22) showing the island amplitudes as the hose parameter is varied. Each section shows the intersections of twenty orbits with the same set of initial conditions for two hundred section crossings. Island amplitudes decrease with the reference circle radius and as half the particles in a Bennett equilibrium are contained within one Bennett radius and nearly all the particles within two Bennett radii, we have computed the islands at one Bennett radius and one-half Bennett radius in the results presented below. These isolated islands agree well with the amplitude estimate given above and with the location of their unstable hyperbolic and stable elliptic points.

8 Resonance Overlap of the Lowest Order Islands

In this section we investigate the pairwise overlapping of the three lowest resonances discussed in the previous section. We are interested in resonant modification of the transverse invariant J_r due to interaction of the islands generated by the weak hose. For fixed energy, diffusion of J_r in action space results in diffusion of J_θ as well. Since J_θ is related in a one-to-one fashion to the radial postion of the reference circle orbit diffusion in action is equivalent to diffusion in radius.

Neglect of collisionally driven extrinsic diffusion is an admittedly severe limitation on the significance of these results which involve only intrinsic diffusion, yet for heavy ion beams such as in the context of heavy-ion fusion for example, one may reasonably expect collisional effects to be nondominant for a coasting beam. Even for a light ion beam comprised of ions and counterflowing electrons one may argue that the collisions will have a negligible effect upon the ions. For a particle beam propagating through a plasma, however, it is much more difficult to ignore the effect of beam ion collisions with plasma ions. In all cases what is important is a comparison of the time scale of the intrinsic diffusion with that of the extrinsic diffusion which is related to the frequency of important collisions.

As stated in the introduction, we feel that such an investigation of nonaxisymmetric beam phenomena from the point of view of Hamiltonian system theory, or more generally, modern dynamical system theory,, is potentially a fruitful approach to the physics of such phenomena. Since to our knowledge this has not been examined from our point of view we present these results as a preliminary attempt to develop some of the theory of particle beams in fusion in terms of modern mechanics. It is particularly natural to study the implications of the adiabatic assumptions involved in the fluid-kinetic hybrid approach to simulating particle beam propagation, in the same sense as one examines the breakdown of adiabaticity in gyro-kinetic models of mirror or tokamak plasmas.

We present a series of sections in Figures 4–22 depicting the overlapping of the lowest order islands and the resultant locally stochastic orbit behavior. Each of the sections depicts the crossings of nine orbits with the same set of initial conditions in each plot, each orbit crossing the section two hundred times in these calculations. Rather than discussing the plots in the text we refer the reader to the captions beneath each figure. In the next section we present the conclusions we have drawn from the work presented in this and previous sections.

9 Conclusions

Two oscillators coupled nonlinearly together may exchange energy. If this occurs there is a good chance the more energetic degree of freedom will feed energy to the less energetic degree of freedom. Transverse particle dynamics may be thought of, as we have demonstrated, as two coupled oscillators. We have also shown that a very small anisotropy such as that engendered by a weak hose motion can result in appreciable transfer of energy between the circular drift and the vortex gyration. Is it possible to have a good gyration invariant in such a situation? The sections we have displayed reveal that very small lateral displacements are sufficient to cause stochastisation of the orbits, yet the orbits remain confined by tori preventing the radial action from growing without limit. That is, radial motion cannot absorb unlimited energy from the circular motion. Thus although the oscillators transfer energy due to the hose driven coupling resonances it is not entirely unreasonable to build a drift-kinetic description upon the assumption of a transverse radial invariant. One might, however, believe that the questions of collisionally driven extrinsic diffusion render the entire question addressed in this report moot. However, so also do collisions render any concept of good invariants moot! Our work is applicable to just those situations in which one would physically be justified in speaking of action invariants in the first place. What we have done is to examine a non-collisional mechanism for the modification of such invariants in a plausible beam configuration, a slight nonaxisymmetry, a situation that one is interested in studying, particularly since the particle codes do not work so well for this case. We do not believe that our results show that adiabatic models for nonaxisymmetric beams will not work; on the contrary, we personally feel that such models should be developed further.

10 Appendix A

In this appendix we will present the essentials of Dewar's canonical perturbation theory for the time dependent generating functions necessary to deal with time dependent Hamiltonians. One selects a Lie generating function w the pullback by which is denoted T and the pushout T^{-1}. The Lie derivative L is defined in terms of the canonical Poisson bracket as:

$$L = \{w, \ \} = \frac{\partial w}{\partial q}\frac{\partial}{\partial p} - \frac{\partial w}{\partial p}\frac{\partial}{\partial q}$$

The derivative of the pullback T with respect to a parameter ϵ which labels the generating function w_ϵ is:

$$\frac{dT}{d\epsilon} = -TL$$

and the derivative of the pushout T^{-1} is:

$$\frac{dT^{-1}}{d\epsilon} = LT$$

Integrating the derivative of the pullback with respect to ϵ one arrives at the formal expression for the pullback:

$$T(\epsilon) = exp[-\int_0^\epsilon d\epsilon' L(w_{\epsilon'})]$$

where we must think of this as an ϵ-ordered product as the Lie derivatives do not necessarily commute at different values of ϵ. Here we get the element T in the Lie group of transformations as the exponential of the element L of the Lie algebra. In terms of the pullback the new variables $Z = (Q, P)$ are:

$$Z = Tz$$

where $z = (q, p)$ are the old variables. The new Hamiltonian K is related to the old Hamiltonian H by Dewar's formula:

$$K = T^{-1}H + T^{-1}\int_0^\epsilon d\epsilon' T(\epsilon')\frac{\partial w}{\partial t}(\epsilon', t)$$

For a derivation, which unfortunately we have no time to present, one could see Dewar [22]. Also, for the differential geometry one should see [18].

11 Appendix B

Deprit formulated a canonical perturbation theory in 1969 in terms of Lie transforms in the case in which one has a small parameter ϵ in which one is able to expand. Dewar's treatment in 1976 amounts to a generalization of Deprit's to the case of possible nonanalytic functions, i.e., where series developments may not exist. We may most efficiently motivate things by going chronologically in reverse. Here we expand all the objects of Appendix A in the small parameter ϵ. In terms of the expansion of the generating function w the derivative along the flow of w is:

$$\frac{d}{d\epsilon} = -\sum_{n=0}^{\infty}\{w_n, \ \} = -\sum_{n=0}^{\infty} L_n$$

Likewise the formulae for the pullback and pushout are given recursively:

$$T_n = -\frac{1}{n}\sum_{m=0}^{n-1} T_m L_{n-m}$$

$$T_n^{-1} = \frac{1}{n}\sum_{m=0}^{n-1} L_{n-m} T_m^{-1}$$

Explicitly, the formulae employed in the report are:

$$T_0^{-1} = I$$

$$T_1^{-1} = L_1$$

$$T_2^{-1} = \frac{1}{2}L_2 + \frac{1}{2}L_1^2$$

At each order the Lie generating function is the solution of the following equation where the new Hamiltonian K_n is chosen to eliminate secularities:

$$\left(\frac{\partial}{\partial t} - \{H_0, \ \}\right)w_n = n(K_n - H_n) - \sum_{m=1}^{n-1}(L_{n-m}K_m + mT_{n-m}^{-1}H_m)$$

Writing out explicitly the two formulae used in the report we have:

$$\left(\frac{\partial}{\partial t} - \{H_0, \ \}\right)w_1 = K_1 - H_1$$

$$\left(\frac{\partial}{\partial t} - \{H_0, \ \}\right)w_2 = 2(K_2 - H_2) - L_1(K_1 + H_1)$$

The differential operator on the left hand side is inverted by integrating along the flow lines of H_0. The explicit formula for this inversion, in terms of some functions f and g is:

$$\left(\frac{\partial}{\partial t} - \{H_0, \ \}\right)f = g$$

$$f = (F^*)^{-1}\int_{t_0}^{t} dt'' F^* g(t'') + f(t_0)$$

We have included just enough of this formalism to enable one to follow the argument in the report; the beautiful geometry involved must remain undescribed.

12 Acknowledgment

Support for this work has been provided by Lawrence Livermore National Laboratory, through the auspices of the U.S. Department of Energy.

