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COLLISIONAL BOUNDARY LAYER EFFECTS ON THE THERMAL BARRIER ION TRAPPING RATE IN A TANDEM MIRROR

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ABSTRACT

In a classical analysis of the particle distribution function in a thermal barrier tandem mirror the expression for the ion trapping rate in the thermal barrier grows unboundedly in the limit of strong pumping. This singularity is removed by considering a boundary layer problem around the boundary contours in velocity space. The problem is converted into a Wiener-Hopf functional equation in the complex plane along a boundary line in phase space whose solution yields the desired barrier distribution function. Then, an expression for the thermal barrier ion trapping rate valid for fairly strong pumping rates (trapped density << passing density) is obtained.

1. INTRODUCTION

The thermal barrier[1] in a tandem mirror is a region of depressed magnetic field and particle density in which a dip in the ambipolar electrostatic potential is maintained by removing those ions that become trapped in the magnetic and electrostatic well[1,2] and creating a population of hot electrons which are magnetically and electrostatically trapped in the barrier region.[1,3] The thermal barrier isolates central cell electrons from plug electrons and then allows tandem mirror operation with plugs which are less dense than the central cell while having plug electrons of higher temperature than central cell electrons.[1,4] A sketch of the variation of magnetic field and electrostatic potential along field lines in a tandem mirror is given in Fig. 1. In our model thermal barrier the ion pumping is obtained by charge exchange with neutral beams. In the thermal barrier there are two ion species: ions that are trapped in the magnetic and electrostatic well which are referred to as trapped; and ions which originate in the central cell, pass over the barrier and are reflected back into the central cell by the plug, which are called passing. The pumping mechanism replaces a trapped ion by a passing ion.

In a classical analysis[2] of the ion distribution function in a tandem mirror the kinetic equation is solved in the various phase space regions by an expansion in powers of $\lambda = \nu_s/\omega_b$ (collision frequency/bounce frequency) and this procedure is applied to each of the relevant ion species, namely, ions magnetically trapped in the central cell, central cell ions passing over the barrier and ions magnetically and electrostatically trapped in the barrier. As a result of this analysis the following expression for the ion trapping rate is obtained[2]

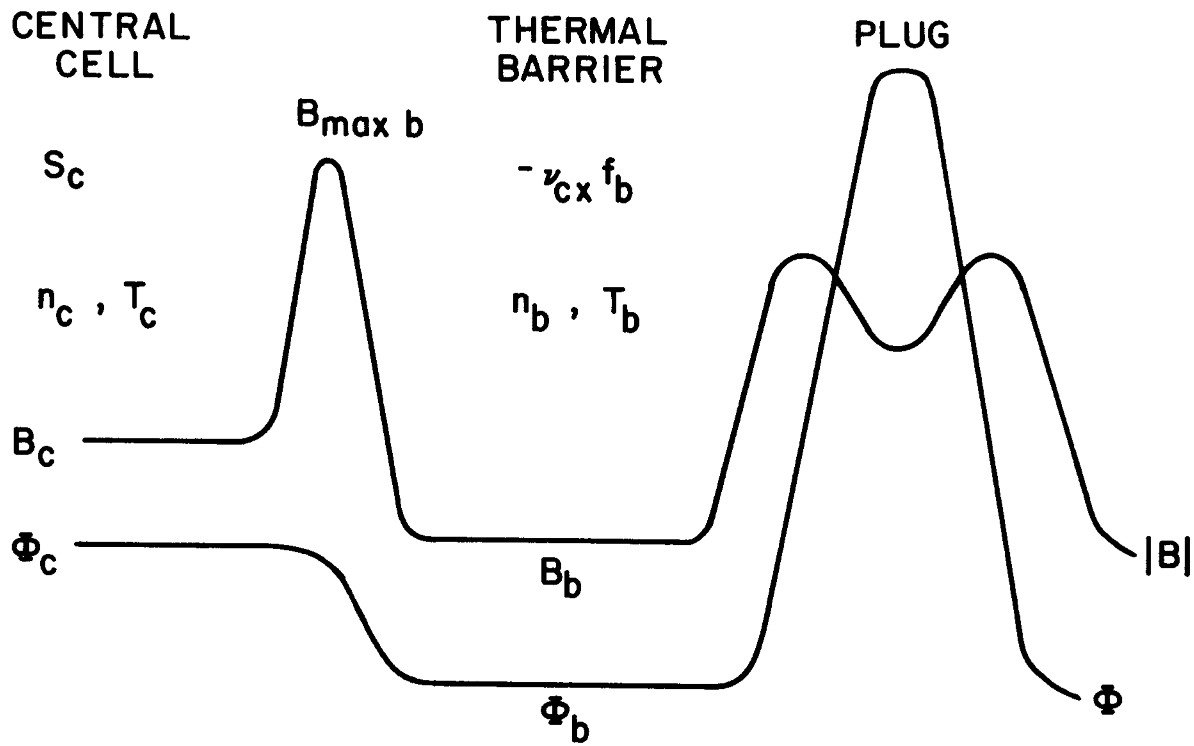

Fig. 1. Sketch of the axial magnetic field and electrostatic potential profiles for a tandem mirror with thermal barriers. Indicated are the densities, temperatures and sources/sinks.

