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J.D. Callen and K.C. Shaing

Fusion Technology Institute
University of Wisconsin
1500 Engineering Drive
Madison, WI 53706

http://fti.neep.wisc.edu

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J.D. Callen and K.C. Shaing*

Fusion Engineering Program
Nuclear Engineering Department
University of Wisconsin-Madison
Madison, WI 53706

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*Present Address: Fusion Energy Division, Oak Ridge National Laboratory, Oak Ridge, TN 37830.

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J.D. Callen
Fusion Engineering Program, Nuclear Engineering Department
University of Wisconsin, Madison, WI 53706

and

K.C. Shaing
Fusion Energy Division, Oak Ridge National Laboratory
Oak Ridge, TN 37830

ABSTRACT

The moment equation approach to neoclassical processes is used to derive the linearized electrostatic perturbed flows, currents and resistive MHD-like equations for a tokamak plasma. The new features of the resultant "neoclassical MHD," which requires a multiple length scale analysis for the parallel eigenfunction but is valid in the experimentally relevant banana-plateau regime of collisionality, are: (1) a global Ohm's law that includes a fluctuating bootstrap current resulting from the "parallel" electron viscous damping (at rate μ_e) of the poloidal flow due to the perturbed radial pressure gradient; (2) reduction of the curvature effects to their flux surface average because Pfirsch-Schlüter currents cancel out the lowest order geodesic curvature effects; (3) an increased polarization drift contribution with B^{-2} replaced by B_Θ^{-2} where B_Θ is the poloidal magnetic field component. An electrostatic eigenmode equation is determined from $\underline{\nabla}\cdot\underline{\tilde{J}} = 0$. For the unstable fluid-like eigenmodes the new viscous damping effects dominate (by $\varepsilon^{-3/2}$) over the curvature effects, but the growth rates still scale roughly like resistive-g or resistive-ballooning modes -- $\gamma_\mu \tau_A \sim n^{2/3}\, S_N^{1/3}\, \beta_T^{2/3}\, (\mu_e/\nu_e)^{1/3}$. Diamagnetic drift frequency corrections to these new modes are also discussed.

I. Introduction

Nonlinear resistive MHD models of low mode number instabilities in tokamaks have had considerable success in explaining many of the macroscopic phenomena in tokamak discharges[1] -- internal disruptions (m/n = 1/1), Mirnov oscillations (m/n = 2/1), major disruptions (2/1-3/2 coupling), etc. Also, nonlinear models of medium mode number resistive ballooning modes in high β_p resistive tokamaks have been advanced to explain the deterioration of plasma confinement with high power neutral beam injection.[2] Most of the tokamak plasmas to which these theoretical models have been applied operate in a "long mean free path" regime in which the Coulomb collision mean free path λ exceeds the magnetic connection length $R_0 q$, i.e. the so-called banana-plateau regime of collisionality[3] where $\nu < \omega_b$. However, the resistive MHD equations[4] upon which the theory is based are usually derived in a short mean free path limit [$\lambda(\underline{b}\cdot\underline{\nabla}) \sim \lambda/R_0 q \ll 1$ where the two-fluid equations[5] apply] -- the so-called Pfirsch-Schlüter regime of collisionality where $\nu > \omega_b$. In this paper we explore the major modifications of the resistive MHD equations in the long mean free path limit (banana-plateau regime) and find a new type of "neoclassical MHD" instability.

The usual resistive MHD equations[4] differ from the ideal MHD equations primarily by the addition of the parallel (to the magnetic field $\underline{B}$) component of Ohm's law. From the neoclassical transport theory of tokamaks[3,6] we know that, at least in equilibrium, in the long mean free path regime the parallel component of Ohm's law is changed from its usual $E_\parallel = J_\parallel/\sigma_\parallel$ form in two important respects. First, it is no longer valid locally, but rather must be averaged over the periodic poloidal structure of the tokamak via a flux surface average. Second, the "parallel" viscous drag on the equilibrium poloidal flow carried by the untrapped particles, which is caused by collisions with the immobile toroidally trapped particles, leads to a parallel neoclassical or bootstrap[6,7] current. The bootstrap current is given approximately in the banana-plateau collisionality regime by $J_{\parallel} \simeq -(\mu_e/\nu_e)(c/B_p)\, dP/dr$, where ν_e is the electron collision frequency and $\mu_e \simeq \sqrt{\varepsilon}\, \nu_e/(1 + \nu_{*e})$ is the electron viscous drag frequency. (Throughout the paper we will neglect order unity constants in the viscosity coefficient definitions so as to be able to emphasize the dominant physical scalings.) Here $\varepsilon \equiv r/R_0 << 1$ is the inverse aspect ratio of the tokamak and $\nu_{*e} = \nu_e/\varepsilon^{3/2}\omega_{be}$ (with $\omega_{be} = v_{Te}/R_0 q$, the electron bounce frequency) is the electron collisionality parameter such that $\nu_{*e} = 1$ at the boundary between the banana and plateau collisionality regimes.

In view of these changes in the equilibrium parallel Ohm's law, we can thus anticipate the following modifications of the resistive MHD instability equations in the banana-plateau collisionality regime ($\nu_{*e} \sim 1$). First, since the perpendicular flows and currents are not significantly changed from ideal MHD by the addition of resistivity effects, they should remain relatively unaffected. However, the parallel flows and currents should be significantly modified, and now their net contributions will emerge only after averaging over a flux surface. Finally, and most importantly, for modes which evolve slowly compared to the poloidal flow viscous damping rate (i.e., $\omega < \mu_e$), we expect that the ideal equation of state $d/dt(P/\rho_m^{\Gamma}) = 0$ will no longer be valid and that there will be a fluctuating bootstrap current type modification to the parallel Ohm's law and hence to the resistive MHD instability equations. We will refer to the new equations so derived as "neoclassical MHD" equations.

A critical element in this work involves the use of a multiple length scale analysis in the ballooning mode representation of the perturbed potential along a magnetic field line. Specifically, the small, short scale variations (Pfirsch-Schlüter type effects) within one poloidal transit are averaged over to obtain perturbed flows and currents (i.e., the neoclassical flows and bootstrap current) that vary only slowly on the long scale, which represents the variation along a field line as it circumnavigates the torus many times. Thus, the lowest order density and momentum conservation equations become nearly the same as the corresponding neoclassical equations, except for their parametric dependence on the long scale variable. After determining the long scale perturbed currents, we combine them in $\underline{\nabla}\cdot\tilde{\underline{J}} = 0$, which gives an electrostatic eigenmode equation whose solution yields new unstable modes due to the fluctuating viscous damping (bootstrap current) effects.

This paper is organized as follows. In Section II we utilize the usual small gyroradius, perpendicularly localized mode ordering to derive density and momentum conservation equations in various orders. Also, the perpendicular (to $\underline{B}$) flows are solved for in both first order (diamagnetic and $\underline{E} \times \underline{B}$) and second order (classical, neoclassical diffusion and polarization drift). Next, in Section III we review how the moment equations are utilized[6] in neoclassical transport theory to determine the parallel flows and currents (both Pfirsch-Schlüter and bootstrap). In Section IV we apply this same methodology to determine the perturbed parallel flows and currents. It is also in this section that we first introduce the multiple length scale approximation alluded to above. Then in Section V we combine the perturbed currents into the charge continuity equation $\underline{\nabla}\cdot\tilde{\underline{J}} = 0$ and analyze the resultant eigenmode equation to obtain the new pressure-gradient-driven resistive MHD-like instability. In Section VI we discuss the modifications to our fluidlike analysis that occur when the mode number increases to the point where the mode frequency can become comparable to the ion viscous damping rate and/or the diamagnetic drift frequency. The results of this paper are discussed and summarized in Section VII. Finally, the Appendix presents a simplified derivation of resistive interchange or resistive-g modes in the notation and style of this paper to facilitate comparison of our results with the usual ones in resistive MHD.

II. Finite Gyroradius, Perpendicularly Localized Mode Ordering

In order to develop these neoclassical effect ideas more mathematically we derive long mean free path resistive MHD-like equations for tokamaks through a special ordering scheme on moment equations in a form similar to those utilized in neoclassical transport theory.[6] A more rigorous kinetic analysis being developed[8] will be published in a companion paper (see also Ref. 9). Assuming constant temperature (but $T_e \neq T_i$) throughout the plasma for simplicity so as to eliminate the need for considering heat balance equations, the moment equations for any given species of plasma particles can be written as[6]

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot n\underline{V} = 0 \tag{1}$$

$$mn \frac{d\underline{V}}{dt} = nq(\underline{E} + \frac{1}{c}\underline{V} \times \underline{B}) - \underline{\nabla} p - \underline{\nabla} \cdot \underline{\underline{\Pi}} - \underline{R} \tag{2}$$

where $\underline{\underline{\Pi}}$ is the viscous stress tensor and $\underline{R} = nq(\underline{J}_\perp/\sigma_\perp + \underline{J}_\parallel/\sigma_\parallel)$ is the friction between the electron and ion species in the plasma, with $\sigma_\perp = ne^2/m_e\nu_e$ and $\sigma_\parallel = ne^2/m_e\nu_e\alpha_e$, where α_e is[5] a constant of order unity that depends on the ion charge Z and which we will now set to unity for simplicity. (In axisymmetric neoclassical transport theory $\sigma_\parallel$ becomes reduced by a factor $(1 + \mu_e/\alpha_e\nu_e)^{-1}$ because of the viscous damping of the untrapped electrons as they flow over the trapped electrons.) For the usual small gyroradius expansion with $\delta \sim \rho/\ell \lesssim 10^{-2} << 1$, we expand the density in the series

$$n = n_0(\psi) + \delta\tilde{n}_1 + \ldots \qquad (3)$$

and similarly for the potential ϕ. Here, $n_0(\psi)$ indicates the equilibrium radial density profile, which is only a function of the (radial) poloidal magnetic flux coordinate ψ. The first order density has no equilibrium contribution[3,6] and is due only to the perturbation. Further, we expand the flow $\underline{V}$ and viscous stress tensor $\underline{\underline{\Pi}}$ in the form

$$\underline{V} = \delta(\overline{\underline{V}}_1 + \tilde{\underline{V}}_1) + \ldots \qquad (4)$$

where $\overline{\underline{V}}_1$ indicates the (neoclassical) equilibrium flow and $\tilde{\underline{V}}_1$ is the perturbed flow. We allow $\tilde{\underline{V}}_1$ to be on the order of the equilibrium neoclassical flow $\overline{\underline{V}}_1$ so as to be able to treat neoclassical and instability effects on an equal footing. For simplicity we will concentrate on electrostatic effects and thus assume that $\tilde{\underline{B}}_1 \sim \delta^2 B_0$ or smaller; hence $\tilde{\underline{B}}_1$ will give negligible contributions to the order we calculate and we will henceforth set $\underline{B}_0 = \underline{B}$ and $\underline{E} = -\underline{\nabla}\phi + E_\parallel^A\underline{b}$ $= -\underline{\nabla}(\phi_0 + \delta\tilde{\phi}_1) + \delta E_\parallel^A\underline{b}$, where $E_\parallel^A$ is the ohmic heating transformer induced $\underline{E}$ field in the plasma.

Next, we adopt a high mode number, perpendicularly localized mode ordering scheme whereby all perpendicular (to the $\underline{B}$ field) gradients of perturbed quantities are of order δ^{-1}, whereas gradients of equilibrium quantities are of order unity:

$$\underline{\nabla}_\perp(n_o + \delta\tilde{n}_1) = \delta^o\underline{\nabla}_\perp n_o + \frac{1}{\delta}\,\underline{\nabla}_1(\delta\tilde{n}_1) \sim \delta^o \ . \tag{5}$$

Thus, perturbations can significantly distort the local density gradient. Parallel (to $\underline{B}$) gradients are assumed to be comparable to equilibrium gradients:

$$(\underline{b}\cdot\underline{\nabla})(n_o + \delta\tilde{n}_1) = \delta(\underline{b}\cdot\underline{\nabla})\tilde{n}_1 \ , \tag{6}$$

in which $\underline{b} \equiv \underline{B}/B$ is the unit vector along the magnetic field. Finally, in concert with the rest of our small gyroradius ordering scheme, we assume temporal derivatives are on the order of the sound wave frequencies and hence of order unity:

$$\frac{\partial}{\partial t}(n_o + \delta\tilde{n}_1) = \delta\,\frac{\partial}{\partial t}\,\tilde{n}_1 \sim \delta \ . \tag{7}$$

With this customary ordering scheme, the linearized, lowest (δ^o) and first (δ) order density and momentum conservation equations become

$$\delta^o: \qquad n_o\{\underline{\nabla}_\perp\cdot\underline{\tilde{V}}_{1\perp}\} \equiv n_o\left[\frac{\partial}{\partial\psi}\,(\underline{\tilde{V}}_{1\perp}\cdot\underline{\nabla}\psi) + \frac{\partial}{\partial\beta}\,(\underline{\tilde{V}}_{1\perp}\cdot\underline{\nabla}\beta)\right] = 0 \tag{8}$$

$$0 = n_o q[-\underline{\nabla}_\perp(\phi_o + \tilde{\phi}_1) + \frac{1}{c}\,\underline{V}_{1\perp} \times \underline{B}] - \underline{\nabla}_\perp(p_o + \tilde{p}_1) \ , \tag{9}$$

$$\delta': \qquad \frac{\partial \tilde{n}_1}{\partial t} + \underline{V}_1 \cdot \underline{\nabla} n_o + \underline{\bar{V}}_{1\perp} \cdot \underline{\nabla} \tilde{n}_1 + (\underline{B} \cdot \underline{\nabla})(n_o V_{1\parallel}/B) - n_o \underline{V}_{1\perp} \cdot \underline{\nabla} \ln B^2 \tag{10}$$

$$+ n_o \{\underline{\nabla}_\perp \cdot \underline{\bar{V}}_{1\perp}\} + n_o \{\underline{\nabla}_\perp \cdot \underline{\tilde{V}}_{2\perp}\} = 0 ,$$

$$mn_o \left(\frac{\partial \underline{\tilde{V}}_1}{\partial t} + \underline{\bar{V}}_1 \cdot \underline{\nabla}_\perp \underline{\tilde{V}}_1\right) = n_o q[-\underline{b}(\underline{b} \cdot \underline{\nabla}) \tilde{\phi}_1 - \underline{\nabla}_\perp \tilde{\phi}_2 + E_\parallel^A \underline{b} + \frac{1}{c} \underline{V}_{2\perp} \times \underline{B}] \tag{11}$$

$$+ \tilde{n}_1 q[-\underline{\nabla}_\perp \phi_o + \frac{1}{c} \underline{\bar{V}}_{1\perp} \times \underline{B}] - \underline{b}(\underline{b} \cdot \underline{\nabla}) \tilde{p}_1 - \underline{\nabla}_\perp \tilde{p}_2 - \underline{\nabla} \cdot \underline{\underline{\Pi}}_1 - \underline{R}_1 .$$

Here, to order the various parts of the compressibility term $\underline{\nabla} \cdot \underline{V}_1$, we have utilized a Clebsch representation of the magnetic field for which

$$\underline{B} = \underline{\nabla}\psi \times \underline{\nabla}\beta = B\underline{b} , \tag{12}$$

with contra- and covariant basis vectors $\underline{\nabla}\psi$, $\underline{\nabla}\beta$, $\underline{b}$ and $\underline{u}_\psi$, $\underline{u}_\beta$, $\underline{b}$, respectively, where

$$\underline{u}_\psi = - \underline{b} \times \underline{\nabla}\beta/B , \qquad \underline{u}_\beta = \underline{b} \times \underline{\nabla}\psi/B . \tag{13}$$