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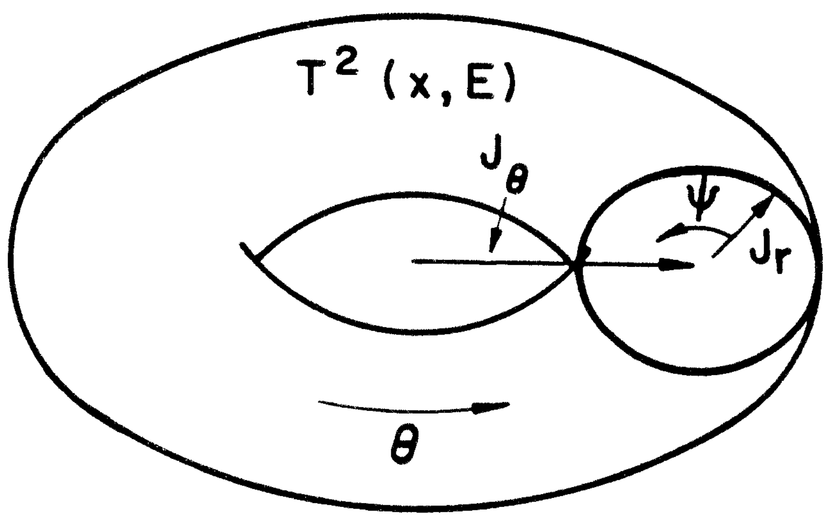

Figure 1 Ground State Tori

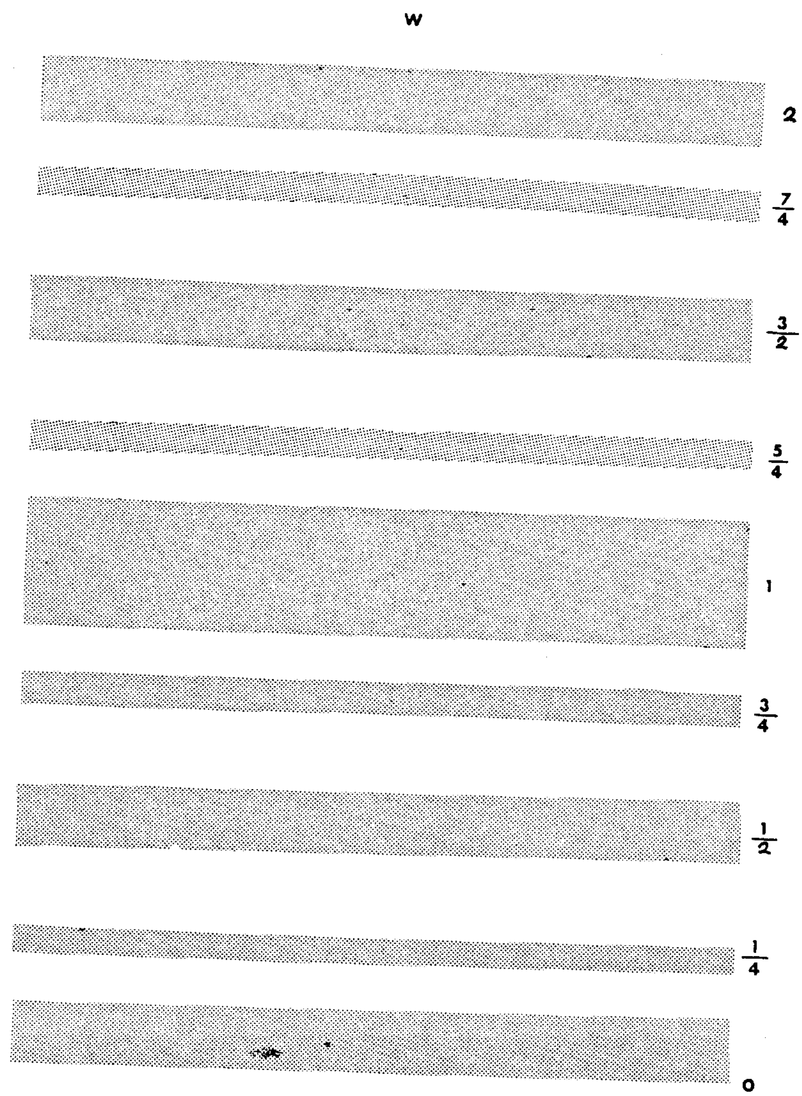

Figure 2 Secondary resonances arrayed on either side of the primary resonance at $w = 1$. There is a dense distribution of such secondary resonances, one at each rational w, however, the island amplitude decreases rapidly as the numerator and denominator of the fraction increase. The KAM theorem requires that the secondary resonances of appreciable amplitude be well separated in order that invariant tori may exist.

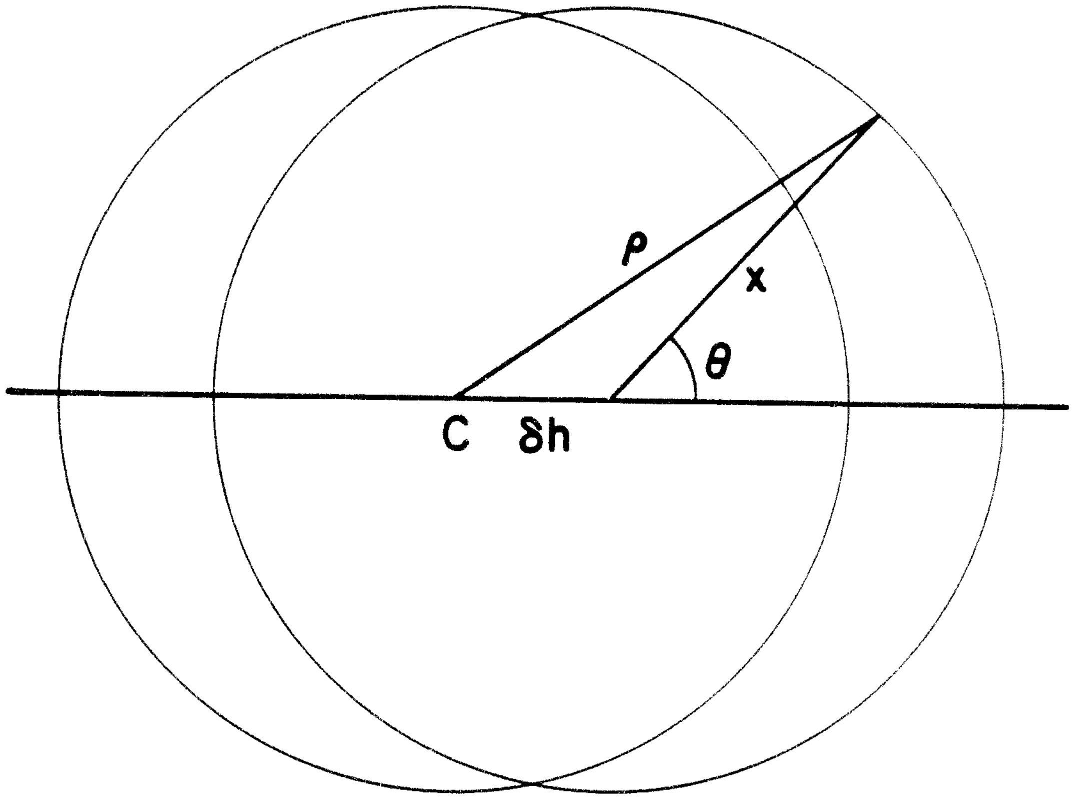

Figure 3 Lateral Hose Deflection of a Beam

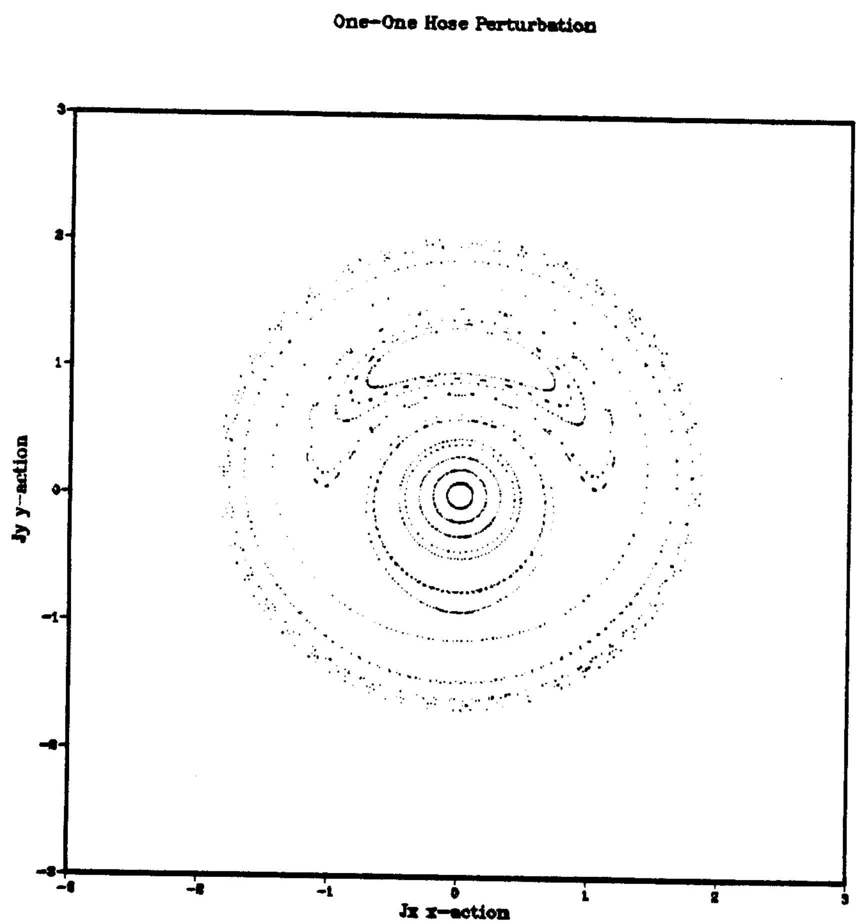

Figure 4 $x_o = 0.5$ and $s = 0.01$; depicting the growth of a primary island with elliptic point at $\psi = \pi/2$ and hyperbolic point at $\psi = -3\pi/2$. The separatrix orbit is not shown.