$$j_t = \frac{2n_b^2 R_b (T_p/\Phi_b)^{1/2}}{(n_b\tau_{ip}) g(g-1)} , \quad (1)$$

where j_t is the ion trapping rate (number of ions being trapped into the thermal barrier per second); n_b = barrier density (cm^{-3}); $R_b = B_{maxb}/B_b$; T_p = passing ion temperature (keV); Φ_b = barrier potential dip (keV); $n_b\tau_{ip} = M^{1/2}T_p^{3/2}/\sqrt{2}\ \pi e^4 \ln \Lambda$, M = ion mass, $\ln \Lambda$ = Coulomb logarithm; and $g = n_b/n_{bp}$ is the filling ratio with n_{bp} = passing ion density in the barrier(cm^{-3}). The filling ratio, g, is given by

$$g = 1 + \left[\frac{2n_b R_b (T_p/\Phi_b)^{1/2}}{(n_b\tau_{ip})\nu_{cx}}\right]^{1/2} , \quad (2)$$

where ν_{cx} = charge-exchange pumping rate (sec^{-1}). Equation (1) is singular when $g \to 1$, i.e. the ion trapping rate increases unboundedly as the pumping rate is increased. This is because for fairly strong pumping, the gradient of the distribution function of barrier ions close to the boundary contour in velocity space becomes very sharp. This singular behavior is a consequence of the assumption $\lambda \ll 1$ since then any passing particle in the barrier requires essentially no time to reach the central cell. This in turn causes a discontinuity in the derivative of the distribution function to be obtained at the boundary separating trapped from passing particles in velocity space.

The expansion procedure utilized in a classical analysis fails in regions of velocity space where there exist large derivatives of the distribution function.(5) Therefore, a boundary layer is necessary around the boundary contours in velocity space in order for the trapped and passing particle distribution functions to join smoothly. A boundary layer analysis should match onto the bulk analysis results that are valid far away from the velocity space boundaries. The effects due to this boundary layer are expected to modify significantly the barrier distribution function for filling ratios $g < 2$.[2] In order for the ion trapping rate to remain finite as the pumping rate is increased the derivative of the barrier distribution function must be continuous. In a boundary layer analysis small angle collisions will smooth out any discontinuity of the distribution function at the boundaries and thus it will remove the singularity in the ion trapping rate when $g \to 1$.

We consider a thermal barrier model in which n_b is assumed to be known by quasineutrality, Φ_b is determined by electron flux conservation in velocity space[3,4] and R_b, T_p and ν_{cx} are given quantities. The central cell particles have temperature T_{ic} ($= T_p$) and density n_c. We wish to find the ion distribution function modifications due to boundary layer effects and from them obtain an expression for the ion trapping rate valid for fairly strong pumping rates in the thermal barrier ($g \sim 1$).

In the following sections we will use the quantities: η = number of particles in the barrier region/number of particles in the central cell (typically $\eta << 1$, i.e., "large" central cell), and $\gamma = \nu_{cx}/\omega_b$ = charge exchange frequency/bounce frequency in barrier. Typically, in a bulk analysis (like that of Ref. 2) $\gamma \simeq 0$ whereas in our boundary layer analysis $0 < \gamma < 1$. In Section 2 of this work we introduce the general boundary layer analysis. The thermal barrier distribution function is obtained in Section 3 and then from it the ion trapping rate is calculated in Section 4. The conclusions are given in Section 5.