(We will later set $\beta = q\Theta - \zeta$ to be tokamak specific.) Then, writing the velocity $\underline{V}$ as

$$\underline{V} = \underline{u}_\psi(\underline{\nabla}\psi \cdot \underline{V}) + \underline{u}_\beta(\underline{\nabla}\beta \cdot \underline{V}) + \underline{b}(\underline{b} \cdot \underline{V}) = \underline{\underline{I}} \cdot \underline{V} , \tag{14}$$

where $\underline{\underline{I}}$ is the identity tensor and utilizing $\underline{\nabla} \cdot \underline{B} = 0$ and the fact that in equilibrium $(\underline{\nabla} \times \underline{B})_\perp = (4\pi/c)\underline{J}_\perp = 4\pi(\underline{B} \times \underline{\nabla}p/B^2) = \underline{B} \times [(\underline{b} \cdot \underline{\nabla})\underline{b} - \frac{1}{B} \underline{\nabla}B]$, so that $\underline{\nabla} \cdot \underline{u}_\psi = -\underline{u}_\psi \cdot [(\underline{b} \cdot \underline{\nabla})\underline{b} + \frac{1}{B} \underline{\nabla}B]$, we find

$$\underline{\nabla}\cdot\underline{V} = (\underline{B}\cdot\underline{\nabla})(\underline{B}\cdot\underline{V}/B^2) + (\underline{u}_\psi\cdot\underline{\nabla})(\underline{V}\cdot\underline{\nabla}\psi) + (\underline{u}_\beta\cdot\underline{\nabla})(\underline{V}\cdot\underline{\nabla}\beta) \tag{15}$$

$$- [(\underline{V}\cdot\underline{\nabla}\psi)\underline{u}_\psi + (\underline{V}\cdot\underline{\nabla}\beta)\underline{u}_\beta]\cdot[(\underline{b}\cdot\underline{\nabla})\underline{b} + \frac{1}{B}\underline{\nabla}B]$$

$$\simeq (\underline{B}\cdot\underline{\nabla})(\underline{B}\cdot\underline{V}/B^2) + \{\underline{\nabla}_\perp\cdot\underline{V}_\perp\} - \underline{V}_\perp\cdot\underline{\nabla}\ln B^2$$

$$= \frac{1}{B^2}(\underline{B}\cdot\underline{\nabla})(\underline{B}\cdot\underline{V}) + \{\underline{\nabla}_\perp\cdot\underline{V}_\perp\} - \underline{V}\cdot\underline{\nabla}\ln B^2 .$$

In the last expressions we have made use of the fact that for the low β plasma equilibria in tokamaks, to lowest order in β we have $(\underline{b}\cdot\underline{\nabla})\underline{b} \simeq \frac{1}{B}\underline{\nabla}B$. Also, in Eqs. (9) and (10) we have ordered the perpendicular component of the viscosity one order smaller than might be indicated in Eq. (4) because the perpendicular viscosity coefficients are one or more orders smaller in the gyroradius expansion than the parallel viscosity coefficients.[5]

The lowest order momentum balance given in Eq. (9) can be readily solved to yield the perpendicular flow velocity

$$\underline{V}_{1\perp} = \frac{c}{B}\underline{b} \times [\underline{\nabla}(\phi_o + \tilde{\phi}_1) + \frac{1}{n_o q}\underline{\nabla}(p_o + \tilde{p}_1)] \equiv \underline{\bar{V}}_{1\perp} + \underline{\tilde{V}}_{1\perp} , \tag{16}$$

which indicates both $\underline{E} \times \underline{B}$ and diamagnetic flow contributions, and the perpendicular diamagnetic current

$$\underline{J}_{1\perp} = \frac{c}{B^2}\underline{B} \times \underline{\nabla}(P_o + \tilde{P}_1) , \tag{17}$$

where $P_o \equiv \sum_{\text{species}} p_o$, $\tilde{P}_1 = \sum_{\text{species}} \tilde{p}_1$. Note that, because of our localized mode ordering $\underline{\nabla}_\perp \sim \delta^{-1}$ [cf., Eq. (5)], these lowest order flows include contributions due to the perpendicular derivatives of the perturbations as well as the usual ones due to the gradients of the equilibrium potential and pressure. Also, note that since

$$\{\underline{\nabla}_\perp \cdot \tilde{\underline{V}}_{1\perp}\} = + c\left(\frac{\partial}{\partial\beta}\frac{\partial}{\partial\psi} - \frac{\partial}{\partial\psi}\frac{\partial}{\partial\beta}\right)\left[\tilde{\phi}_1 + \frac{1}{n_o q}\tilde{p}_1\right] = 0 , \tag{18}$$

the $\tilde{\underline{V}}_{1\perp}$ in Eq. (16) automatically satisfies the lowest order density conservation or incompressibility equation given in Eq. (8). Similarly, the $\{\underline{\nabla}_\perp \cdot \overline{\underline{V}}_{1\perp}\}$ contribution to Eq. (10) vanishes.

Next, we consider solving the first order density and momentum conservation equations given in Eqs. (10) and (11) for the required quantities $V_{1\parallel}$, $\tilde{n}_1$ and $\underline{V}_{2\perp}$. The second order perpendicular flow $\underline{V}_{2\perp}$ is obtained by taking $\underline{B}\times$ the momentum balance equation and is found to be (after cancelling the diamagnetic flow term part of $mn_o\overline{\underline{V}}_1 \cdot \underline{\nabla}_\perp \tilde{\underline{V}}_1$ off against part of $\underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}_1$ -- see Ref. 10)

$$\underline{V}_{2\perp} = \frac{c}{n_o qB}\,\underline{b} \times \Big[n_o q\underline{\nabla}\tilde{\phi}_2 + \tilde{n}_1 q\underline{\nabla}\phi_o + \underline{\nabla}\tilde{p}_2 + \underline{\nabla}\cdot\underline{\underline{\Pi}}_1$$

$$+ \underline{R}_1 + mn_o\left(\frac{\partial}{\partial t} + \overline{\underline{V}}_E \cdot \underline{\nabla}\right)\tilde{\underline{V}}_1\Big] - \frac{n_1}{n_o}\overline{\underline{V}}_{1\perp} , \tag{19}$$

in which $\overline{\underline{V}}_E = (c/B)(\underline{b} \times \underline{\nabla}\phi_o)$ is the equilibrium $\underline{E}_o \times \underline{B}$ flow. This leads to the second order perpendicular current

$$\tilde{\underline{J}}_{2\perp} \equiv \sum_{\text{species}} q\left(n_o\tilde{\underline{V}}_{2\perp} + \tilde{n}_1\overline{\underline{V}}_{1\perp}\right)$$

$$= \frac{c}{B}\,\underline{b} \times \left[\underline{\nabla}\tilde{P}_2 + \rho_m\left(\frac{\partial}{\partial t} + \overline{\underline{V}}_E \cdot \underline{\nabla}\right)\tilde{\underline{V}}_{1i} + \underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}_e + \underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}_i\right] , \tag{20}$$

where $\tilde{P}_2$ and ρ_m are the total second order pressure and mass density (dominated by the ions) defined by

$$\tilde{P}_2 = \sum_{species} \tilde{p}_2 , \qquad \rho_m = \sum_{species} n_o m \simeq n_o m_i . \tag{21}$$

Note that the perpendicular friction force component $\underline{b} \times \underline{R}_1$ in Eq. (19) just yields the classical, ambipolar diffusive flow due to Coulomb collisions:

$$\frac{c}{n_o q B^2} \underline{B} \times \underline{R}_1 = \frac{c}{\sigma_\perp B} \underline{b} \times \underline{J}_{1\perp} = - D \underline{\nabla}_\perp (n_o + \tilde{n}_1) , \; D = \frac{c^2 (T_e + T_i)}{B^2 \sigma_\perp} = \nu_e \rho_e^2 \left(\frac{T_e + T_i}{2T_e}\right) \tag{22}$$

where $\rho_e \equiv \sqrt{2T_e/m_e}/\Omega_e$ is the electron gyroradius and $\Omega_e \equiv eB/m_e c$ is the electron gyrofrequency. Further, the $\underline{\nabla}\psi \cdot \underline{b} \times (\underline{\nabla} \cdot \underline{\underline{\Pi}})$ component of Eq. (19) can be shown [see Ref. 6 and the discussion after Eq. (44)], at least in its flux surface average form, to lead to the Pfirsch-Schlüter and banana-plateau regime crossfield transport, which is also ambipolar and so does not contribute to Eq. (20). While the flows induced by $\tilde{\phi}_2$ and $\tilde{p}_2$ are nonzero, they do not contribute to the $\{\underline{\nabla}_\perp \cdot \tilde{\underline{V}}_{2\perp}\}$ term in Eq. (10) because, in analogy with Eq. (18), they are proportional to gradients of perturbed quantities and thus, to lowest order, are incompressible flows.

To determine the remaining quantities $V_{1\parallel}$ and $\tilde{n}_1$, we utilize Eqs. (16), (19) in Eqs. (10) and the parallel $(\underline{B}\cdot)$ component of Eq. (11) to obtain

$$\frac{\partial \tilde{n}_1}{\partial t} + \underline{V}_1 \cdot \underline{\nabla} n_o + \overline{\underline{V}}_{1\perp} \cdot \underline{\nabla} \tilde{n}_1 + (\underline{B} \cdot \underline{\nabla})(n_o V_{1\parallel}/B) - n_o \underline{V}_{1\perp} \cdot \underline{\nabla} \ln B^2 \tag{23}$$

$$- \frac{c n_o}{B\Omega} \left(\frac{\partial}{\partial t} + \overline{\underline{V}}_E \cdot \underline{\nabla}\right) \nabla_\perp^2 \left(\tilde{\phi}_1 + \frac{1}{n_o q} \tilde{p}_1\right) + \frac{1}{m\Omega} \frac{\partial}{\partial \psi} \left(\underline{\nabla}\psi \cdot \underline{b} \times (\underline{\nabla} \cdot \tilde{\underline{\underline{\Pi}}}_1)\right) \equiv 0 ,$$

$$mn_o\left(\frac{\partial}{\partial t} + \underline{\overline{V}}_1 \cdot \underline{\nabla}_\perp\right)(\underline{B} \cdot \underline{\tilde{V}}_1) = -(\underline{B} \cdot \underline{\nabla})(\tilde{p}_1 + n_o q \tilde{\phi}_1) + n_o q E_\parallel^A B - \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\Pi}}_1 - \underline{B} \cdot \underline{R} \tag{24}$$

in which $$\nabla_\perp^2 \equiv \{\underline{\nabla}_\perp \cdot \underline{\nabla}_\perp\} \equiv \frac{\partial}{\partial \psi} |\underline{\nabla}\psi|^2 \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \beta} |\underline{\nabla}\beta|^2 \frac{\partial}{\partial \beta} .$$

A $(1/m\Omega)(\partial/\partial\beta)(\underline{b} \times \underline{\nabla}\beta \cdot (\underline{\nabla} \cdot \underline{\underline{\tilde{\Pi}}}_1))$ term has been neglected in Eq. (23) since it represents a radial viscosity contribution that is higher order in the gyroradius expansion than the viscous stress effects within the flux surface which are being retained. To proceed further it is necessary to take into account more explicitly the tokamak geometry and to develop a subsidiary expansion based upon large aspect ratio and greater radial than azimuthal localization of the modes. Before doing this, we discuss the solution of Eqs. (23) and (24) for the equilibrium neoclassical flow $\overline{V}_{1\parallel}$, which we need to determine the convective flow derivative $\underline{\overline{V}}_1 \cdot \underline{\nabla}_\perp$ in the perturbed equation.

III. Neoclassical Equilibrium Flows and Currents

Taking an average of Eqs. (23), (24) over time and length scales longer than those of the fluctuating modes, we obtain for the neoclassical equilibrium equations

$$(\underline{B} \cdot \underline{\nabla})(\overline{V}_{1\parallel}/B) - \underline{\overline{V}}_{1\perp} \cdot \underline{\nabla} \ln B^2 = 0 \tag{25}$$

$$0 = + n_o q E_\parallel^A B - \underline{B} \cdot \underline{\nabla} \cdot \underline{\underline{\overline{\Pi}}}_1 - \underline{B} \cdot \underline{\overline{R}}_1 . \tag{26}$$

Note that, in contrast to the usual MHD equations, Eq. (25) shows that the parallel velocity and perpendicular flow velocity components in the magnetic flux surface are linked in neoclassical equilibria. We now focus our attention on an axisymmetric tokamak equilibrium for which we define

$$\underline{B} = I\underline{\nabla}\zeta + \underline{\nabla}\zeta \times \underline{\nabla}\psi = \underline{\nabla}\psi \times \underline{\nabla}(q\Theta - \zeta) \tag{27}$$

where $I = I(\psi) \equiv RB_\zeta$, $q = q(\psi)$ (the toroidal winding number, or inverse of the rotational transform), and Θ and ζ are the poloidal and toroidal angles. Note also that for this coordinate system $\beta = q\Theta - \zeta$ and that $\underline{\nabla}\zeta = \hat{\zeta}/R$. Then, we find

$$\begin{aligned}
\underline{\overline{V}}_{1\perp}\cdot\underline{\nabla}\ \ln B^2 &= \underline{\overline{V}}_{1\perp}\cdot\left(\underline{\nabla}\zeta\frac{\partial}{\partial\zeta} + \underline{\nabla}\Theta\frac{\partial}{\partial\Theta} + \underline{\nabla}\psi\frac{\partial}{\partial\psi}\right)\ln B^2 \\
&= \frac{c}{B}\,\underline{b}\times\underline{\nabla}\psi\cdot\underline{\nabla}\Theta\left(\frac{\partial}{\partial\Theta}\ln B^2\right)\left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o q}\frac{dp_o}{d\psi}\right) \\
&= +\,c\,\frac{I^2(\psi)}{q(\psi)R^2}\left(\frac{1}{B^2}\frac{\partial}{\partial\Theta}\ln B^2\right)\left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o q}\frac{dp_o}{d\psi}\right) \\
&= -(\underline{B}\cdot\underline{\nabla})\left[\frac{cI(\psi)}{B^2}\left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o q}\frac{dp_o}{d\psi}\right)\right],
\end{aligned} \tag{28}$$

since $(\underline{B}\cdot\underline{\nabla})f = (\underline{B}\cdot\underline{\nabla}\psi\frac{\partial}{\partial\psi} + \underline{B}\cdot\underline{\nabla}\Theta\frac{\partial}{\partial\Theta} + \underline{B}\cdot\underline{\nabla}\zeta\frac{\partial}{\partial\zeta})f = (I/qR^2)(q\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\Theta})f$. Utilizing this result in Eq. (25), we find that it can be written as

$$(\underline{B}\cdot\underline{\nabla})\left[\frac{\overline{V}_{1\parallel}}{B} + \frac{cI(\psi)}{B^2}\left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o q}\frac{dp_o}{d\psi}\right)\right] = 0\ .$$