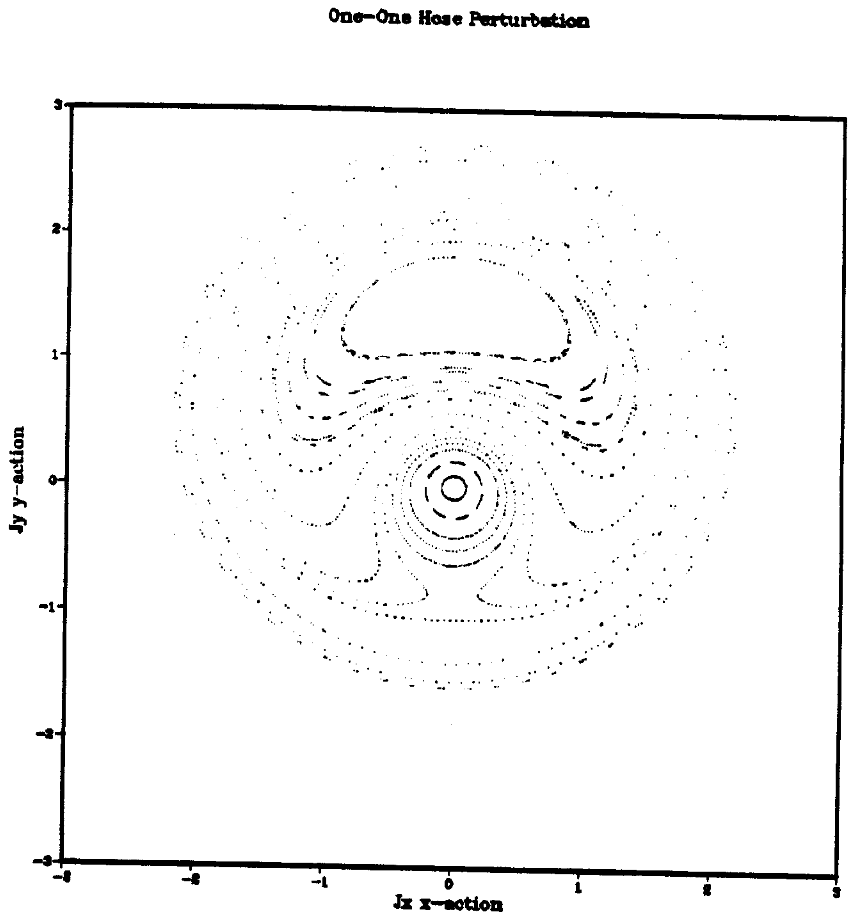

Figure 5 $x_o = 1.0$ **and** $s = 0.01$. **At one Bennett radius the primary island has grown to an appreciable amplitude for this very small lateral displacement.**

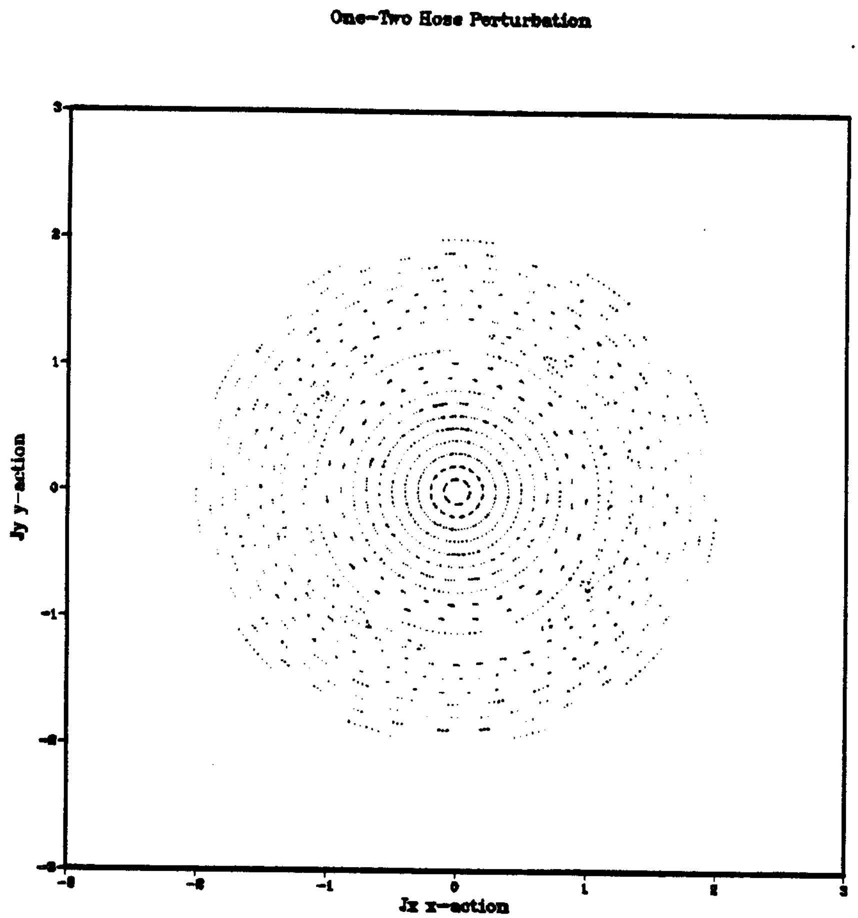

Figure 6 $x_o = 0.5$ and $s = 0.01$. The two thin islands are visible.

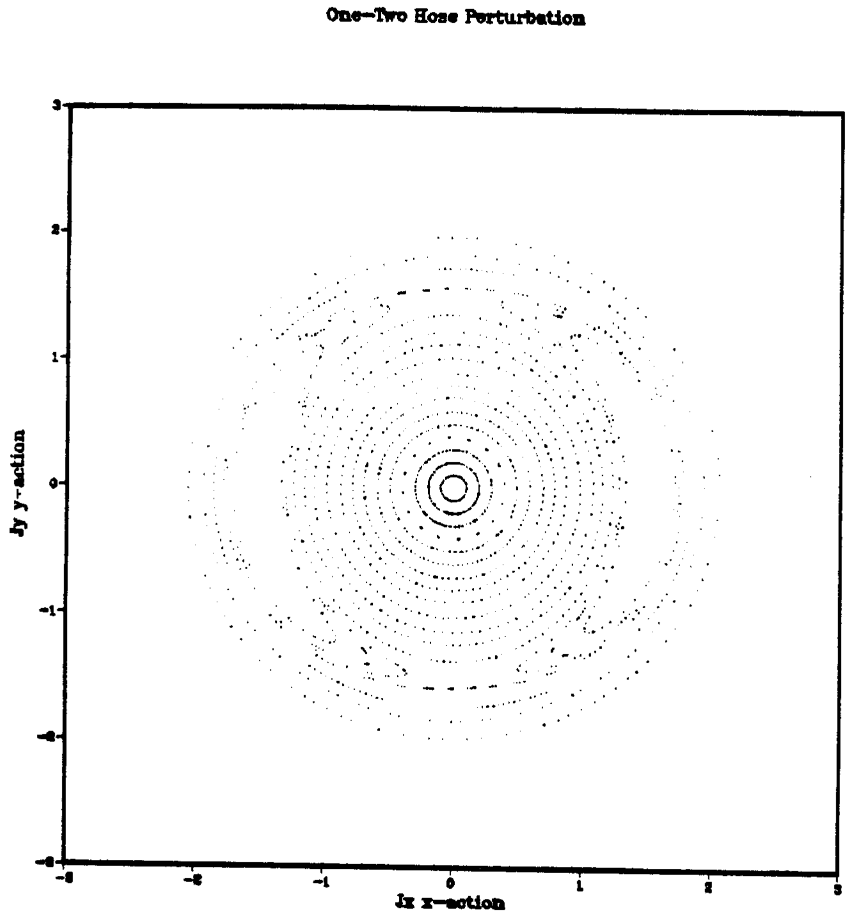

Figure 7 $x_o = 1.0$ and $s = 0.01$. The section shows again the increase in amplitude as compared to one-half the Bennett radius.