2. BOUNDARY LAYER ANALYSIS

The particle distribution function f for a plasma confined in a mirror trap is the solution of a kinetic equation of the form df/dt = C(f) + S where d/dt represents the total derivative along particle trajectories, C(f) is the Fokker-Planck collision operator and S is any source or sink intervening in the problem. For a steady state situation with azimuthal symmetry we can write $f = f(v,\zeta,s)$ where v = particle speed, $\zeta = \vec{v} \cdot \hat{b}/v$, $\hat{b}$ = unit vector in the direction of the magnetic field, s = axial variable along field lines and where a parametric field line dependence in f has been omitted. This equation can be simplified by using the midplane (s = 0) variables v_0, ζ_0 in velocity space which are related to the local variables v,ζ through the adiabatic relations

$$v^2 = v_o^2 + \frac{2\left[\Phi(o) - \Phi(s)\right]}{M} ,$$

$$v^2(1 - \zeta^2) = \frac{B(s)}{B(o)} v_o^2 (1 - \zeta_o^2) , \tag{3}$$

where $\Phi(s)$ is the ambipolar electrostatic potential and B(s) the magnetic field. Then, the kinetic equation reduces to

$$v_{\parallel} \frac{\partial f}{\partial s} = C(f) + S , \tag{4}$$

where $v_{\parallel} = v\zeta$ and both sides of the equation are expressed in terms of the velocity space variables at the midplane of the mirror cell (this is a point such that all particles pass through it during a bounce). For simplicity we consider that the square-well approximation is always applicable in our problem so that $v \equiv v_0$ and $\zeta \equiv \zeta_0$.

We are interested in the distribution function behavior in a region close to the boundary contours in velocity space where the nature of the particles as trapped or passing is indistinguishable due to the small angle Coulomb collisions and therefore, even in the square-well approximation we retain the s dependence of the distribution function. We note that in a boundary layer analysis the distribution function, when expressed as a function of the constants of motion, is not constant along field lines. In the region of velocity space under consideration the higher order angular derivatives of the distribution function are dominant in the expression for the collision operator[5] so that Eq. (4) takes the form

$$v_\parallel \frac{\partial f}{\partial s} \simeq \frac{\Gamma}{2v^3} \frac{\partial g}{\partial v} (1 - \zeta_*^2) \frac{\partial^2 f}{\partial \zeta^2} , \qquad (5)$$

where $\Gamma = 4\pi e^4 \ln \Lambda/M^2$, $g(\vec{v}) = \int d^3v' \; f(\vec{v}')|\vec{v} - \vec{v}'|$ is the Rosenbluth potential, and $\zeta = \zeta_*(v)$ represents the boundary contour in velocity space separating trapped from passing particles. Particle sources do not appear in Eq. (5) since they act on a much slower time scale. In circumstances of interest to our problem ($\eta << 1$, $\Phi_b/T_p > 1$, $R_b >> 1$),[2] we can approximately write $\partial g/\partial v \simeq n[(1 - v_t^2/2v^2) \times \text{erf}(v/v_t) + 1/\sqrt{\pi}\,(v_t/v) \exp(-v^2/v_t^2)]$ in the central cell ($v_t = (2T_{ic}/M)^{1/2}$) and $\partial g/\partial v \simeq n$ in the thermal barrier. Then, we obtain the following boundary layer equations

$$v_{\parallel} \frac{\partial f}{\partial s} \simeq \frac{\Gamma}{2v^3} \frac{\partial g}{\partial v} (1 - \zeta_*^2) \frac{\partial^2 f}{\partial \zeta^2} ,$$

$$v_{\parallel} \frac{\partial f}{\partial s} \simeq \frac{n\Gamma}{2v^3} (1 - \zeta_*^2) \frac{\partial^2 f}{\partial \zeta^2} - \nu_{cx} f , \tag{6}$$

for the central cell and thermal barrier, respectively. In the central cell and in the barrier particles diffuse around the boundary contour while streaming along field lines but in the thermal barrier region an additional alteration of the distribution function occurs because of the strong pumping. The pumping mechanism eliminates particles from the trapped and boundary layer region in the thermal barrier and places them into the bulk of the passing region where they leak to the central cell in a bounce time. We assume that ν_{cx} is constant over the barrier phase space. A sketch of the distribution function for ions in the central cell and barrier is given in Fig. 2. Indicated are the type of distortions expected because of boundary layer effects. The boundary layer width is $\Delta\zeta \sim \lambda^{1/2}$ ($\lambda = \nu_s/\omega_b$). In the central cell we neglect the influence of the plug loss-boundary. In the thermal barrier velocity space we assume a separable boundary contour $\zeta_* \equiv \zeta_b$ = constant(2) and we neglect that part of the trapping not due to pitch-angle scattering which occurs through a small boundary layer in $v_{\parallel}$. The boundary contours in velocity space at the central cell and barrier are then defined by $\zeta_* = \zeta_c$ and $\zeta_* = \zeta_b$, $v \geq [2\Phi_b/M]^{1/2}$, respectively, where $1 - \zeta_c^2 = R_b^{-1} = B_c/B_{maxb}$ and $1 - \zeta_b^2 = R_b^{-1} (1 + \Phi_b/T_p)^{-1}$.(2)

Next, we define the boundary layer variables $x = (\zeta - \zeta_*)/(D\lambda)^{1/2}$ and $z = s/L$ where $D = 1/n_0\ \partial g/\partial v\ [(1 - \zeta_*^2)/2]$, n_0 = density at the cell midplane, $\lambda = \nu_s/\omega_b$, $\nu_s = n_0\Gamma/v^3$, $\omega_b = v\zeta_*/L$ and L = cell length. Thus we may write Eqs.