This equation can be integrated along a field line to yield

$$\overline{V}_{1\parallel} = \overline{U}_\parallel(\psi)/B + \overline{U}_p(\psi)B \tag{29}$$

where
$$\overline{U}_\parallel(\psi) \equiv -\,cI(\psi)\left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o q}\frac{dp_o}{d\psi}\right) \tag{30}$$

is B times the parallel flow component that results in no net poloidal flow and $\bar{U}_p(\psi)$ is a constant of integration on a flux surface, which is[6] the poloidal flow velocity divided by the poloidal component of $\underline{B}$ ($\bar{U}_p \equiv \underline{\bar{V}}_1 \cdot \nabla\Theta / \underline{B} \cdot \nabla\Theta$). The total parallel flow can be written in terms of its Pfirsch-Schlüter ($\langle B\bar{V}_{1\parallel}\rangle = 0$) and flux surface average ($\langle B\bar{V}_{1\parallel}\rangle \neq 0$) parts in the form

$$B\bar{V}_{1\parallel} = \left[1 - \frac{B^2}{\langle B^2\rangle}\right] \bar{U}_\parallel(\psi) + B^2 \left[\bar{U}_p(\psi) + \frac{\bar{U}_\parallel(\psi)}{\langle B^2\rangle}\right] \tag{31}$$

in which $\langle B^2\rangle$ is the flux surface average of B^2 defined by

$$\langle B^2\rangle \equiv \oint \frac{d\ell}{B} B^2 \Big/ \oint \frac{d\ell}{B} = \oint \frac{d\Theta}{\underline{B}\cdot\nabla\Theta} B^2 \Big/ \oint \frac{d\Theta}{\underline{B}\cdot\nabla\Theta} . \tag{32}$$

To determine the neoclassical poloidal flow velocity coefficient $\bar{U}_p(\psi)$, we consider the momentum balance equations in Eq. (26) for both electrons and ions:

$$0 = - n_o e E_\parallel^A B - \underline{B}\cdot\nabla\cdot\bar{\underline{\underline{\Pi}}}_{1e} - \underline{B}\cdot\bar{\underline{R}}_{1e} \tag{33}$$

$$0 = n_o e E_\parallel^A B - \underline{B}\cdot\nabla\cdot\bar{\underline{\underline{\Pi}}}_{1i} - \underline{B}\cdot\bar{\underline{R}}_{1i} . \tag{34}$$

For roughly comparable electron and ion $\underline{E} \times \underline{B}$ and diamagnetic flows $\underline{\bar{V}}_1$, the dominant term in these equations is the ion viscosity term, which upon flux surface averaging yields[6]

$$0 = \langle \underline{B}\cdot\nabla\cdot\bar{\underline{\underline{\Pi}}}_i\rangle = \langle B^2\rangle\, m_i n_i \mu_i \bar{U}_{pi}(\psi) \tag{35}$$

in which μ_i is a parallel or poloidal flow viscous damping rate given approximately by

$$\mu_i \simeq \frac{\sqrt{\epsilon}\ \nu_i}{1 + \nu_{*i}} \ . \tag{36}$$

The solution of Eq. (35) is

$$\overline{U}_{pi} = 0 \ , \qquad \overline{V}_{1\parallel i} = \frac{\overline{U}_{\parallel i}}{B} = - \frac{cI(\psi)}{B} \left(\frac{d\phi_o}{d\psi} + \frac{1}{n_o e} \frac{dp_{oi}}{d\psi}\right) \ . \tag{37}$$

This indicates that the parallel ion viscosity, which is due to the viscous force exerted by the trapped ions on the toroidally (and poloidally) passing ions, damps the poloidal flow of the ion species to zero. Further, for $\overline{U}_{pi} = 0$, it is easily shown that the total equilibrium ion flow velocity $\overline{V}_{1\parallel}\underline{b} + \underline{\overline{V}}_{1\perp}$ is purely toroidal and given by

$$\overline{V}_{\zeta i}/R \equiv \underline{\nabla}\zeta\cdot\underline{\overline{V}}_{1i} = - c\left[\frac{d\phi_o}{d\psi} + \frac{1}{n_o e} \frac{dp_{oi}}{d\psi}\right] \simeq - \frac{c}{RB_p} \left[\frac{d\phi_o}{dr} + \frac{1}{n_o e} \frac{dp_{oi}}{dr}\right] \ . \tag{38}$$

Next, we make a subsidiary expansion in the smallness Δ of the electron viscosity and the electric field $E_\parallel^A$ compared to the electron collision rate:

$$\Delta \sim \mu_e/\nu_e \ , \ e\sigma_\parallel E_\parallel^A/m_e \nu_e \overline{V}_{1\parallel} \ll 1 \ . \tag{39}$$

Then, expanding the parallel electron and ion flows as

$$\overline{V}_{1\parallel} = \overline{V}_{1\parallel}^{(0)} + \Delta\overline{V}_{1\parallel}^{(1)} + \dots \ , \tag{40}$$

we find from the lowest order form of Eq. (33) that

$$0 = -\underline{B}\cdot\overline{R}_{1e} = n_e e\underline{B}\cdot\overline{\underline{J}}_1^{(0)}/\sigma_\parallel = -m_e n_e \nu_e B(\overline{V}_{1\parallel e}^{(0)} - \overline{V}_{1\parallel i}^{(0)}) , \tag{41}$$

which requires that

$$\overline{V}_{1\parallel e}^{(0)} = \overline{V}_{1\parallel i}^{(0)} , \quad \langle B(\overline{V}_{1\parallel e}^{(0)} - \overline{V}_{1\parallel i}^{(0)})\rangle = 0 . \tag{42}$$

Since the parallel ion viscosity forces $\overline{U}_{pi}$ to zero, in order to keep the parallel electron and ion flows equal, as required by Eq. (35), we must have

$$\overline{U}_{pe} = \frac{\overline{U}_{\parallel i} - \overline{U}_{\parallel e}}{\langle B^2\rangle} = -\frac{cI(\psi)}{n_o e\langle B^2\rangle}\frac{dP_o}{d\psi} . \tag{43}$$

Note that while the equilibrium ion flow is purely toroidal, the electron flow is not; the viscous damping of the residual electron poloidal flow will lead to the bootstrap current. Utilizing the results of Eqs. (35), (36) and a form for the parallel electron viscosity analogous to Eqs. (35), (36), we find that the flux-surface-averaged first order parallel electron momentum balance equation becomes

$$\langle\underline{B}\cdot\overline{\underline{R}}_{1e}^{(1)}\rangle \equiv -n_o e\langle B\overline{J}_{1\parallel}^{(1)}\rangle/\sigma_\parallel = -n_o e\langle E_\parallel^A B\rangle - m_e n_o \mu_e \overline{U}_{pe}\langle B^2\rangle ,$$

or

$$\langle B\overline{J}_{1\parallel}^{(1)}\rangle = \sigma_\parallel\langle E_\parallel^A B\rangle + \left(\frac{\mu_e}{\nu_e}\right) n_o e\overline{U}_{pe}\langle B^2\rangle = \sigma_\parallel\langle E_\parallel^A B\rangle - \left(\frac{\mu_e}{\nu_e}\right) cI(\psi)\frac{dP_o}{d\psi} . \tag{44}$$

This is the standard[6] flux-surface-averaged parallel Ohm's law for a neoclassical, axisymmetric tokamak, with the second term representing the bootstrap current,[7] since $(\mu_e/\nu_e)cI(\psi)\, dP_o/d\psi \simeq \sqrt{\varepsilon}\,(1 + \nu_{*e})^{-1}(c/B_p)(dP_o/dr)$.

Having obtained the first order equilibrium neoclassical flows, we can now return to Eq. (19) to work out the second order neoclassical transport flows $(\underline{\nabla}\psi\cdot\overline{\underline{V}}_{2\perp})$. Namely, we calculate $\underline{\nabla}\psi\cdot\overline{\underline{V}}_{2\perp}$ utilizing the geometric identity

$$\frac{\underline{b} \times \underline{\nabla}\psi}{B} = R^2\underline{\nabla}\zeta - \frac{I\underline{B}}{B^2}$$

to yield, in flux surface average form,

$$n_o\langle\underline{\nabla}\psi\cdot\overline{\underline{V}}_{2\perp}\rangle_{NC} \equiv \langle\frac{c}{eB^2}\,\underline{\nabla}\psi\cdot\underline{B}\times(\underline{\nabla}\cdot\overline{\underline{\underline{\Pi}}}_1)\rangle = \frac{c}{e}\,I\langle\left(\frac{1}{B^2} - \frac{1}{\langle B^2\rangle}\right)\underline{B}\cdot\underline{\nabla}\cdot\overline{\underline{\underline{\Pi}}}_1\rangle + \frac{cI}{e\langle B^2\rangle}\langle\underline{B}\cdot\underline{\nabla}\cdot\overline{\underline{\underline{\Pi}}}_1\rangle ,$$

in which we have utilized the fact that $\langle R^2\underline{\nabla}\zeta\cdot\underline{\nabla}\cdot\overline{\underline{\underline{\Pi}}}_1\rangle = 0$ because of the conservation of toroidal angular momentum to the order (in the small gyroradius expansion) being calculated.[6] The first term here gives the Pfirsch-Schlüter transport contribution while the second indicates the banana-plateau transport.[6] As can be seen from Eqs. (33) and (34), these electron and ion radial particle transport flows are ambipolar.

Before proceeding to analyze the perturbed flows, we summarize what we have learned from the first order neoclassical equilibrium flows as follows:

1. The parallel ion viscosity couples the parallel and $\underline{u}_\beta = \frac{1}{B}\,\underline{b} \times \underline{\nabla}\psi$ components of the ion flow in such a way as to damp out the poloidal ion flow and yield only a toroidal flow given by Eq. (38).
2. Collisional friction between electrons and ions causes the parallel electron flow to be equal to the parallel ion flow in lowest order, which leads to a poloidal electron flow given by Eq. (43). The viscous damping

of this poloidal electron flow leads to the bootstrap contribution to the parallel Ohm's law.

3. The parallel electron momentum balance or Ohm's law in the banana-plateau collisionality regime is only meaningful in a flux surface averaged sense, as indicated in Eq. (44).

4. In addition to the flux surface average $\bar{J}_{1\parallel}$ there is a Pfirsch-Schlüter component that averages to zero ($\langle \bar{J}_{1\parallel PS} B \rangle = 0$) and is given by

$$\bar{J}_{1\parallel PS} = \frac{n_o e(\bar{U}_{\parallel i} - \bar{U}_{\parallel e})}{B} \left(1 - \frac{B^2}{\langle B^2 \rangle}\right) = \frac{cI(\psi)}{B} \left(\frac{B^2}{\langle B^2 \rangle} - 1\right) \frac{dP_o}{d\psi}. \qquad (45)$$

The Pfirsch-Schlüter current is the zero average current whose parallel derivative is just what is required to cancel the geodesic curvature [$\partial B^2/\partial \beta \rightarrow (\underline{B}\cdot\underline{\nabla})(1/B^2)$ in an axisymmetric tokamak, cf. Eq. (28)] driven charge imbalance $\underline{\nabla}\cdot\bar{\underline{J}}_{\perp}$. That is, we have

$$(\underline{B}\cdot\underline{\nabla})(\bar{J}_{1\parallel PS}) = -\int \frac{d\ell}{B} \underline{\nabla}\cdot\bar{\underline{J}}_{1\perp} = -2 \int \frac{d\ell}{B} \frac{\partial P_o}{d\psi} \frac{\partial \ln B^2}{\partial \beta} .$$

Note that this cancellation was insured by the first order incompressibility constraint conditions for each plasma species, as given by Eq. (25).

IV. Analysis of Perturbed Flows and Currents

We now analyze the perturbed flows in a manner analogous to this development of the neoclassical flows. The first step in this procedure is to make a subsidiary expansion (in Δ) so that the lowest order flows are poloidal and toroidal, with the radial perturbed flow being first order in Δ. To do this we need to order the poloidal ($\partial/\partial\beta$) gradients of perturbations to be one order in Δ smaller than the radial $(\partial/\partial\psi)$ gradients. Also, we order the wave frequency $(\partial/\partial t \sim -i\omega)$ to be of order the parallel gradients $[v_T(\underline{b}\cdot\underline{\nabla}) \sim k_\parallel v_T \lesssim \omega]$ and diamagnetic drift frequency ω_*, with all three being order Δ smaller than the bounce frequency $\omega_b = v_T/R_0 q$. To make this explicit, we utilize a local ballooning mode representation for the perturbed quantities:

$$\tilde{\phi}_1 = \sum_n e^{-in\beta-i\omega t}\,\tilde{\phi}_n(\psi,y) = \sum_{n,k} e^{in\zeta-ik\Theta-i\omega t}\int_{-\infty}^{\infty} dy\; e^{i(k-nq)y}\,\tilde{\phi}_n(\psi,y) \tag{46}$$

$$= \sum_{n,k} e^{in\zeta-ik\Theta-i\omega t}\,\tilde{\phi}_{nk}(\psi)$$

in which $d\ell = R_0 q\, dy$. Then, we order

$$v_{Te}(\underline{b}\cdot\underline{\nabla}),\ \omega,\omega_* \ll \mu_e \sim \Delta\omega_{be} \ll \nu_e,\ \omega_{be}\ , \tag{47}$$

$$v_{Ti}(\underline{b}\cdot\underline{\nabla}),\ \omega,\omega_* \sim \mu_i \sim \Delta\omega_{bi} \ll \nu_i,\ \omega_{bi}\ ,$$

$$\frac{\partial}{\partial\psi} \sim \Delta^0\ , \qquad \frac{\partial}{\partial\beta} \sim \Delta\ , \qquad \underline{b}\cdot\underline{\nabla} \sim \Delta\ , \tag{48}$$

and expand $\tilde{n}_1$, $\underline{\tilde{V}}_1$ in the small parameter Δ:

$$\begin{aligned} \underline{\tilde{V}}_1 &= \underline{\tilde{V}}_1^{(0)} + \Delta\underline{\tilde{V}}_1^{(1)} + \dots \\ \tilde{n}_1 &= \tilde{n}_1^{(0)} + \Delta\tilde{n}_1^{(1)} + \dots \\ \tilde{\phi}_1 &= \tilde{\phi}_1^{(0)} + \Delta\tilde{\phi}_1^{(1)} + \dots\ . \end{aligned} \tag{49}$$

Utilizing these orderings, we find that the lowest order perpendicular flow is in the $\underline{u}_\beta$ direction, and the first order flow is radial:

$$\tilde{\underline{V}}_{1\perp}^{(0)} = \frac{c}{B}\,\underline{b}\times\underline{\nabla}\psi\left[\frac{\partial\tilde{\phi}_1^{(0)}}{\partial\psi} + \frac{1}{n_oq}\frac{\partial\tilde{p}_1^{(0)}}{\partial\psi}\right] \rightarrow -iny\,\frac{dq}{d\psi}\,c\underline{u}_\beta\left[\tilde{\phi}_n^{(0)} + \frac{1}{n_oq}\,\tilde{p}_n^{(0)}\right]$$

$$\tilde{\underline{V}}_{1\perp}^{(1)} = \frac{c}{B}\,\underline{b}\times\underline{\nabla}\beta\left[\frac{\partial\tilde{\phi}_1^{(0)}}{\partial\beta} + \frac{1}{n_oq}\frac{\partial\tilde{p}_1^{(0)}}{\partial\beta}\right] + \frac{c}{B}\,\underline{b}\times\underline{\nabla}\psi\left[\frac{\partial\tilde{\phi}_1^{(1)}}{\partial\psi} + \frac{1}{n_oq}\frac{\partial\tilde{p}_1^{(1)}}{\partial\psi}\right] \tag{50}$$

$$\rightarrow inc\underline{u}_\psi\left[\tilde{\phi}_n^{(0)} + \frac{1}{n_oq}\,\tilde{p}_n^{(0)}\right] - iny\,\frac{\partial q}{\partial\psi}\,c\underline{u}_\beta\left[\tilde{\phi}_n^{(1)} + \frac{1}{n_oq}\,\tilde{p}_n^{(1)}\right] .$$