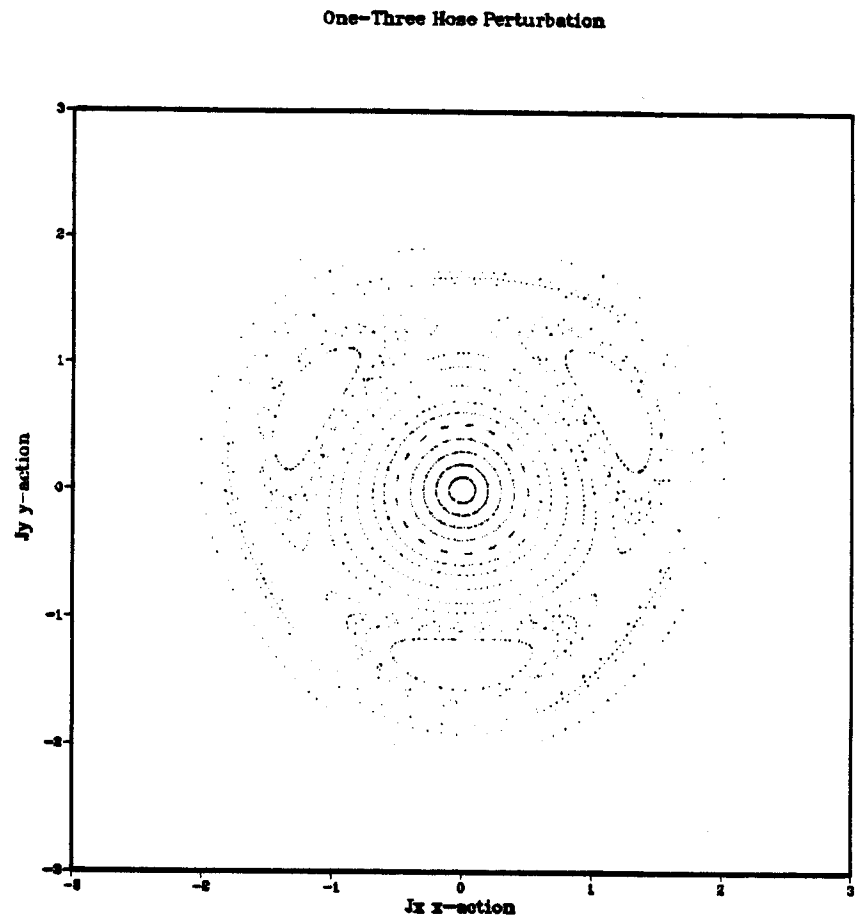

Figure 8 One–Three Isolated Resonance: $x_o = 0.5$ and $s = 0.01$.

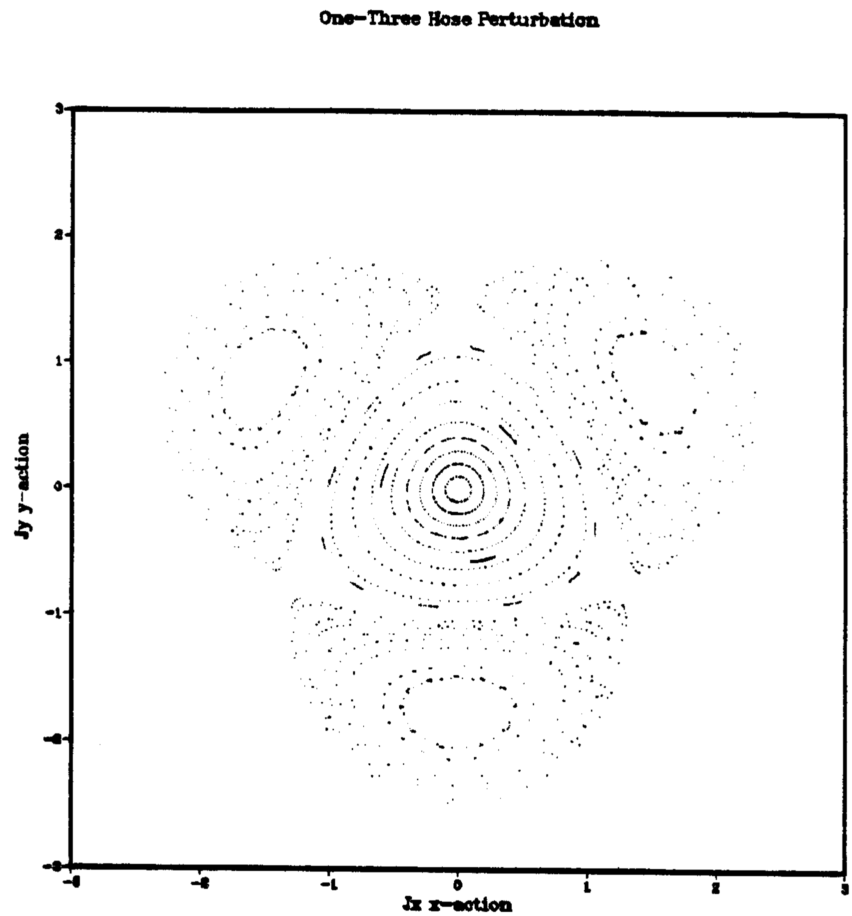

Figure 9 One–Three Isolated Resonance: $x_o = 1.0$ and $s = 0.01$.

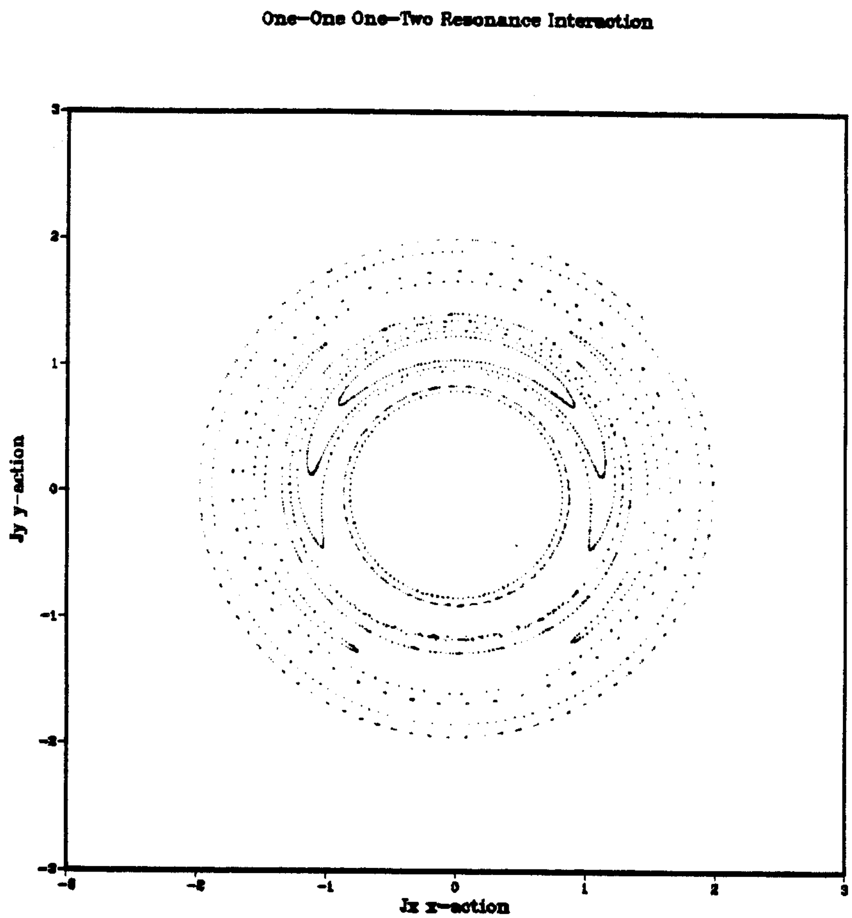

Figure 10 $x_o = 1.0$ **and** $s = 0.0005$. **A thin stochastic layer around the separatrix near the unstable hyperbolic point is emerging.**