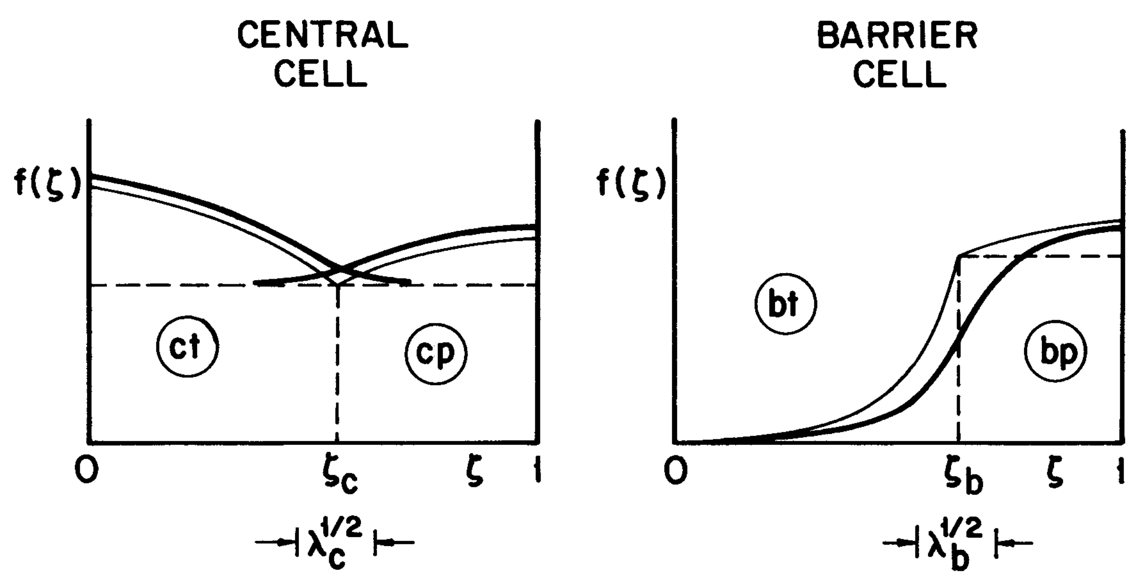

Fig. 2. Angular distribution function in the central cell and the thermal barrier. Indicated are the bulk analysis results of Ref. 2 (thin lines) and the boundary layer modifications to them (heavy lines). The boundary layer widths are given in terms of $\lambda = \nu_s/\omega_b$. The various phase space regions for particles in the central cell and barrier are shown: cp = central cell passing; bp = barrier passing; ct = central cell trapped; and bt = barrier trapped.

(6) in the form

$$\text{central cell:} \quad \frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x^2} , \tag{7}$$

$$\text{thermal barrier:} \quad \frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x^2} - \gamma f , \tag{8}$$

where $\gamma = \nu_{cx}/\omega_b$. Since $\lambda << 1$ we consider the boundaries in x to be effectively at $\pm\infty$; z varies from 0 to 1 in each of the traps. It is assumed that $\lambda^{1/2} << 1 - \zeta_b$ in the barrier region. The sub-boundary layers of width $\Delta\zeta \sim \lambda$ due to the lower order angular derivatives of the distribution function are negligible since they extend over a much smaller region in velocity space than the boundary layers considered here.

In principle, for our tandem mirror model the boundary layers due to the trapped central cell particles (with distribution function $f = t_+ + f_o$, $\zeta > 0$; $f = t_- + f_o$, $\zeta < 0$), passing central cell particles ($f = p_+ + f_o$, $\zeta > 0$; $f = p_- + f_o$, $\zeta < 0$), particles in the right-side thermal barrier ($f = r$, $\zeta > 0$; $f = \ell$, $\zeta < 0$) and particles in the left-side thermal barrier ($f = r'$, $\zeta > 0$; $f = \ell'$, $\zeta < 0$) need to be considered (see Fig. 3). The isotropic component of the distribution function in the central cell is $f_o \simeq (n_c/\pi^{3/2} v_t^3) \exp(-v^2/v_t^2)$. Because of the symmetry of the device, half of these boundary layer problems are decoupled and then we need only solve the following system of equations,