The first order perpendicular current is thus of the form

$$\tilde{\underline{J}}_{1\perp} = \tilde{\underline{J}}_{1\perp}^{(0)} + \Delta\tilde{\underline{J}}_{1\perp}^{(1)} = c\left[\underline{u}_\beta\,\frac{\partial\tilde{p}_1^{(0)}}{\partial\psi} - \Delta\underline{u}_\psi\,\frac{\partial\tilde{p}_1^{(0)}}{\partial\beta} + \Delta\underline{u}_\beta\,\frac{\partial\tilde{p}_1^{(1)}}{\partial\psi} + \ldots\right]$$

$$\rightarrow c\left[\underline{u}_\beta\left(-iny\,\frac{dq}{d\psi}\right) + \Delta\underline{u}_\psi(in) + \Delta\underline{u}_\beta\left(-iny\,\frac{dq}{d\psi}\right)\frac{\tilde{p}_1^{(1)}}{\tilde{p}_1^{(0)}} + \ldots\right]\tilde{p}_1^{(0)} . \tag{51}$$

Then, with all these orderings, we find that the lowest and first order components of the density conservation and parallel momentum equations given in Eqs. (23), (24) become

$$\Delta^o:\quad (\underline{B}\cdot\underline{\nabla})(\tilde{V}_{1\parallel}^{(0)}/B) - \tilde{\underline{V}}_{1\perp}^{(0)}\cdot\underline{\nabla}\ln B^2 = 0 \tag{52}$$

$$0 = -\,\underline{B}\cdot\tilde{\underline{R}}_1^{(0)} \tag{53}$$

$$\Delta:\quad \frac{\partial\tilde{n}_1^{(0)}}{\partial t} + \tilde{\underline{V}}_{1\perp}^{(1)}\cdot\underline{\nabla}\psi\,\frac{dn_o}{d\psi} + \overline{\underline{V}}_{1\perp}\cdot\underline{\nabla}\tilde{n}_1 + (\underline{B}\cdot\underline{\nabla})(n_o\tilde{V}_{1\parallel}^{(1)}/B) - n_o\tilde{\underline{V}}_{1\perp}^{(1)}\cdot\underline{\nabla}\ln B^2 \tag{54}$$

$$-\frac{cn_o}{B\Omega}\left(\frac{\partial}{\partial t} + \overline{\underline{V}}_E\cdot\underline{\nabla}\right)\nabla_\perp^2\left(\tilde{\phi}_1^{(0)} + \frac{1}{n_oq}\,\tilde{p}_1^{(0)}\right) + \frac{1}{m\Omega}\frac{\partial}{\partial\psi}\left(\underline{\nabla}\psi\cdot\underline{b}\times(\underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}_1^{(0)})\right) = 0$$

$$mn_o\left(\frac{\partial}{\partial t} + \underline{\bar{V}}_1\cdot\underline{\nabla}_\perp\right)(B\tilde{V}^{(0)}_{1\parallel}) = -\ (\underline{B}\cdot\underline{\nabla})(\tilde{p}^{(0)}_1 + n_o q\tilde{\phi}^{(0)}_1) - \underline{B}\cdot\underline{\nabla}\cdot\underline{\underline{\tilde{\Pi}}}^{(0)}_1 - \underline{B}\cdot\underline{\tilde{R}}^{(1)}_1\ . \tag{55}$$

To solve these equations for $\tilde{V}^{(0)}_{1\parallel}$ and $\tilde{n}^{(0)}_1$, we proceed in a manner analogous to the neoclassical flow determination discussed above. First, we consider Eq. (53) which in analogy with Eq. (41) indicates that

$$\tilde{V}^{(0)}_{1\parallel e} = \tilde{V}^{(0)}_{1\parallel i}\ , \qquad \tilde{J}^{(0)}_{1\parallel}B = 0\ . \tag{56}$$

Next, we consider the terms in Eq. (52). Utilizing the ballooning mode representation in Eq. (46), we can write

$$\begin{aligned}(\underline{B}\cdot\underline{\nabla})(\tilde{V}^{(0)}_{1\parallel}/B) &= \frac{I}{qR^2}\left(q\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\Theta}\right)(\tilde{V}^{(0)}_{1\parallel}/B) \\ &= (\underline{B}\cdot\nabla\Theta)\sum_{k,n} e^{in\zeta-ik\Theta-i\omega t}\int_{-\infty}^{\infty} dy\ e^{i(k-nq)y}\frac{\partial}{\partial y}(\tilde{V}^{(0)}_{\parallel n}/B)\ .\end{aligned} \tag{57}$$

In analogy with Eq. (28), we write the other term in Eq. (52) as follows:

$$\begin{aligned}\underline{\tilde{V}}^{(0)}_{1\perp}\cdot\underline{\nabla}\ln B^2 &= -\ cI\left(\frac{\partial\tilde{\phi}^{(0)}_1}{d\psi} + \frac{1}{n_o q}\frac{\partial\tilde{p}^{(0)}_1}{d\psi}\right)(\underline{B}\cdot\underline{\nabla})\left(\frac{1}{B^2}\right) \\ &= \frac{I}{qR^2}\sum_{k,n} e^{in\zeta-ik\Theta-i\omega t}\int_{-\infty}^{\infty} dy\ e^{i(k-nq)y}(-cI)\left(-iny\frac{dq}{d\psi}\right) \\ &\quad\times\left(\tilde{\phi}^{(0)}_n + \frac{1}{n_o q}\tilde{p}^{(0)}_n\right)\frac{\partial}{\partial y}\left(\frac{1}{B^2}\right)\ .\end{aligned} \tag{58}$$

Substituting Eqs. (57) and (58) into Eq. (52) and operating on the resultant equation with $\frac{1}{2\pi}\int_0^{2\pi} d\zeta\ e^{-in\zeta}$ to pick out the nth toroidal mode, we find

$$\frac{\partial}{\partial y}\,(\tilde{V}^{(0)}_{\parallel n}/B) + (cI)(-iny\,\frac{dq}{d\psi})(\tilde{\phi}^{(0)}_n + \frac{1}{n_0 q}\,\tilde{p}^{(0)}_n)\,\frac{\partial}{\partial y}\,(\frac{1}{B^2}) = 0 \;. \tag{59}$$

Some further assumptions are required to solve this equation. Namely, we assume, as can be validated a postiori in a kinetic treatment of this problem,[8,9] that $\tilde{\phi}^{(0)}_n$ varies only slowly with y, and that rapid fluctuations of $\tilde{\phi}_n$ with y (over 2π intervals) are present only in $\tilde{\phi}^{(1)}_n$ -- see Fig. 1. (In particular, we find from kinetic theory[8] that $\tilde{\phi}^{(1)}_n \sim (\omega_D/\omega_*)\phi^{(0)}_n \sim \varepsilon \cos\theta\, \tilde{\phi}^{(0)}_n$.) Thus, we are led to a multiple length scale approximation. Since we anticipate modes that are highly extended along magnetic field lines (many times around the torus), we expect $y \sim 1/\Delta$. Therefore, we order

$$y = y_s + \Delta\Theta \;, \qquad \frac{\partial}{\partial y} = \frac{1}{\Delta}\frac{\partial}{\partial\Theta} + \frac{\partial}{\partial y_s} \;, \tag{60}$$

where y_s is the slow variation and Θ indicates the variation within one poloidal transit of a field line. Then, from our physical discussion above, we assume

$$\tilde{\phi}_n(y) = \tilde{\phi}^{(0)}_n(y_s) + \Delta\tilde{\phi}^{(1)}_n(\Theta,y_s) + \ldots \tag{61}$$

and similarly for $\tilde{n}_n$. Rigorously speaking, $\tilde{\phi}^{(0)}_n(y_s)$ is a local flux surface average of $\tilde{\phi}_n$ defined by $\tilde{\phi}^{(0)}_n(y_s) \equiv \int_{y_s-\pi}^{y_s+\pi} \frac{d\Theta}{\underline{B}\cdot\nabla\Theta}\,\tilde{\phi}_n \,/\, \oint \frac{d\Theta}{\underline{B}\cdot\nabla\Theta}$. Utilizing these orderings, we find that to lowest order in Δ, Eq. (59) can be simplified to

$$\frac{\partial}{\partial\Theta}\,[\frac{\tilde{V}^{(0)}_{\parallel n}}{B(\Theta)} + \frac{cI}{B^2(\Theta)}\,(-iny_s\,\frac{dq}{d\psi})(\tilde{\phi}^{(0)}_n(y_s) + \frac{1}{n_0 q}\,\tilde{p}^{(0)}_n(y_s))] = 0 \;.$$

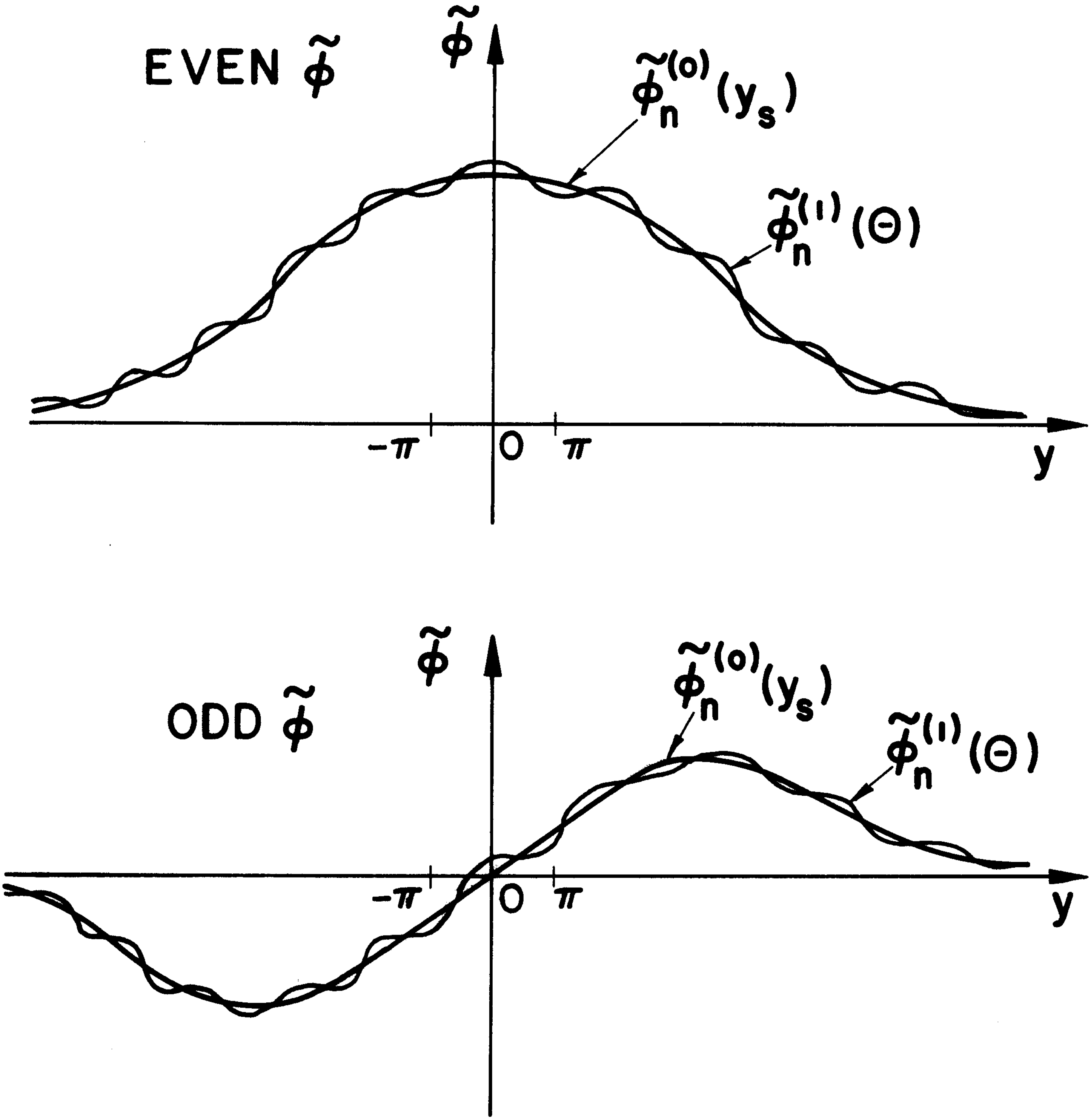

Fig. 1. Schematic illustration of the long (y_s) and short (Θ) scale variations of the eigenfunction $\tilde{\phi}_n(y)$ in the multiple scale length approximation given in Eq. (61).

In analogy with Eq. (29), the solution of this equation can be written as

$$\tilde{V}^{(0)}_{\parallel n} = \tilde{U}_{\parallel}(y_s)/B + \tilde{U}_n(y_s)B \tag{62}$$

where $$\tilde{U}_{\parallel}(y_s) \equiv -cI\left(-iny_s\frac{dq}{d\psi}\right)\left(\tilde{\phi}^{(0)}_n(y_s) + \frac{1}{n_o q}\tilde{p}^{(0)}_n(y_s)\right), \tag{63}$$

and $\tilde{U}_n(y_s)$ is a constant of integration on a flux surface, which is the perturbed poloidal flow velocity divided by the poloidal component of $\underline{B}$ ($\tilde{U}_n \equiv \tilde{\underline{V}}_n \cdot \underline{\nabla}\Theta/\underline{B}\cdot\underline{\nabla}\Theta$). Note that this poloidal flow velocity constant $\tilde{U}_n$ depends parametrically on the long scale or "slow" ballooning mode variable y_s. Also note that Eq. (63) is analogous to the neoclassical result given by Eq. (29) with $d/d\psi$ replaced by $-iny_s\, dq/d\psi$, because of the ballooning mode representation.