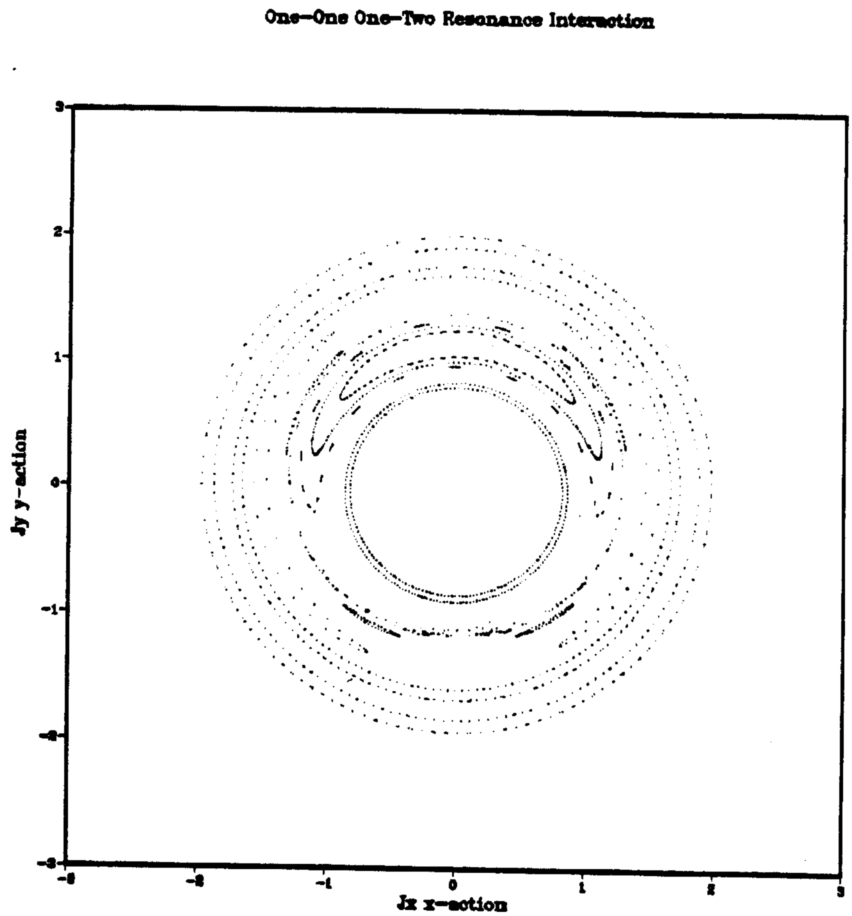

Figure 11 $x_o = 1.0$ and $s = 0.0006$. Island interaction has generated visible satellite islands and the stochastic layer width has increased.

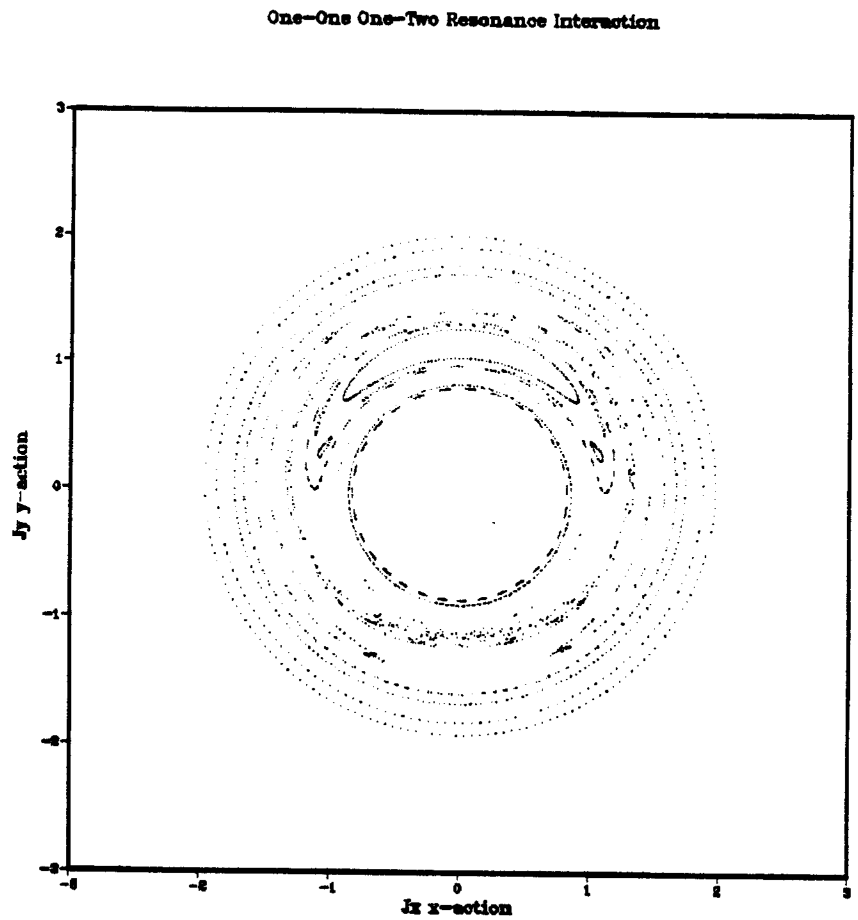

Figure 12 $x_o = 1.0$ **and** $s = 0.0007$. **Further widening of the stochastic layer and more satellite island structure is visible.**

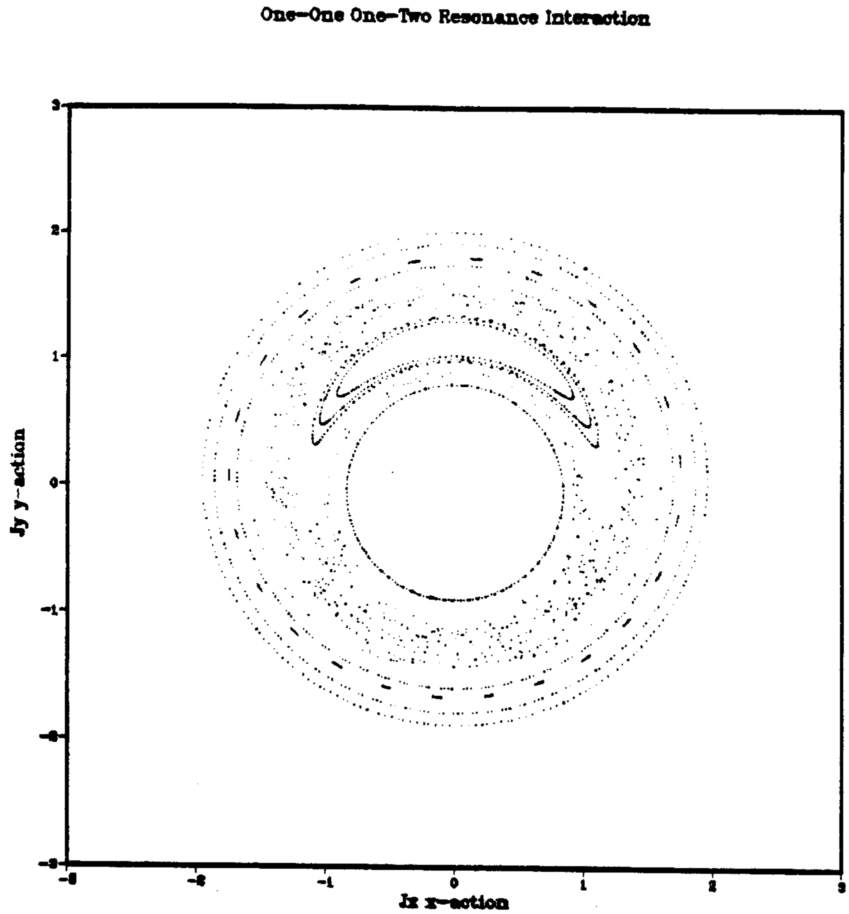

Figure 13 $x_o = 1.0$ and $s = 0.001$. The one–two islands have shrunken out of visibility; a wide stochastic layer surrounds the one–one island but is still well bounded by KAM tori.

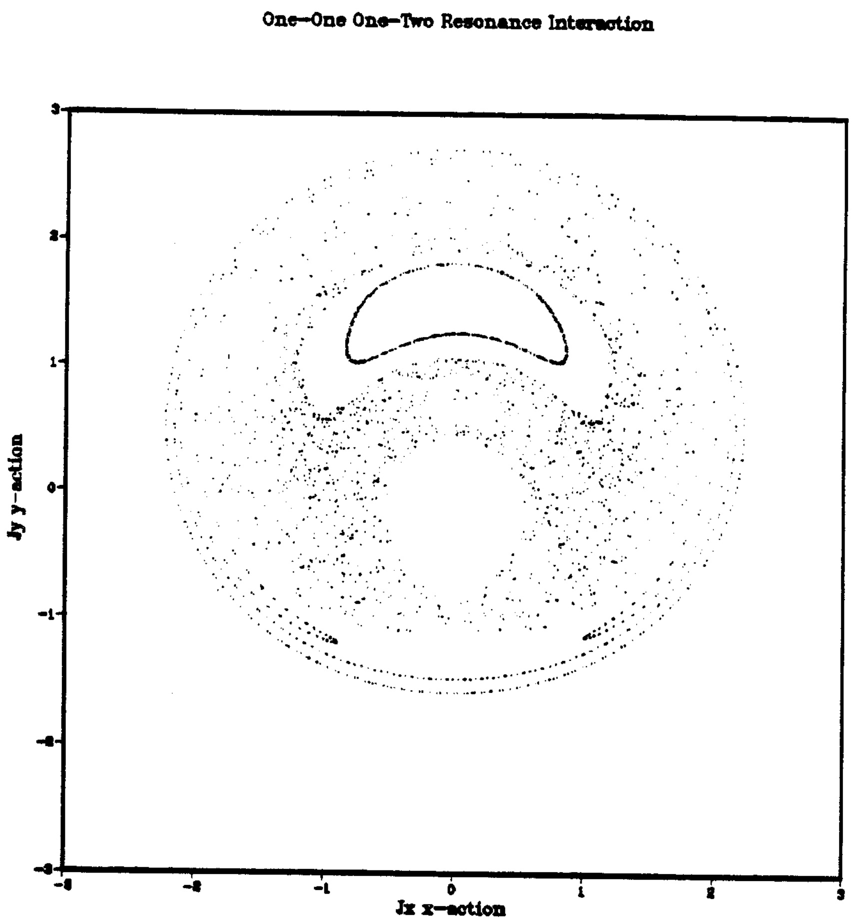

Figure 14 $x_o = 1.0$ **and** $s = 0.01$. **The inner tori are shrinking as the stochastic region encompasses more and more of the section. One can see very thin remnants of the one–two islands.**

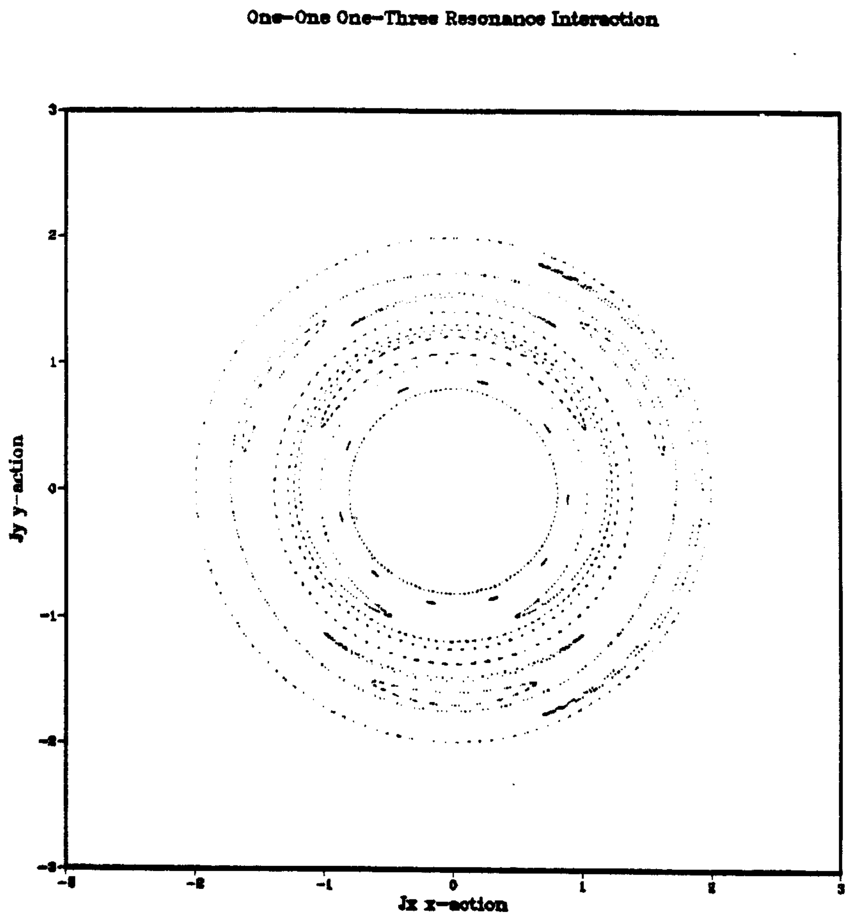

Figure 15 $x_o = 1.0$ and $s = 0.001$. Interaction between the one–one and one–three islands has generated satellites but there is no apparent instability yet.

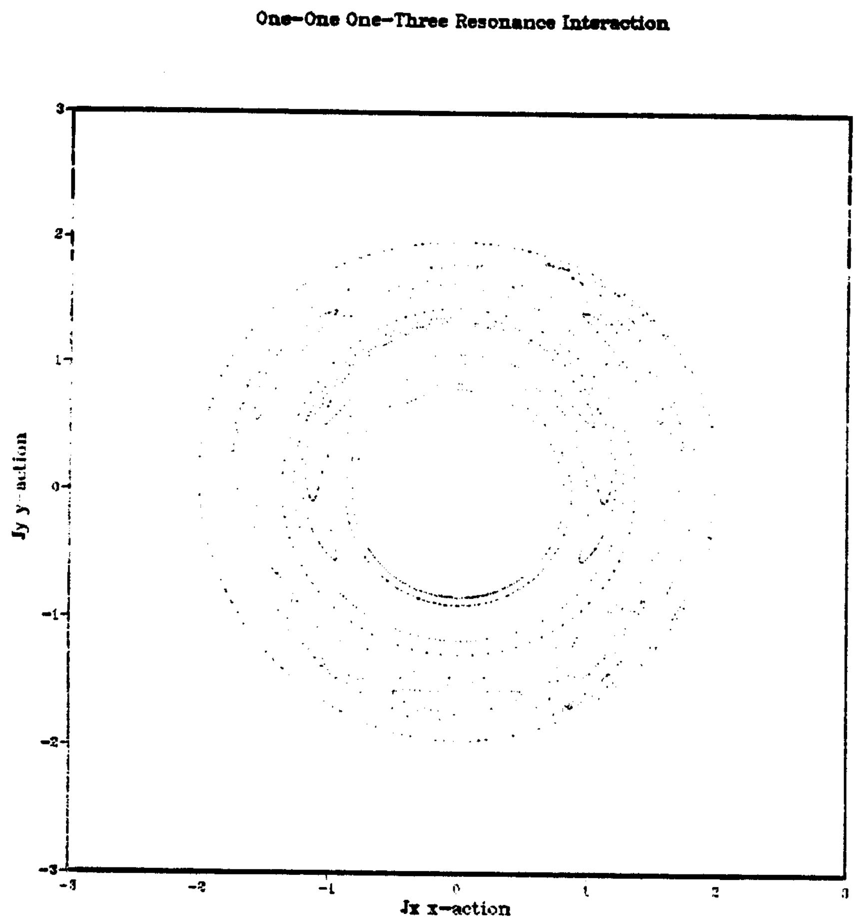

Figure 16 $x_o = 1.0$ and $s = 0.0003$. One can see some instability around the separatrices of the one–three island chain bounded by as yet unaffected tori.