As in the determination of the neoclassical flows we determine the constant $\tilde{U}_n$ by considering the first order (in Δ) parallel momentum balance, which is Eq. (55). Rather than considering this entire equation, we want to select out the nth toroidal Fourier component and take the flux surface average of the kth poloidal Fourier component. To do this we utilize the operator

$$[A] \equiv \left\langle \frac{1}{2\pi}\int_0^{2\pi} d\zeta\, e^{-in\zeta+ik\Theta} A(\zeta,\Theta,\psi)\right\rangle \tag{64}$$

on the equation to obtain

$$mn_o(-i\omega + in\overline{\underline{V}}_1\cdot\underline{\nabla}\zeta - ik\overline{\underline{V}}_1\cdot\underline{\nabla}\Theta)\langle B\tilde{V}^{(0)}_{\parallel n}\rangle = -\langle\underline{B}\cdot\underline{\nabla}\rangle(\tilde{p}^{(0)}_n + n_o q\tilde{\phi}^{(0)}_n) - \langle\underline{B}\cdot\underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}^{(0)}_n\rangle - \langle\underline{B}\cdot\tilde{\underline{R}}^{(1)}_n\rangle, \tag{65}$$

in which $$\underline{\langle B\cdot\nabla\rangle} \equiv \frac{2\pi}{\oint d\ell/B}\frac{d}{dy_s} \tag{66}$$

with $\oint d\ell/B = V'(\psi)/2\pi$ being the normalization factor for the $\oint d\ell/B$ average. Now, since the perturbed flows $\underline{\tilde{V}}_{1\perp}^{(0)}$ and $\tilde{V}_{1\parallel}^{(0)}$ are closely analogous to their neoclassical counterparts, the parallel viscosity term in Eq. (65) is analogous to that given in Eq. (35) except that part of it cancels the equilibrium diamagnetic flow contribution to the $\underline{V}_1\cdot\underline{\nabla}\langle B\tilde{V}_{\parallel n}^{(0)}\rangle$ term (see Refs. 8, 10). Thus, Eq. (65) can be written as

$$- i\bar{\omega}mn_o\langle B\tilde{V}_{\parallel n}^{(0)}\rangle = - \underline{\langle B\cdot\nabla\rangle}(\tilde{p}_n^{(0)} + n_o q\tilde{\phi}_n^{(0)}) - mn_o\mu\tilde{U}_n\langle B^2\rangle - \langle \underline{B}\cdot\underline{\tilde{R}}_n^{(1)}\rangle \tag{67}$$

in which $$\bar{\omega} \equiv \omega - \omega_E \tag{68}$$

with $$\omega_E \equiv n\underline{\nabla}\zeta\cdot\underline{V}_E = - nc\frac{d\phi_o}{d\psi} \simeq - \left(\frac{nq}{r}\right)\left(\frac{c}{B}\right)\left(\frac{d\phi_o}{dr}\right) \tag{69}$$

is the frequency Doppler-shifted by the equilibrium $\underline{E}_0 \times \underline{B}$ flow effect.

Writing separate averaged parallel momentum balances for electrons and ions utilizing Eq. (67), the specification of $\tilde{V}_{\parallel n}^{(0)}$ from Eq. (62) and the form of $\underline{B}\cdot\underline{R}$ given in Eq. (41), and then solving for the poloidal flow components for ions and electrons, we obtain

$$\tilde{U}_{ni} = \frac{i\bar{\omega}m_i n_o\hat{U}_{\parallel i} - \underline{\langle B\cdot\nabla\rangle}(\tilde{p}_{ni}^{(0)} + n_o e\tilde{\phi}_n^{(0)}) - \underline{\langle B\cdot\nabla\rangle}(\tilde{p}_{ne}^{(0)} - n_o e\tilde{\phi}_n^{(0)})/(1 + \mu_e/\nu_e)}{m_i n_o(\mu_i - i\bar{\omega})\langle B^2\rangle} \tag{70}$$

$$\tilde{U}_{ne} = - \frac{\hat{U}_{\parallel e} - \hat{U}_{\parallel i}}{(1 + \mu_e/\nu_e)\langle B^2\rangle} + \tilde{U}_{ni} - \frac{\underline{\langle B\cdot\nabla\rangle}(\tilde{p}_{ne}^{(0)} - n_o e\tilde{\phi}_n^{(0)})}{m_e n_e \nu_e(1 + \mu_e/\nu_e)\langle B^2\rangle} \tag{71}$$

in which we have neglected terms proportional to $\sqrt{m_e/m_i}$, $\bar{\omega}/\nu_e$, and $(m_e/m_i)\nu_e/\bar{\omega}$. Note that for $\bar{\omega} \ll \mu_i$ and neglecting the $\langle \underline{B}\cdot\underline{\nabla}\rangle$ parallel derivative terms we would have $\tilde{U}_{ni} \simeq 0$ (no perturbed poloidal ion flow) and $\tilde{U}_{ne} \simeq -(\tilde{U}_{\parallel e} - \tilde{U}_{\parallel i})/(1 + \mu_e/\nu_e)\langle B^2\rangle$, which are analogous to the neoclassical equilibrium results given in Eqs. (37) and (43).

The lowest order parallel electron flow velocity $\tilde{V}^{(0)}_{\parallel ne}$ is now completely determined. However, since it does not give rise to any net parallel current directly [cf. Eqs. (53), (56)], what we need to determine is the first order current $\langle B\tilde{J}^{(1)}_{\parallel n}\rangle$. This can be obtained from the electron component of Eq. (65), which for $\bar{\omega} \ll \mu_e \sim \sqrt{\varepsilon}\,\nu_e/(1 + \nu_{*e})$ can be simplified to

$$0 = n_o e\langle \underline{B}\cdot\underline{\nabla}\rangle\left(\tilde{\phi}^{(0)}_n - \frac{1}{n_o e}\,\tilde{p}^{(0)}_{ne}\right) - m_e n_o \mu_e \tilde{U}_{ne}\langle B^2\rangle + n_o e\langle B\tilde{J}^{(1)}_{\parallel n}\rangle/\sigma_\parallel\ . \qquad (72)$$

Solving this equation for $\langle B\tilde{J}^{(1)}_{\parallel n}\rangle$, we obtain the global Ohm's law appropriate for "slow" perturbations ($\bar{\omega} \ll \mu_e$) in a ballooning mode representation:

$$\langle B\tilde{J}^{(1)}_{\parallel n}\rangle = -\sigma_\parallel\langle \underline{B}\cdot\underline{\nabla}\rangle\left(\tilde{\phi}^{(0)}_n - \frac{1}{n_o e}\,\tilde{p}^{(0)}_{ne}\right) + \frac{\mu_e}{\nu_e}\,n_o e\tilde{U}_{ne}\langle B^2\rangle\ . \qquad (73)$$

The first term is the appropriate combination of the parallel electrostatic electric field and electron pressure gradient terms. The last term, which is proportional to $\tilde{U}_{ne}$, indicates the fluctuating bootstrap current analogous to the neoclassical result given in Eq. (44).

Finally, in our determination of the lowest order perturbed quantities, we need to solve for the perturbed density $\tilde{n}^{(0)}_1$ from Eq. (54). Since from our multiple length scale expansion in Eq. (61) we have $\tilde{n}^{(0)}_n$ independent of the short scale length variation Θ and only dependent on the long scale length variation y_s, we need only utilize the flux surface average of Eq. (54). Thus, operating on this equation with the operator indicated in Eq. (64), we find that to lowest order in Δ the first order perturbed density equation for electrons (so that the polarization drift type contributions that are inversely proportional to BΩ can be neglected) becomes simply

$$-i\omega\tilde{n}_{ne}^{(0)} + inc\left(\tilde{\phi}_n^{(0)} - \frac{1}{n_o e}\tilde{p}_{ne}^{(0)}\right)\frac{dn_o}{d\psi} - inc\left(\frac{d\phi_o}{d\psi} - \frac{1}{n_o e}\frac{dp_{oe}}{d\psi}\right)\tilde{n}_{ne}^{(0)} = 0 \ . \quad (74)$$

Since for our constant temperature plasma $p_{oe} = n_o T_e$ and $\tilde{p}_{ne}^{(0)} = \tilde{n}_{ne}^{(0)} T_e$, the $\tilde{n}_{ne}^{(0)}$ terms proportional to $dn_o/d\psi$ cancel and we obtain simply

$$- i(\omega - \omega_E)\ \tilde{n}_{ne}^{(0)} = - inc\tilde{\phi}_n^{(0)} \frac{dn_o}{d\psi}$$

or,

$$\tilde{n}_{ne}^{(0)} = \frac{nc}{\bar{\omega}} \tilde{\phi}_n^{(0)} \frac{dn_o}{d\psi} = \frac{\omega_{*e}}{\bar{\omega}} \left(\frac{e\tilde{\phi}_n^{(0)}}{T_e}\right) n_o \ , \quad (75)$$

in which the electron diamagnetic drift frequency ω_{*e} and a corresponding ion diamagnetic drift frequency are defined by

$$\omega_{*e} = n \frac{cT_e}{e} \frac{1}{n_o} \frac{dn_o}{d\psi} \ , \qquad \omega_{*i} = - n \frac{cT_i}{e} \frac{1}{n_o} \frac{dn_o}{d\psi} \ . \quad (76)$$

Note also that the perturbed electron density in Eq. (75) is just the usual perturbed MHD convective flow density response

$$\tilde{n} = (\underline{\tilde{V}} \cdot \underline{\nabla} n_o)/i\omega = c(\underline{B} \times \underline{\nabla}\tilde{\phi} \cdot \underline{\nabla}\psi)(dn_o/d\psi)/i\omega B^2 \rightarrow \frac{nc}{\omega} \tilde{\phi}_n \frac{dn_o}{d\psi} \ . \quad (77)$$

Finally, we note that the combination of terms $\tilde{p}_{ni}^{(0)} + n_o e\tilde{\phi}_n^{(0)}$ and

$\tilde{p}_{ne}^{(0)} - n_o e\tilde{\phi}_n^{(0)}$ appearing in Eqs. (70) and (71) can be written, utilizing our convective response in Eq. (75), in the form

$$\tilde{p}_{ni}^{(0)} + n_o e\tilde{\phi}_n^{(0)} = n_o e\tilde{\phi}_n^{(0)}\left(1 - \omega_{*i}/\bar{\omega}\right)$$

$$\tilde{p}_{ne}^{(0)} - n_o e\tilde{\phi}_n^{(0)} = - n_o e\tilde{\phi}_n^{(0)}\left(1 - \omega_{*e}/\bar{\omega}\right) . \tag{78}$$

Before proceeding to synthesize these results into an eigenmode equation determining instability, we summarize what we have found for these neoclassical-like perturbed flows:

1. The parallel ion viscosity couples the parallel and $\underline{u}_\beta$ ion perturbed flow components so as to determine a net poloidal ion flow velocity $\tilde{U}_{ni}$ given in Eq. (70), which depends on the ratio of the net wave frequency $\bar{\omega} \equiv \omega - \omega_E$ to the poloidal ion flow viscous damping rate μ_i.
2. Collisional friction causes the lowest order perturbed parallel electron and ion flows to be equal [cf. Eq. (56)]. The viscous damping of the perturbed poloidal electron flow given in Eq. (71) leads to the bootstrap contribution to the global parallel Ohm's law given in Eq. (73).
3. The parallel electron momentum balance or Ohm's law in the banana-plateau collisionality regime is only meaningful in a flux surface averaged sense, as indicated in Eq. (73).
4. In addition to the flux surface average $\langle \tilde{J}_{\parallel n}^{(1)} B\rangle \neq 0$, there is a Pfirsch-Schlüter component that averages to zero $\left(\langle \tilde{J}_{\parallel nPS}^{(0)} B\rangle = 0\right)$ and is given by

$$\tilde{J}_{\parallel n}^{(0)} \equiv \tilde{J}_{\parallel nPS}^{(0)} = \frac{n_o e\left(\tilde{U}_{\parallel i} - \tilde{U}_{\parallel e}\right)}{B}\left(1 - \frac{B^2}{\langle B^2\rangle}\right) = \frac{cI}{B}\left(\frac{B^2}{\langle B^2\rangle} - 1\right)\left(-iny_s \frac{dq}{d\psi}\right)\tilde{p}_n^{(0)} . \tag{79}$$

Note that as in the theory of neoclassical flows, the parallel derivative of this current is just what is required to cancel the geodesic curvature driven contribution to $\underline{\nabla}\cdot\tilde{J}_{1\perp}$ in the quasineutrality condition $\underline{\nabla}\cdot\tilde{\underline{J}} = 0$.

V. Eigenmode Equation and Analysis: Resistive MHD-Like Modes

These perturbed flow results will now be combined to obtain an eigenmode equation governing instability of resistive MHD-like modes in the banana-plateau regime of collisionality. They will be combined through a quasi-neutrality condition $\underline{\nabla}\cdot\tilde{\underline{J}} = 0$, which from the next to last form of Eq. (15), can be written as

$$0 = \underline{\nabla}\cdot\tilde{\underline{J}} = (\underline{B}\cdot\underline{\nabla})(\underline{B}\cdot\tilde{\underline{J}}_1/B^2) + \{\underline{\nabla}_\perp\cdot\tilde{\underline{J}}_{2\perp}\} - \tilde{\underline{J}}_{1\perp}\cdot\underline{\nabla}\ \ln B^2 \ . \tag{81}$$

Utilizing the form of $\tilde{\underline{J}}_{2\perp}$ from Eq. (20) with $\tilde{\underline{V}}_{1\perp}$ from Eq. (16), and the form of $\{\underline{\nabla}_\perp\cdot\tilde{\underline{J}}_\perp\}$ from Eq. (18), we find that

$$\{\underline{\nabla}_\perp\cdot\tilde{\underline{J}}_{2\perp}\} = -\frac{c^2\rho_m}{B^2}\left(\frac{\partial}{\partial t} + \underline{V}_E\cdot\underline{\nabla}\right)\nabla_\perp^2\left(\tilde{\phi}_1 + \frac{1}{n_0 e}\tilde{p}_{1i}\right) + c\frac{\partial}{\partial\psi}\left(\frac{\nabla\psi\times\underline{b}}{B}\cdot\left(\underline{\nabla}\cdot(\tilde{\underline{\underline{\Pi}}}_e + \tilde{\underline{\underline{\Pi}}}_i)\right)\right) \tag{82}$$

where we have omitted $(\underline{\nabla}\beta\times\underline{b})\cdot(\underline{\nabla}\cdot\tilde{\underline{\underline{\Pi}}}_1)$ terms since they represent radial viscosity effects and are apparently one order higher in the gyroradius expansion than the viscous stress effects within the magnetic flux surface that are retained. Here, the first term is just the usual ion polarization drift contribution, with the finite ion gyroradius correction and equilibrium $\underline{E}_0\times\underline{B}$ drift included. The cross (i.e., $\underline{b}\times\underline{\nabla}\psi/B = \underline{u}_\beta$ component) viscosity terms in Eq. (82) will yield a neoclassical enhancement of the polarization drift and some $\langle\underline{B}\cdot\underline{\nabla}\rangle\tilde{p}$ terms. To determine them, we multiply these terms by B^2 and operate on them with the averaging operator given in Eq. (64). Then, proceeding as in the determination of the radial neoclassical transport flows [see discussion after Eq. (44)], neglecting order Δ terms arising from the $R^2\underline{\nabla}\zeta$ or toroidal component of viscosity, and utilizing the $\tilde{U}_{ni}$ and $\tilde{U}_{ne}$ poloidal flow components from Eqs. (70) and (71), we obtain

$$[B^2 c \frac{\partial}{\partial \psi} (\frac{\underline{b} \times \underline{\nabla}\psi}{B} \cdot \underline{\nabla} \cdot (\tilde{\underline{\underline{\Pi}}}_e + \tilde{\underline{\underline{\Pi}}}_i))] \simeq - c \frac{\partial}{\partial \psi} I [\underline{B} \cdot \underline{\nabla} \cdot (\tilde{\underline{\underline{\Pi}}}_e + \tilde{\underline{\underline{\Pi}}}_i)]$$

$$= - c \frac{\partial}{\partial \psi} I \langle B^2 \rangle (m_e n_o \mu_e \tilde{U}_{ne} + m_i n_o \mu_i \tilde{U}_{ni}) \tag{83}$$

$$\rightarrow - i(\bar{\omega} - \omega_{*i}) \, M(\bar{\omega}) \, \rho_m c^2 n^2 y_s^2 I^2 (\frac{dq}{d\psi})^2 \, \tilde{\phi}_n^{(0)} + cIL(\bar{\omega}) (-iny_s \frac{dq}{d\psi}) \langle \underline{B} \cdot \underline{\nabla} \rangle \tilde{P}_n^{(0)}$$

in which we have neglected terms of order m_e/m_i and the frequency factors $L(\bar{\omega})$ and $M(\bar{\omega})$ involved in the $\underline{B} \cdot \underline{\nabla} \tilde{P}$ and neoclassical polarization drift terms are defined by

$$L(\bar{\omega}) \equiv M(\bar{\omega}) + \frac{\mu_e/\nu_e}{1 + \mu_e/\nu_e} (1 - M(\bar{\omega})) \frac{\omega - \omega_{*e}}{\omega_{*i} - \omega_{*e}} , \tag{84}$$

$$M(\bar{\omega}) \equiv \frac{\mu_i}{\mu_i - i\omega} . \tag{85}$$

In the limit $\mu_i >> \bar{\omega}$ these frequency coefficients are both unity. Then, the first term in the last form of Eq. (83) is a neoclassical polarization drift term which is a factor of $B_\zeta^2/B_\Theta^2 \simeq q^2/\varepsilon^2$ larger than the usual finite gyroradius polarization term given in the first part of Eq. (82). This neoclassical polarization drift term reflects the fact[11] that in an axisymmetric tokamak the perpendicular dielectric constant has the B^2 in the denominator replaced by B_Θ^2 -- the square of the poloidal magnetic field strength.

Thus, to obtain the total $\{\underline{\nabla}_\perp \cdot \underline{\tilde{J}}_{2\perp}\}$ contribution, we multiply Eq. (82) by B^2 and average with the operator in Eq. (64) to obtain

$$[B^2\{\underline{\nabla}_\perp \cdot \underline{J}_{2\perp}\}] = -\,i(\overline{\omega} - \omega_{*i})\left(M(\overline{\omega}) + 2\langle|\underline{\nabla}\psi|^2\rangle/I^2\right)\,\rho_m c^2 n^2 y_s^2 I^2 \left(\frac{dq}{d\psi}\right)^2 \tilde{\phi}_n^{(0)}$$

$$+\; cIL(\overline{\omega})\left(-iny_s\,\frac{dq}{d\psi}\right)\langle \underline{B}\cdot\underline{\nabla}\rangle \tilde{P}_n^{(0)} \;. \tag{86}$$

Here, the $M(\overline{\omega})$ part represents the neoclassical polarization drift effect and the $\langle|\underline{\nabla}\psi|^2\rangle/I^2 \sim B_\Theta^2/B_\zeta^2$ term represents the usual finite gyroradius polarization drift contribution. Note that in terms of our small Δ or subsidiary expansion, all significant terms in $\underline{\nabla}_\perp \cdot \underline{\tilde{J}}_{2\perp}$ are of order Δ. Since the usual finite gyroradius polarization drift is of order ε^2/q^2 smaller, and hence not significant in this equation, we will neglect it henceforth.

For the first order perturbed current contributions to Eq. (81), we see from Eqs. (51), (73) and (79) that $\underline{\tilde{J}}_1$ can be written as

$$\underline{\tilde{J}}_1 = c\left(\underline{u}_\beta \frac{\partial \tilde{P}_1^{(0)}}{\partial\psi} - \Delta\underline{u}_\psi \frac{\partial \tilde{P}_1^{(0)}}{\partial\beta} + \Delta\underline{u}_\beta \frac{\partial \tilde{P}_1^{(1)}}{\partial\psi}\right) + \underline{B}\left(\tilde{J}_{\parallel PS}^{(0)} + \Delta\tilde{J}_\parallel^{(1)}\right)/B + \;(\Delta^2)\;, \tag{87}$$

in which we have spelled out the form of the terms in the small Δ or subsidiary expansion. While we do not have an explicit form for $\tilde{J}_\parallel^{(1)}$, the flux surface average $\langle B\tilde{J}_\parallel^{(1)}\rangle$ has been obtained in Eq. (73). Utilizing the lowest order components of $\underline{\tilde{J}}_1$ from Eq. (87), we find that to lowest order in Δ

$$\underline{\tilde{J}}_1^{(0)}\cdot\underline{B} = B\tilde{J}_{1\parallel PS}^{(0)} = cI\left(\frac{B^2}{\langle B^2\rangle} - 1\right)\frac{\partial P_1^{(0)}}{\partial\psi} + cI\left(\frac{B^2}{\langle B^2\rangle} - 1\right)\left(-iny_s\,\frac{dq}{d\psi}\right)\tilde{P}_n^{(0)}\;,$$

and hence

$$(\underline{B}\cdot\underline{\nabla})(\underline{\tilde{J}}_1^{(0)}\cdot\underline{B}/B^2) = -\, cI \frac{\partial\tilde{p}_1^{(0)}}{\partial\psi} (\underline{B}\cdot\underline{\nabla}) \frac{1}{B^2} \rightarrow -\, cI\left(-iny_s \frac{dq}{d\psi}\right) \tilde{p}_n^{(0)} \frac{I}{qR^2} \frac{\partial}{\partial\Theta} \frac{1}{B^2} \; .$$

As with the neoclassical first order currents, this parallel derivative of the perturbed Pfirsch-Schlüter current cancels the geodesic curvature driven contribution to $\underline{\nabla}\cdot\underline{\tilde{J}}$ that is given by

$$\underline{\tilde{J}}_{1\perp}^{(0)}\cdot\underline{\nabla}\ \ln B^2 = \frac{c}{B^2}\frac{\partial B^2}{\partial\beta}\frac{\partial\tilde{p}_1^{(0)}}{\partial\psi} = -cI \frac{\partial\tilde{p}_1^{(0)}}{\partial\psi} (\underline{B}\cdot\underline{\nabla}) \frac{1}{B^2}$$

$$\rightarrow (-cI)\left(-iny_s \frac{dq}{d\psi}\right) \tilde{p}_n^{(0)} \frac{I}{qR^2} \frac{\partial}{\partial\Theta} \frac{1}{B^2} \; .$$

Thus, the lowest order (Δ^0) geodesic curvature driven terms are cancelled out in Eq. (81).

To next order in Δ there are a number of $\underline{\tilde{J}}_1$ terms that contribute, both directly from $\underline{\tilde{J}}_1^{(1)}$ and indirectly from higher order oscillatory corrections to the cancellation of the $\underline{\tilde{J}}_1^{(0)}$ terms. Since all we are interested in are the flux surfaced averaged quantities, we utilize the operator in Eq. (64) to project out only the averaged part. Then, writing $(\underline{B}\cdot\underline{\nabla})(\underline{J}_1^{(1)}\cdot\underline{B}/B^2) = (1/B^2) \times \left((\underline{B}\cdot\underline{\nabla})(\underline{\tilde{J}}_1^{(1)}\cdot\underline{B}) - (\underline{\tilde{J}}_1^{(1)}\cdot\underline{B})(\underline{B}\cdot\underline{\nabla})\ \ln B^2\right)$, we find that the [] average of B^2 times the quasineutrality condition of Eq. (81) becomes simply

$$0 = [(\underline{B}\cdot\underline{\nabla})(\underline{\tilde{J}}_1^{(1)}\cdot\underline{B})] + [B^2\{\underline{\nabla}_\perp\cdot\underline{\tilde{J}}_{2\perp}\}] - [\underline{\tilde{J}}_{1\perp}^{(1)}\cdot\underline{\nabla}B^2] \; . \tag{88}$$

Or, performing these averages, we obtain

$$0 = \langle \underline{B}\cdot\underline{\nabla}\rangle \langle \tilde{J}^{(1)}_{\parallel n} B\rangle + ic^2\rho_m(\overline{\omega} - \omega_{*i})\, M(\overline{\omega})\, n^2 y_s^2 \left(I \frac{dq}{d\psi}\right)^2 \tilde{\phi}^{(0)}_n$$

$$+ cIL(\overline{\omega})\left(-iny_s \frac{dq}{d\psi}\right)\langle \underline{B}\cdot\underline{\nabla}\rangle \tilde{P}^{(0)}_n - inc\tilde{P}^{(0)}_n \left(\langle \frac{\partial B^2}{\partial\psi}\rangle - \frac{1}{q}\frac{dq}{d\psi}\langle \Theta \frac{\partial B^2}{\partial\Theta}\rangle\right) . \tag{89}$$

Note that while the flux surface average of the normal curvature $\partial B^2/\partial\psi$ enters directly, the lowest order geodesic curvature term $y_s \partial B^2/\partial\beta = (y_s/q)(\partial B^2/\partial\Theta)$ does not -- because the perturbed Pfirsch-Schlüter current contribution cancelled out the lowest order term. However, the $\oint d\ell/B$ average first order geodesic curvature $\langle \Theta\, \partial B^2/\partial\Theta\rangle$ does contribute. Rigorously speaking, Eq. (89) is supposed to be an order Δ equation. However, for low β ($\sim \varepsilon^2/q^2$) tokamak equilibria $\langle \partial B^2/\partial\psi\rangle \sim \langle \Theta\, \partial B^2/\partial\Theta\rangle \sim \varepsilon^2 \sim \Delta^2$. Thus, formally, the last terms in Eq. (89) vanish to the order Δ that we are calculating in Eq. (89); to obtain the complete curvature driven contributions to Eq. (89) correct to order Δ^2 we would have to go to higher order in our solutions. However, since the main emphasis in this paper is to elucidate the parallel viscosity effects, which we do retain in the order Δ equation, we will be content to simply identify these last terms as indicating the curvature effects, realizing that they may not contain all the order Δ^2 terms.

Upon utilizing Eqs. (73), (75) and (78) to relate all perturbed quantities to $\tilde{\phi}^{(0)}_n$ (except $\tilde{U}_{ne}$, which will be dealt with later), we find Eq. (89) can be written as

$$0 = -\left(1 - \frac{\omega_{*e}}{\bar{\omega}}\right)\sigma_{\parallel} \langle \underline{B}\cdot\underline{\nabla}\rangle\langle \underline{B}\cdot\underline{\nabla}\rangle \tilde{\phi}_n^{(0)} + \frac{\mu_e}{\nu_e}\langle \underline{B}\cdot\underline{\nabla}\rangle\left(n_o e\tilde{U}_{ne}\langle B^2\rangle\right) + i\bar{\omega}c^2\rho_m\left(1 - \frac{\omega_{*i}}{\bar{\omega}}\right)$$

$$\times M(\bar{\omega})n^2 y_s^2\left(I\,\frac{dq}{d\psi}\right)^2 \tilde{\phi}_n^{(0)} - i\,\frac{n^2c^2}{\bar{\omega}}\,\frac{dP_o}{d\psi}\,L(\bar{\omega})\,\frac{dq}{d\psi}\,y_s\langle \underline{B}\cdot\underline{\nabla}\rangle\tilde{\phi}_n^{(0)} \qquad (90)$$

$$-\frac{in^2c^2}{\bar{\omega}}\,\frac{dP_o}{d\psi}\,\tilde{\phi}_n^{(0)}\left(\left\langle\frac{\partial B^2}{\partial\psi}\right\rangle - \frac{1}{q}\,\frac{dq}{d\psi}\left\langle\Theta\,\frac{\partial B^2}{\partial\Theta}\right\rangle\right) .$$

Dividing Eq. (90) by $[-i\bar{\omega}\, n^2c^2\rho_m(I\ dq/d\psi)^2]$ and utilizing the "natural" definitions

$\tau_R \equiv 4\pi\sigma_{\parallel}/c^2(I\ dq/d\psi)^2$ -- resistive diffusion time (smaller than the usually defined one by a factor $r^2I^2(dq/d\psi)^2 \sim B_\zeta^2/B_\Theta^2 \sim q^2/\varepsilon^2 >> 1$) (91)

$\tau_A \equiv \sqrt{4\pi\rho_m}\oint d\ell/2\pi B$ -- poloidal Alfvén time ($\sim Rq/V_A$, which is different from the one usually defined by a factor of $q \gtrsim 1$) (92)

$\omega_H^2 = -\dfrac{dP_o/d\psi}{\rho_m(I\ dq/d\psi)^2}\left(\left\langle\dfrac{\partial B^2}{\partial\psi}\right\rangle - \dfrac{1}{q}\,\dfrac{dq}{d\psi}\left\langle\Theta\,\dfrac{\partial B^2}{\partial\Theta}\right\rangle\right) \equiv V_s^2/r_p\langle R_c\rangle \sim \varepsilon^2\beta_T/\tau_A^2$ -- an effective hydrodynamic frequency ($\omega_H^2 > 0$ for "ideal" MHD stability), (93)

$$\omega_s^2 = \left(-\frac{1}{\rho_m}\frac{dP_o}{d\psi}\right)\frac{2\pi}{(I\ dq/d\psi)\oint d\ell/B} \text{ -- an effective sound wave frequency} \qquad (94)$$

in the plasma $\left(\omega_s^2 \simeq V_s^2/R^2 q(r_p\ dq/dr) \sim \beta_T/\tau_A^2\right)$

$$N(\bar{\omega}) = 1 + \left(1 - M(\bar{\omega})\right)\frac{\bar{\omega} - \omega_{*i}}{\omega_{*i} - \omega_{*e}} \text{ -- a multiplier indicating the relative} \qquad (95)$$

effect of ion viscous damping on the perturbed poloidal electron

flow ($N = 1$ for $\mu_i \gg \bar{\omega}$),

together with the specification of $\langle \underline{B}\cdot\underline{\nabla}\rangle$ given in Eq. (66), we find our eigenmode equation becomes simply

$$\frac{(1 - \omega_{*e}/\bar{\omega})\tau_R}{i\bar{\omega}n^2\tau_A^2}\frac{d^2\tilde{\phi}_n^{(0)}}{dy_s^2} + M(\bar{\omega})(1 - \omega_{*i}/\bar{\omega})\ y_s^2\tilde{\phi}_n^{(0)} + \frac{\mu_e}{\nu_e}\frac{\omega_s^2}{\bar{\omega}^2}N(\bar{\omega})\frac{d}{dy_s}y_s\tilde{\phi}_n^{(0)}$$

$$- \frac{\omega_s^2}{\bar{\omega}^2}L(\bar{\omega})\ y_s\frac{d\tilde{\phi}_n^{(0)}}{dy_s} - \frac{\omega_H^2}{\bar{\omega}^2}\tilde{\phi}_n^{(0)} = 0\ . \qquad (96)$$