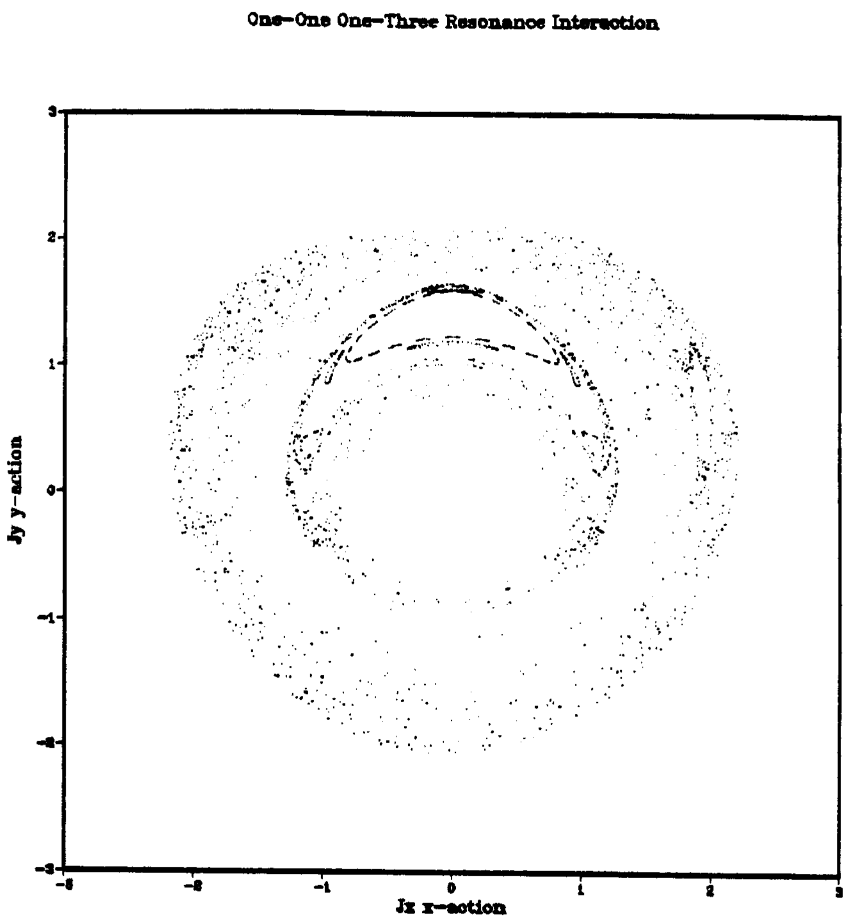

Figure 17 $x_o = 1.0$ **and** $s = 0.01$**. Here the situation has changed dramatically, though good tori are still bounding the action quite effectively.**

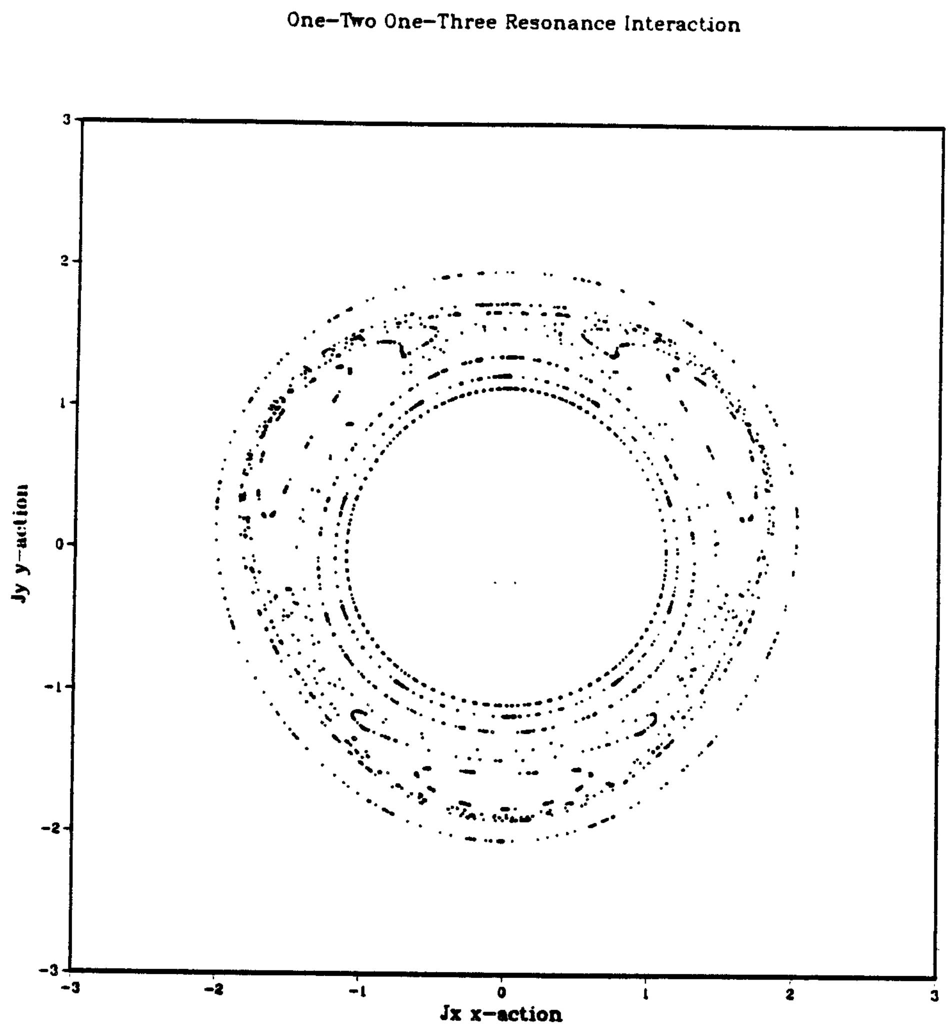

Figure 18 $x_o = 1.0$ **and** $s = 0.007$.

One–Two One–Three Resonance Interaction

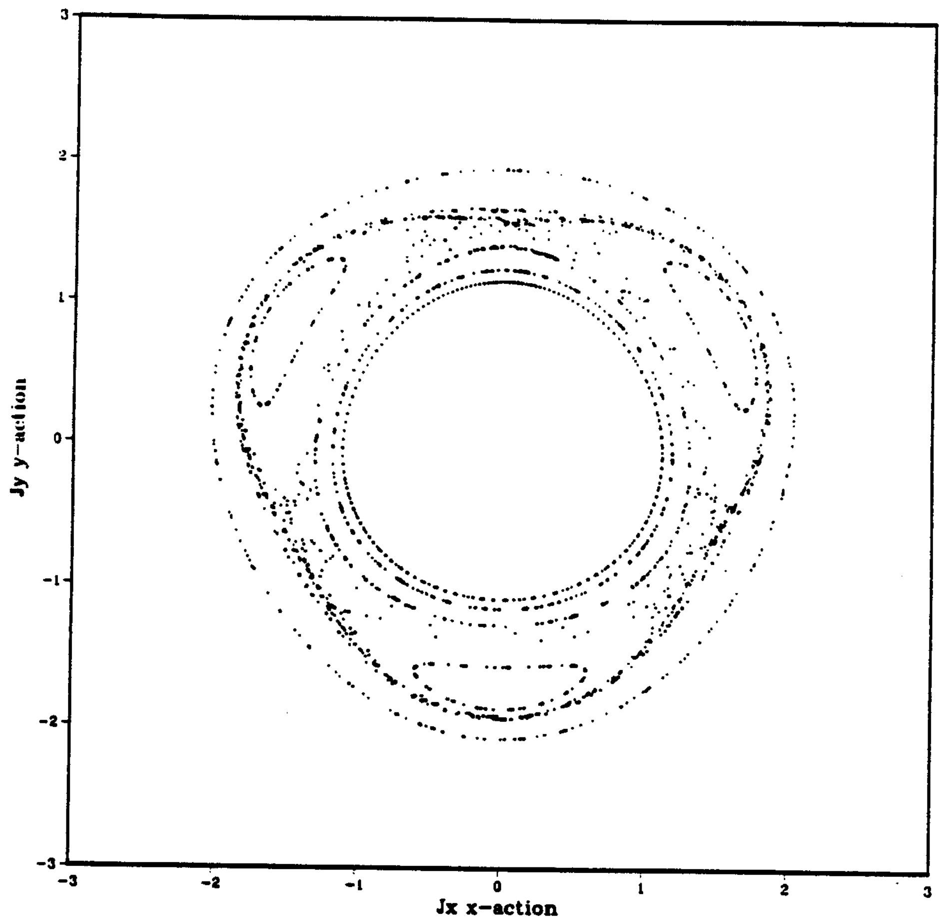

Figure 19 $x_o = 1.0$ **and** $s = 0.01$. **The one–two islands are not even visible; there is some instability near the separatrices of the one–three islands.**