To determine solutions of this differential equation in the ballooning mode variable y_s, we put it in the following convenient form

$$D\frac{d^2\phi_n^{(0)}}{dy_s^2} + 2(C + E)\frac{d}{dy_s}y_s\tilde{\phi}_n^{(0)} + \left(B - 2E - Ay_s^2\right)\tilde{\phi}_n^{(0)} = 0 \qquad (97)$$

in which we have defined

$$A = -(1 - \omega_{*i}/\bar{\omega})M(\bar{\omega}) \;, \qquad B = -\frac{\omega_H^2}{\bar{\omega}^2} \;, \qquad C = (\frac{\mu_e}{\nu_e})\, N(\bar{\omega})\, \frac{\omega_s^2}{2\bar{\omega}^2} \;,$$

$$D = \frac{(1 - \omega_{*e}/\bar{\omega})\tau_R}{i\bar{\omega}\tau_A^2 n^2} \;, \qquad E = -\frac{\omega_s^2}{2\bar{\omega}^2} L(\bar{\omega}) \; . \tag{98}$$

This differential equation can be simplified into the standard harmonic oscillator or Weber equation form with the transformation

$$\tilde{\phi}_n^{(0)}(y_s) = \Psi(y_s) e^{-(C+E)y_s^2/2D} \;, \tag{99}$$

which yields

$$\frac{d^2\Psi}{dy_s^2} + [(\frac{B + C - E}{D}) - (\frac{A}{D} + \frac{(C + E)^2}{D^2})y_s^2]\Psi = 0 \; . \tag{100}$$

For modes localized away from the boundaries at $y_s = \pm\infty$, the solutions of this equation are

$$\Psi_\ell = H_\ell(\sqrt{\sigma}\, y_s) e^{-\sigma y_s^2/2} \;, \qquad \frac{d^2\Psi_\ell}{dy_s^2} = [-(2\ell + 1)\sigma + \sigma^2 y_s^2]\Psi_\ell \;, \tag{101}$$

where H_ℓ is the Hermite polynomial. Substituting the differential form in Eq. (101) into Eq. (100), we find the eigenvalue conditions

$$\sigma^2 = A/D + (C + E)^2/D^2 \tag{102}$$

$$(2\ell + 1)\sigma = (B + C - E)/D \; . \tag{103}$$

Now, we must require $\mathrm{Re}[\sigma + (C + E)/D] > 0$ for the modes to be localized near $y_s = 0$. For $\bar{\omega} \ll \mu_i$ where $L = M = N = 1$, we have $E \gg C, B$. Then, since for the form of σ given in Eq. (103), we have $\sigma + (C + E)/D = (E/D)(1 - 1/(2\ell$

$+ 1)) + (C/D)(1 + 1/(2\ell + 1)) + B/D(2\ell + 1)$ and $E/D \propto - i/\bar{\omega}$, we find that growing modes ($\mathrm{Im}\bar{\omega} > 0$) are localizable and hence possible only for $\ell = 0$. Thus, we take $\ell = 0$ and for purely growing modes in the $\underline{E}_0 \times \underline{B}$ rest frame ($\bar{\omega} \equiv i\gamma \gg \omega_{*e}, \omega_{*i}$) we take the positive square root in Eq. (102) to keep $\mathrm{Re}\sigma > 0$:

$$\sigma = (A/D + (C + E)^2/D^2)^{1/2} . \qquad (104)$$

Hence, the general eigenvalue condition in Eq. (103) can be written as ($\ell = 0$):

$$(A/D + (C + E)^2/D^2)^{1/2} = (B + C - E)/D . \qquad (105)$$

Next, we discuss some limiting cases of this general dispersion relation. First, we consider the case where the viscosity effects become negligible ($\mu_e \omega_s^2/\nu_e \ll \omega_H^2$, or $\mu_e/\nu_e \lesssim \varepsilon^2$ which is in the Pfirsch-Schlüter collisionality regime) where we should recover a form of resistive interchange (or -g) modes. Then, C is much less than B and can be neglected in Eq. (105); whence, our eigenvalue condition reduces to $(A/D + E^2/D^2)^{1/2} = (B - E)/D$ or $A = -2EB(1 + B/2E)D \simeq -2EB/D$:

$$\bar{\omega}(\bar{\omega} - \omega_{*i})(\bar{\omega} - \omega_{*e}) = - i\omega_H^2 \omega_s^2 n^2 \tau_A^2/\tau_R \qquad (106)$$

in which for simplicity we have assumed $\bar{\omega} \ll \mu_i$ so that $L(\bar{\omega}) = M(\bar{\omega}) = 1$. Writing $\bar{\omega} = |\bar{\omega}|e^{i\theta}$, we find that growing modes ($0 < \theta < \pi$) occur only for

$$\omega_H^2 < 0 \text{ -- for resistive interchange instability .} \quad (107)$$

For $\bar{\omega} >> \omega_{*e}, \omega_{*i}$ these modes have $\bar{\omega} = i\gamma$ with a growth rate $\gamma = \gamma_H$ given by

$$\gamma_H = n^{2/3}\tau_A^{-2/3}\tau_R^{-1/3}(-\omega_H^2\omega_s^2\tau_A^4)^{1/3} \propto n^{2/3}S_N^{-1/3}(\varepsilon\beta_T)^{2/3}/\tau_A \quad (108)$$

in which

$S_N \equiv \tau_R/\tau_A$, a neoclassical magnetic Reynolds number ($S_N \sim \varepsilon^2 S/q^3$, where S is the usual magnetic Reynolds number) (109)

$$\beta_T = 8\pi P_o/B^2 \text{ , the total plasma } \beta \text{ .} \quad (110)$$

Note that for these resistive interchange or resistive-g type instabilities the only axial eigenmode ($\ell = 0$) is an even function of y_s that extends (for $\tilde{\phi} \sim e^{-y_s^2/w_H^2}$) a distance along a field line of

$$w_H = (2B/D)^{-1/2} = \frac{(\gamma_H\tau_A S_N)^{1/2}}{\sqrt{2}\, n|\omega_H|\tau_A} \propto S_N^{1/3}n^{-2/3}\varepsilon^{-2/3}\beta_T^{-1/6} \text{ .} \quad (111)$$

For $S_N \sim 10^5$, $n \sim 10$ and $\beta_T \sim 10^{-2}$ this yields $w_H \sim 10$, which is of order $1/\Delta$, as has been assumed in the analysis. Since in the ballooning mode representation $\partial/\partial\psi \to = -iny_s\, dq/d\psi$, this eigenmode is even in the radial coordinate $x \equiv (\psi - \psi_s)/(d\psi/dr)_{r_s}$ about a rational surface ($\psi = \psi_s$, $r = r_s$) and extends (for $\tilde{\phi} \sim e^{-x^2/\delta_H^2}$) radially a distance of

$$\delta_H = 1/[nw_n(dq/dr)_{r_s}] \propto r_s S_N^{-1/3} n^{-1/3} \varepsilon^{2/3} \beta_T^{1/6} . \qquad (112)$$

This mode is similar in scaling to the resistive interchange modes discussed by Glasser, Greene and Johnson,[12] except that our $S_N \equiv \tau_R/\tau_A$ is smaller by a factor of order $B_\Theta^2/B_\zeta^2 q \sim \varepsilon^2/q^3 << 1$, primarily because of the increased polarization drift current contribution discussed after Eq. (82) above.

Next, we consider the effects of the viscous damping of the perturbed flows that are embodied in the coefficient C and dominate for $(\mu_e/\nu_e)\omega_s^2 > \omega_H^2$, which apparently is valid over most of the banana-plateau regime of collisionality in which tokamak plasmas usually operate. As a lowest order effect, we note that for C small but not negligible in Eq. (105), the growth rate of the previously considered resistive interchange mode increases as the viscosity effects are increased. In the limit C >> B $(\mu_e/\nu_e > \varepsilon^2)$ where the viscous damping effects dominate over the hydrodynamic curvature effects, the dispersion relation in Eq. (105) simplifies to

$$A = - 4CE/D , \qquad (113)$$

or

$$\bar{\omega}(\bar{\omega} - \omega_{*i})(\bar{\omega} - \omega_{*e}) = - i(\mu_e/\nu_e)L(\bar{\omega})N(\bar{\omega})\omega_s^4 n^2 \tau_A^2/[\tau_R M(\bar{\omega})] . \qquad (114)$$

Writing $\bar{\omega} = |\bar{\omega}|e^{i\theta}$, we find for the "fluid" limit $\mu_i >> \bar{\omega} >> \omega_{*e},\omega_{*i}$ and $M(\bar{\omega}) \simeq 1$, $N(\bar{\omega}) \simeq 1$ that Eq. (114) has one possible unstable root with $\theta = \pi/2$. Since for localization of the mode in y_s we must require $0 < \mathrm{Re}(\sigma + C + E/D) = \mathrm{Re}(C/D) \propto \omega_s^2 \times \mathrm{Re}(i/\bar{\omega})$, the $\theta = \pi/2$ root is localizable only for $\omega_s^2 > 0$ $(dP_0/d\psi < 0$ with $dq/d\psi > 0)$. (For $\omega_s^2 < 0$ no growing modes can be localized in y_s.) For the "fluid" limit $\mu_i >> \bar{\omega} >> \omega_{*e},\omega_{*i}$ the unstable mode is thus purely growing in the $\underline{E}_0 \times \underline{B}$ rest frame with a growth rate given by

$$\gamma_\mu = n^{2/3}\tau_A^{-2/3}\tau_R^{-1/3}(\omega_s\tau_A)^{4/3}(\mu_e/\nu_e)^{1/3}$$

$$= n^{2/3}S_N^{-1/3}\left(\frac{\mu_e}{\nu_e}\right)^{1/3}\left(\frac{\beta_T}{2}\right)^{2/3}\left|\frac{d\ \ln\ P_o}{d\ \ln\ q}\right|^{2/3} / \tau_A \tag{115}$$

in which

$$|\omega_s^2|\tau_A^2 \simeq \frac{\beta_T}{2}\left|\frac{d\ \ln\ P_o/d\psi}{d\ \ln\ q/d\psi}\right| . \tag{116}$$

Note that for this new electron viscosity driven mode the axial eigenmode ($\ell = 0$) is an even function of y_s that extends [for $\tilde{\Phi} \sim \exp(-y_s^2/w_\mu^2)$] a distance along a field line of

$$w_\mu = (C/D)^{-1/2} = \frac{[2\gamma_\mu\tau_A S_N/(\mu_e/\nu_e)]^{1/2}}{n\omega_s\tau_A}$$

$$= \sqrt{2}\ S_N^{1/3}n^{-2/3}\left(\frac{\beta_T}{2}\right)^{-1/6}\left(\frac{\mu_e}{\nu_e}\right)^{-1/3}\left|\frac{d\ \ln\ P_o}{d\ \ln\ q}\right|^{-1/6} . \tag{117}$$

This mode will also be even in the radial coordinate x about a rational surface and extend [for $\tilde{\Phi} \sim \exp(-x^2/\delta_\mu^2)$] radially a distance of

$$\delta_\mu = \frac{1}{[nw_\mu(dq/dr)_{r_s}]} = n^{-1/3}S_N^{-1/3}\left[\frac{\beta_T}{2}\left(\frac{\mu_e}{\nu_e}\right)^2\left|\frac{d\ \ln\ P_o}{d\ \ln\ q}\right|\right]^{1/6}\left(\sqrt{2}\ \frac{dq}{dr}\right)^{-1} . \tag{118}$$

It is interesting to note that since $\sqrt{\beta_T}/S_N^2 = \sqrt{\beta_\Theta}/S_\Theta$ with $\beta_\Theta \equiv 8\pi P/B_\Theta^2$ and $S_\Theta \equiv (4\pi r^2\sigma_\parallel/c^2)(R_o q\sqrt{4\pi\rho_m}/B_\Theta)$, the growth rate and spatial extent of these new modes depend only on the poloidal magnetic field strength B_Θ -- primarily because of the neoclassical polarization drift effect. Note also that all of the $\ell > 1$ modes are stable in our electrostatic approximation.

Comparing the results for this new viscosity driven mode with those given in Eqs. (108)-(112) for our form of the resistive interchange mode, we find that the new mode:

1. Scales like a resistive MHD instability -- except for the additional (μ_e/ν_e) electron viscosity factor and the fact that it is unstable for either sign of the average curvature but instead requires $(dP_0/d\psi)(dq/d\psi) < 0$ for instability.
2. Has a larger growth rate -- by a factor of order $(\mu_e/\epsilon^2\nu_e)^{1/3} \sim \epsilon^{-2/3}/(1 + \nu_{*e})^{1/3}$.
3. Is also a purely growing mode in the $\underline{E}_0 \times \underline{B}$ rest frame -- its real frequency is given by ω_E as defined in Eq. (69).
4. Does not extend as far along the field lines -- but extends further radially.
5. It also has a pressure or potential perturbation that is even about the rational surface -- "twisting" mode parity.

Compared to usual resistive-g modes,[12] this mode has a larger radial extent (by a factor $\sim (q/\epsilon)^{2/3} \sim 5$), because of the larger polarization drift and consequent smaller value of the relevant S_N. Thus, this new viscosity driven mode exhibits a number of interesting properties that can make it a more virulent instability than the resistive MHD instabilities of the interchange or ballooning type that are usually considered.

VI. Eigenmode Equation and Analysis: Higher Frequency and Drift Type Modes

In the preceding section we have pursued a fluidlike analysis in which we have assumed $\mu_i \gg \overline{\omega} \gg \omega_*$. However, since $\gamma_\mu \sim n^{2/3}$ but $\omega_* \sim n$, for some large n the $\bar{\omega} \gg \omega_*$ approximation breaks down. Further, since $\mu_i \sim \sqrt{\varepsilon}\, \nu_i \sim n^0$, the $\mu_i \gg \bar{\omega}$ approximation also breaks down for some large n. We are thus led to consider the "maximal ordering" situation where $\mu_i \sim \bar{\omega} \sim \omega_*$.

When the frequency and diamagnetic drift frequency become comparable to the viscous drag frequency we can anticipate that the viscous drag frequency μ_i is changed from its static equilibrium value determined in neoclassical transport theory. In Section IV of our companion kinetic paper[8] it is shown that in the banana collisionality regime ($\nu_i - i\bar{\omega} \ll \varepsilon^{2/3}\omega_{bi}$) we have $\mu_i \simeq \sqrt{\varepsilon}\,(\nu_i - i\bar{\omega})$. Thus, the ν_i in Eq. (36) is to be replaced by $\nu_i - i\bar{\omega}$. Similarly, since in plateau regime transport the form of the collision operator is not critically important, even when calculating the viscosity,[13] we can infer from the Krook model operator that ν_i would be replaced by $\nu_i - i\bar{\omega}$ (albeit probably with a different order unity numerical coefficient on the $-i\bar{\omega}$), but that the viscosity coefficient is still of order $\varepsilon^2\omega_{bi}$ in the plateau regime. Hence, we hypothesize that

$$\mu_i \simeq \frac{\sqrt{\varepsilon}\,(\nu_i - i\bar{\omega})}{1 + \varepsilon^{-3/2}(\nu_i - i\bar{\omega})/\omega_{bi}} \ . \tag{119}$$

While this phenomenological form has not been derived rigorously, it is physically reasonable and goes to the proper banana and plateau limiting cases for arbitrary ratios of $\nu_i/\bar{\omega}$.

Having specified a form for μ_i, we can now return to our general eigenmode equation given in Eq. (114), which we now write as

$$\bar{\omega}(\bar{\omega} - \omega_{*i})(\bar{\omega} - \omega_{*e}) = -\, i\gamma_\mu^3 L(\bar{\omega})N(\bar{\omega})/M(\bar{\omega}) \ . \tag{120}$$

Formulae for the multipliers $L(\overline{\omega})$, $M(\overline{\omega})$, $N(\overline{\omega})$ that obtain from utilizing the μ_i in Eq. (119) in the various collisionality regimes are listed in Table I.

In the banana regime but with $\overline{\omega} \gg \nu_i$, Eq. (120) can be solved utilizing the limiting values given in Table I. Of the three possible modes, the diamagnetic drift ones with $\overline{\omega} \simeq \omega_{*e}, \omega_{*i}$ can be shown to be damped. The third root is the higher mode number version of the "fluid" root given in Eq. (114). However, it is still purely growing in the $\underline{E_0} \times \underline{B}$ rest frame and has $|\overline{\omega}| \ll \omega_{*e}, \omega_{*i}$. Its growth rate is given by

$$\gamma = \gamma_\mu^3/|\omega_{*i}\omega_{*e}| = (m_e/m_i)\mu_e(T_e + T_i)^2/T_eT_i \ , \tag{121}$$

which is independent of the mode number n.

In the plateau regime both the ion and electron drift modes still seem to be stable, at least for the form of the viscosity given in Eq. (122) and the resultant lowest order forms of $M(\overline{\omega})$ and $N(\overline{\omega})$ given in Table I. However, some unstable modes are possible if the classical polarization drift effect is included since then $M(\overline{\omega}) \to i\varepsilon^2\omega_{bi}/\overline{\omega} + \varepsilon^2/q^2$ is not purely imaginary for real $\overline{\omega}$. This is apparently how these modes connect up with conventional drift waves. The third root again has $\overline{\omega} \ll \omega_{*e}, \omega_{*i}$ and so again obtains the growth rate given by Eq. (121).

The new mode we have found thus remains a purely growing mode in the $\underline{E_0} \times \underline{B}$ rest frame even in the presence of diamagnetic drift effects. The limiting growth rate formulae given in Eqs. (115) and (121) can be combined into the phenomenological formula

$$\gamma \simeq \gamma_\mu/\left(1 + |\omega_{*i}\omega_{*e}|/\gamma_\mu^2\right) \ . \tag{122}$$

Table I. Viscosity μ_i, Polarization Drift Multiplier $M(\bar{\omega})$, Bootstrap Current Multiplier $N(\bar{\omega})$, and $\langle \underline{B}\cdot\underline{\nabla}\rangle\tilde{P}$ Multiplier $L(\bar{\omega})$ in Various Collisionality Regimes

	Banana Regime		Plateau Regime
	$\bar{\omega} \ll \nu_i$	$\nu_i \ll \bar{\omega} \ll \varepsilon^{3/2}\omega_{bi}$	$\nu_i - i\bar{\omega} \gg \varepsilon^{3/2}\omega_{bi}$
μ_i	$\sqrt{\varepsilon}\ \nu_i$	$-i\sqrt{\varepsilon}\ \bar{\omega}$	$\varepsilon^2\omega_{bi}$
$M(\bar{\omega})$	1	$\sqrt{\varepsilon}$	$i\varepsilon^2\omega_{bi}/\bar{\omega}$
$N(\bar{\omega})$	1	$\dfrac{\bar{\omega} - \omega_{*e}}{\omega_{*i} - \omega_{*e}} - \sqrt{\varepsilon}\,\dfrac{\bar{\omega} - \omega_{*i}}{\omega_{*i} - \omega_{*e}}$	$\dfrac{\bar{\omega} - \omega_{*e}}{\omega_{*i} - \omega_{*e}} - i\,\dfrac{\varepsilon^2\omega_{bi}}{\bar{\omega}}\,\dfrac{\bar{\omega} - \omega_{*i}}{\omega_{*i} - \omega_{*e}}$
$L(\bar{\omega})$	1	$\sqrt{\varepsilon} + \dfrac{\mu_e/\nu_e}{1 + \mu_e/\nu_e}\left(1 - \sqrt{\varepsilon}\right) \times \dfrac{\bar{\omega} - \omega_{*e}}{\omega_{*i} - \omega_{*e}}$	$i\,\dfrac{\varepsilon^2\omega_{bi}}{\bar{\omega}} + \dfrac{\mu_e/\nu_e}{1 + \mu_e/\nu_e} \times \left(1 - i\,\dfrac{\varepsilon^2\omega_{bi}}{\bar{\omega}}\right)\dfrac{\bar{\omega} - \omega_{*e}}{\omega_{*i} - \omega_{*e}}$

This formula underestimates the true growth rate by less than 30% as n increases from a small value where $\gamma \simeq \gamma_\mu \propto n^{2/3}$ to a large value where the growth rate, Eq. (121), becomes independent of n.

VII. Discussion and Summary

In this paper we have shown how resistive MHD can be modified using neoclassical ideas to derive the appropriate flows and currents to develop a "neoclassical MHD," which is appropriate for the present banana-plateau regime of collisionality in tokamaks. For simplicity we have restricted the analysis to electrostatic perturbations in a plasma with constant temperature profiles, but $T_e \neq T_i$. The inclusion of magnetic perturbations[9] and temperature gradients is relatively straightforward but tedious and obfuscates the basic issues involved.

The main modifications of resistive MHD in going to "neoclassical MHD" are: (1) only a global Ohm's law is meaningful and it includes a viscosity driven bootstrap current contribution; (2) the curvature effects are reduced to their flux surface average effects and are negligible compared to the viscosity driven bootstrap current effects; and (3) the polarization drift is increased by a factor of $B^2/B_\Theta^2 \sim q^2/\varepsilon^2 \gg 1$. Of these effects (1) and (3) are purely neoclassical or kinetic effects and (2) is a result of our multiple length scale ordering which could be derived in regular resistive MHD.

Taking account of these modifications a new pressure-gradient-driven fluidlike instability which scales in a similar way to resistive-g modes is obtained for $(dP_0/d\psi)(dq/d\psi) < 0$. The mode is unstable for either sign of the average curvature. The growth rate and spatial extension of the new mode are indicated in Eqs. (115), (117), (118), and, including diamagnetic drift effects, in Eq. (122). The real frequency associated with the mode is the equilibrium $\underline{E}_0 \times \underline{B}$ Doppler shift frequency as defined in Eq. (69).

Finally, one might question how this new type of analysis is transformed, in some limiting case, back to regular resistive or ideal MHD. As a function of collisionality, most of our new effects (namely the bootstrap current and polarization drift enhancement) become negligible in the Pfirsch-Schlüter collisionality regime. However, the cancellation of the lowest order geodesic curvature effects by the Pfirsch-Schlüter currents remains -- unless one orders $\omega \gg \nu \sim \omega_b$, which would imply a very rapidly growing mode, or has larger short scale variations in $\tilde{\phi}$ than indicated in Eq. (61) and Fig. 1. Thus, except for the multiple-length-scale form of the curvature term, we recover the usual resistive MHD limit in the Pfirsch-Schlüter collisionality regime. For ideal (or resistive[14]) MHD modes which are growing sufficiently rapidly so as to be localized within roughly $-\pi \lesssim y \lesssim \pi$, or are highly "ballooning," our multiple-length-scale analysis would not apply because then $\tilde{\phi}_n^{(1)}(\Theta,y_s)$ would not be "small" in Δ compared to $\tilde{\phi}_n^{(0)}(y_s)$. Thus, it appears that the usual resistive and ideal MHD analyses would apply to strongly ballooning modes, but for modes that are highly extended along magnetic field lines, the type of "neoclassical MHD" analysis presented in this paper and kinetic variants of it[8,9] should replace analyses of the compressibility effects based upon the equation of state $d/dt(P/\rho_m^{\Gamma}) = 0$. The synthesis of regular resistive MHD and this new "neoclassical MHD" model into one comprehensive theory, which does not require a multiple length scale approximation, remains an unsolved challenge at the present time.

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APPENDIX A

In this appendix we derive the usual electrostatic resistive MHD eigenmode equation in a short mean free path or Pfirsch-Schlüter regime to facilitate comparisons with the long mean free path results derived in the body of the paper. We begin with the usual resistive MHD equations

$$\rho_m \frac{d\underline{V}}{dt} = \underline{J} \times \underline{B} - \underline{\nabla}P \ , \tag{A.1}$$

$$\underline{E} + \frac{1}{c} \underline{V} \times \underline{B} = \underline{J}_\perp/\sigma_\perp + J_\parallel \underline{b}/\sigma_\parallel \ , \tag{A.2}$$

$$\frac{d}{dt} \left(P/\rho_m^\Gamma\right) = 0 \ . \tag{A.3}$$

Taking an electrostatic, isothermal limit, as in the body of the paper, with $\underline{E} = -\underline{\nabla}\phi$ and $P = \rho_m T$, the perturbed form of these equations becomes simply

$$\rho_m \frac{\partial \underline{\tilde{V}}}{\partial t} = \underline{\tilde{J}} \times \underline{B} - \underline{\nabla}\tilde{P} \tag{A.4}$$

$$-\underline{\nabla}\tilde{\phi} + \frac{1}{c} \underline{\tilde{V}} \times \underline{B} = \underline{\tilde{J}}_\perp/\sigma_\perp + \tilde{J}_\parallel \underline{b}/\sigma_\parallel \tag{A.5}$$

$$-(\Gamma - 1)\left(T/\rho_m^\Gamma\right)\left(\partial\tilde{\rho}_m/\partial t + \underline{\tilde{V}}\cdot\underline{\nabla}\rho_o\right) = 0 \ . \tag{A.6}$$

Solving Eq. (A.4) for the perturbed perpendicular current $\underline{\tilde{J}}_\perp$, we find

$$\underline{\tilde{J}}_\perp = \frac{c}{B} \underline{b} \times \left(\underline{\nabla}\tilde{P} + \rho_m \frac{\partial}{\partial t} \underline{\tilde{V}}\right) \ , \tag{A.7}$$

which indicates both the perturbed diamagnetic current and the polarization drift current. Then, solving Eq. (A.4) for its perpendicular and parallel components, we find

$$\underline{\tilde{V}}_\perp = \frac{c}{B}\,\underline{b} \times \left(\underline{\nabla}\tilde{\phi} + \underline{\tilde{J}}_\perp/\sigma_\perp\right) = \frac{c}{B}\,\underline{b} \times \underline{\nabla}\tilde{\phi} - \frac{c^2}{B^2\sigma_\perp}\,\underline{\nabla}_\perp\tilde{P} \tag{A.8}$$

$$\tilde{J}_\parallel = -\sigma_\parallel(\underline{b}\cdot\underline{\nabla})\tilde{\phi}\ . \tag{A.9}$$

Note that the last term in Eq. (A.6) is just the classical diffusion flow due to the perturbed pressure and can be neglected for the weakly collisional plasma of interest. Thus, substituting in the lowest order part of $\underline{\tilde{V}}_\perp$ into Eq. (A.5), we find

$$\underline{\tilde{J}}_\perp = \frac{c}{B}\,\underline{b} \times \underline{\nabla}\tilde{P} + \frac{c^2\rho_m}{B^2}\,\underline{b} \times \left(\underline{b} \times \frac{\partial}{\partial t}\,\underline{\nabla}\,\tilde{\phi}\right)\ . \tag{A.10}$$

Now, substituting $\tilde{J}_\parallel$ and $\underline{\tilde{J}}_\perp$ from Eqs. (A.7) and (A.8) into the quasi-neutrality equation

$$0 = \underline{\nabla}\cdot\underline{\tilde{J}} = (\underline{B}\cdot\underline{\nabla})(\tilde{J}_\parallel/B) + \{\underline{\nabla}\cdot\underline{\tilde{J}}_\perp\} - \underline{\tilde{J}}_\perp\cdot\underline{\nabla}\ \ln B^2\ , \tag{A.11}$$

we readily obtain

$$0 = -(\underline{B}\cdot\underline{\nabla})\left(\frac{\sigma_\parallel}{B^2}\right)(\underline{B}\cdot\underline{\nabla})\tilde{\phi} - \frac{c^2\rho_m}{B^2}\,\frac{\partial}{\partial t}\,\nabla_\perp^2\tilde{\phi} - \frac{c}{B}\,\underline{b} \times \underline{\nabla}\tilde{P}\cdot\underline{\nabla}\ \ln B^2\ . \tag{A.12}$$

To write this equation completely in terms of $\tilde{\phi}$, we need to utilize Eq. (A.6) in the form (for our isothermal plasma)

$$\frac{\partial \tilde{P}}{\partial t} = -\underline{\tilde{V}}\cdot\underline{\nabla}P_o = -\frac{c}{B}\,\underline{b}\times\underline{\nabla}\tilde{\phi}\cdot\underline{\nabla}P_o \,. \tag{A.13}$$

Considering perturbations of the form $e^{-i\omega t}$ this reduces to

$$\tilde{P} = \frac{c}{i\omega B}\,\underline{b}\times\underline{\nabla}\tilde{\phi}\cdot\underline{\nabla}P_o = \frac{c}{i\omega B}\left(\underline{b}\times\underline{\nabla}\tilde{\phi}\cdot\underline{\nabla}\psi\right)\frac{dP_o}{d\psi} = \frac{ic}{\omega}\frac{\partial\tilde{\phi}}{\partial\beta}\frac{dP_o}{d\psi}\,, \tag{A.14}$$

which is the regular perturbed convective pressure response due to the $\underline{\tilde{E}} \times \underline{B}$ flow. Utilizing this in Eq. (A.12), we readily obtain

$$0 = -(\underline{B}\cdot\underline{\nabla})\left(\frac{\sigma_\parallel}{B^2}\right)(\underline{B}\cdot\underline{\nabla})\tilde{\phi} + \frac{i\omega c^2\rho_m}{B^2}\nabla_\perp^2\tilde{\phi} - \frac{ic^2}{\omega}\frac{dP_o}{d\psi}\left(\frac{\partial \ln B^2}{\partial\beta}\frac{\partial}{\partial\psi} - \frac{\partial \ln B^2}{\partial\psi}\frac{\partial}{\partial\beta}\right)\frac{\partial\tilde{\phi}}{\partial\beta}\,. \tag{A.15}$$

This is the usual eigenmode equation for electrostatic resistive interchange (or g -- gravity) modes in a plasma, with the first term representing the magnetic field line diffusion or slippage due to plasma resistivity, the second indicating the polarization drift effect, and the last term the magnetic field curvature effects. (In our low β approximation $\partial \ln B^2/\partial\psi = 2\kappa_\psi \equiv 2\underline{\kappa}\cdot\underline{\nabla}\psi = 2\underline{\nabla}\psi\cdot(\underline{b}\cdot\underline{\nabla})\underline{b}$.) Note that both the normal curvature $\partial \ln B^2/\partial\psi$ and the geodesic curvature $\partial \ln B^2/\partial\beta$ give contributions to this equation. Note also the similarity of this equation to Eq. (90), except for there even in the limit $\omega_* \to 0$: (1) there are added fluctuating bootstrap current and $(\underline{B}\cdot\underline{\nabla})\tilde{p}$ terms; (2) the lowest order geodesic curvature effects are absent; and (3) the polarization drift contribution is larger by a factor $B^2/B_\theta^2 = q^2/\varepsilon^2 \gg 1$. Of these differences (1) and (3) are neoclassical or kinetic effects while (2) would also occur in regular resistive MHD if, for example, the multiple length scale expansion with the $\tilde{\phi}$ ordering indicated in Eq. (61) and Fig. 1 were carried out on Eq. (A.15).