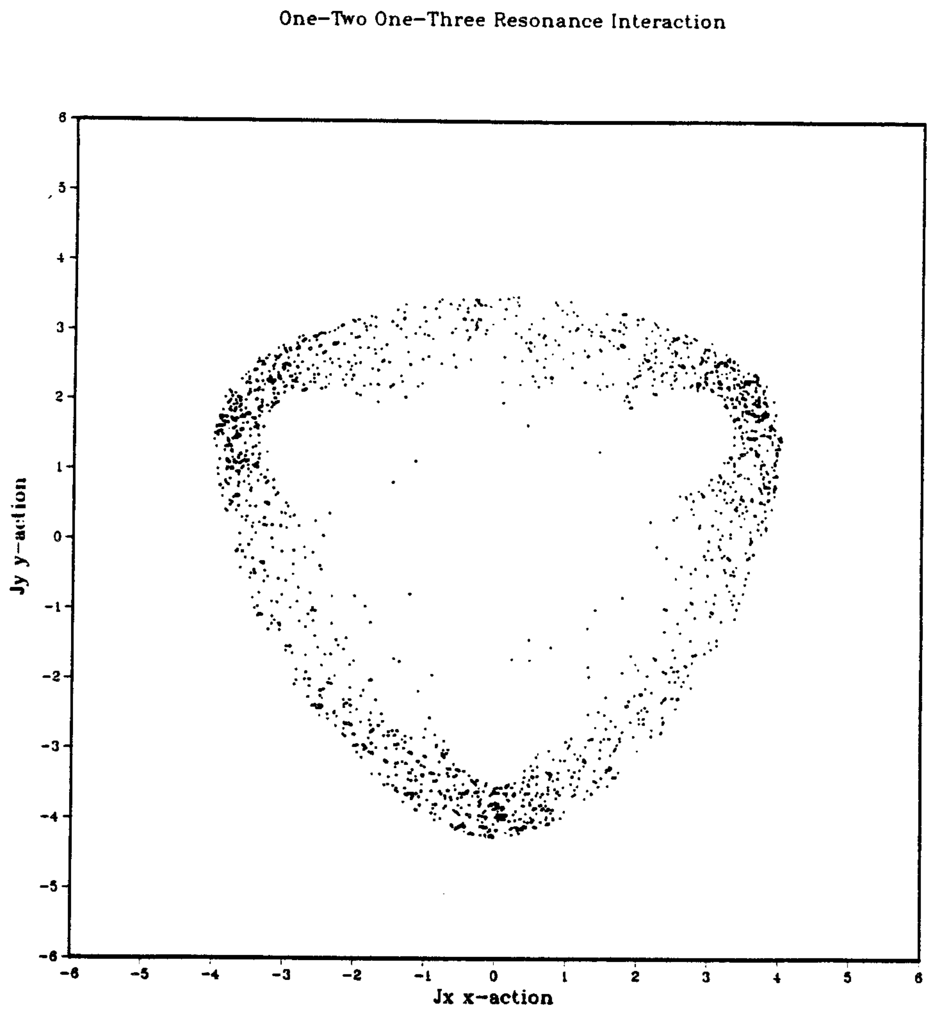

Figure 20 $x_o = 1.0$ **and** $s = 0.1$. **The nine orbits are very unstable but, and this is the significant point, are still bounded within a region of action space.**

One-One One-Two One-Three Resonance Interaction

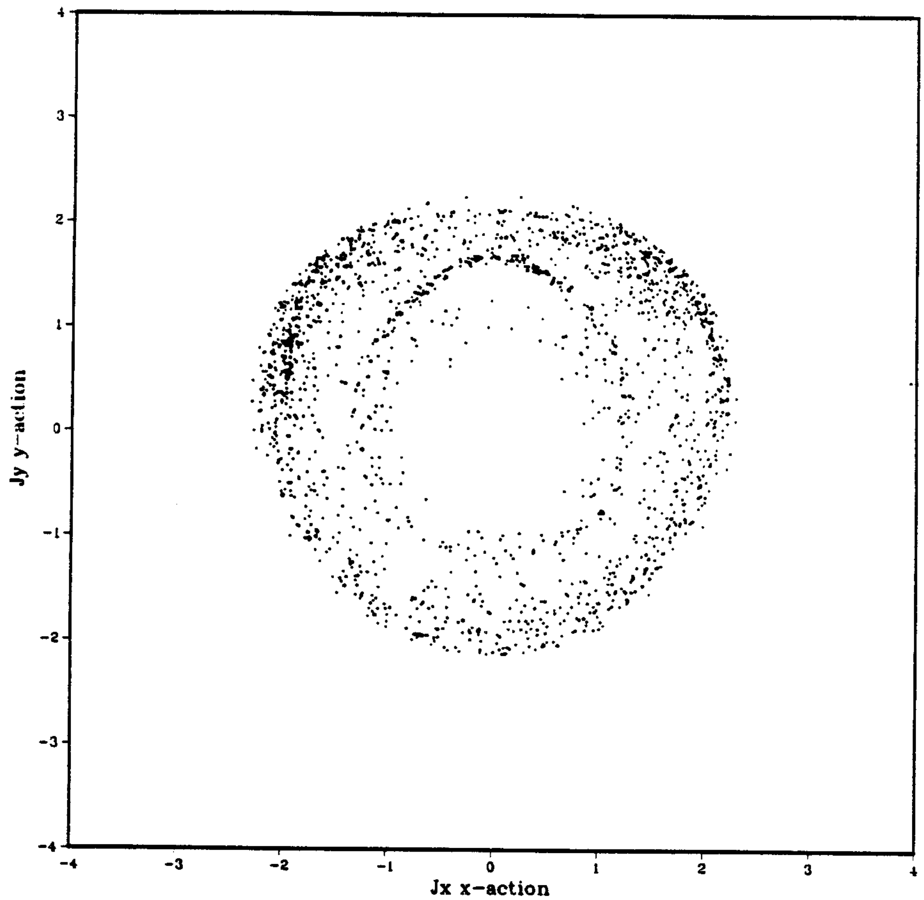

Figure 21 $x_o = 1.0$ **and** $s = 0.01$. **Orbits are locally unstable but well bounded by tori.**

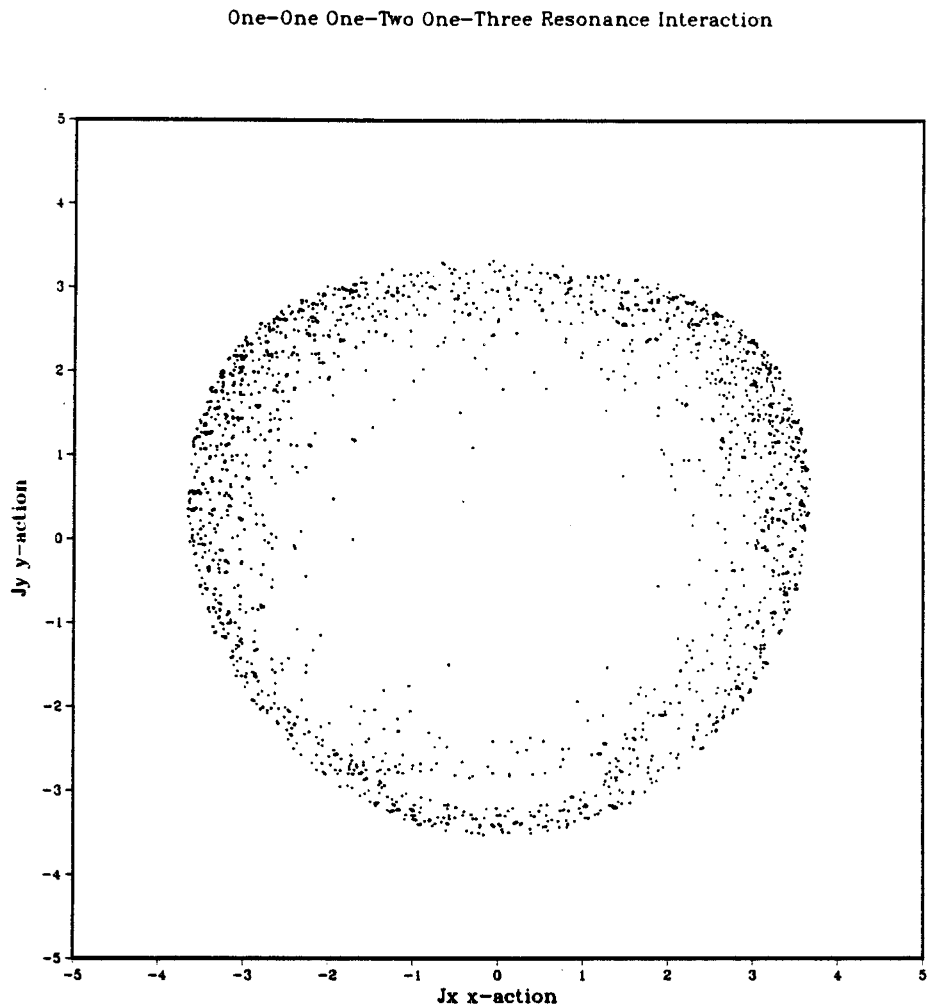

Figure 22 $x_o = 1.0$ and $s = 0.03$. The outer tori are expanding, indicating increasing transfer of energy from the azimuthal to the radial motion. We have found that this continues as the lateral displacement increases further, with no qualitative change in the orbit behavior within the torus.