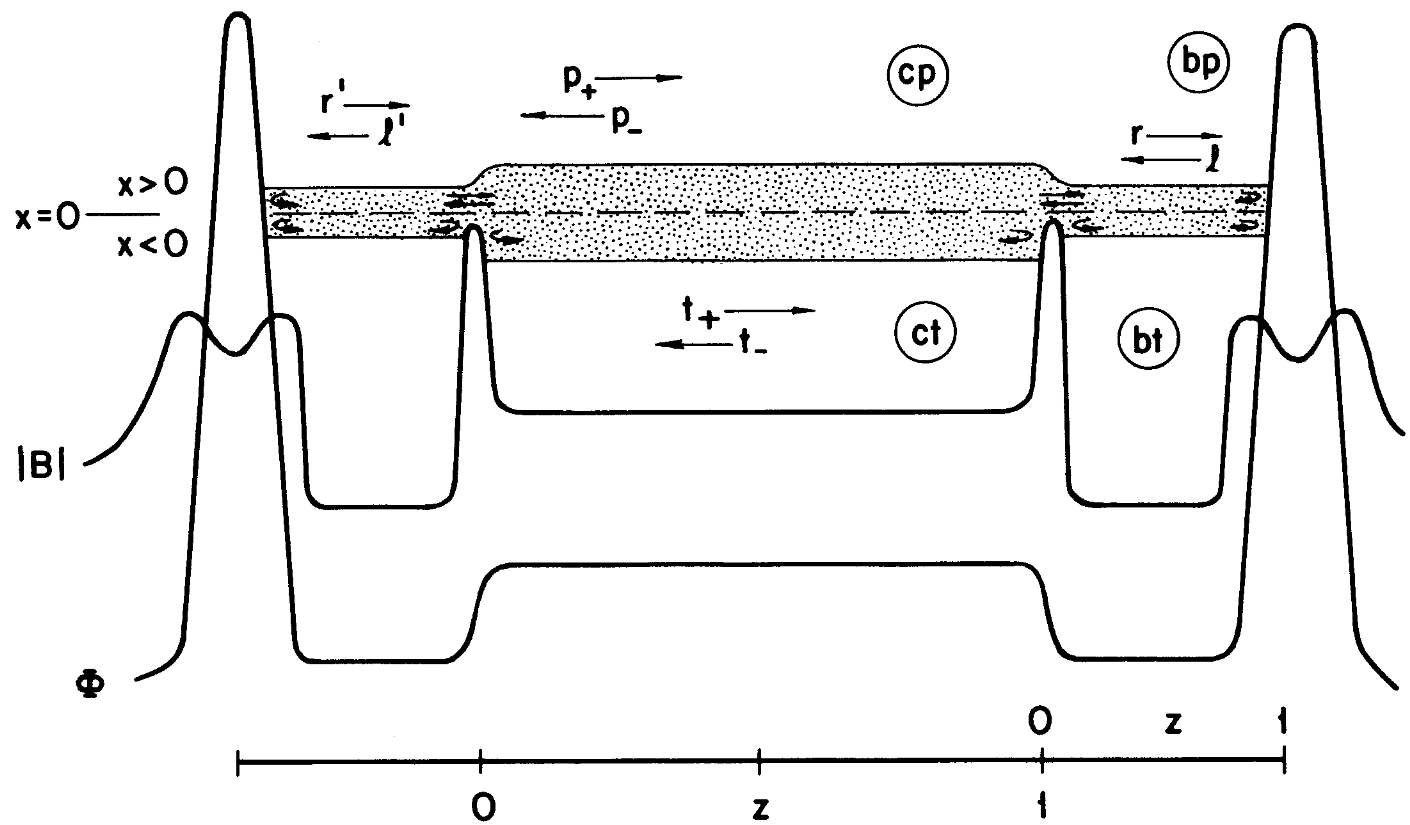

Fig. 3. Diagram representing the various boundary layers around the boundary contours in the phase space of a tandem mirror. Indicated are the relevant distribution functions as given in the text and their boundary conditions.

$$\frac{\partial t_+}{\partial z} = \frac{\partial^2 t_+}{\partial x^2} , \tag{9}$$

$$\frac{\partial p_+}{\partial z} = \frac{\partial^2 p_+}{\partial x^2} ,$$

$$\frac{\partial r}{\partial z} = \frac{\partial^2 r}{\partial x^2} - \gamma r ,$$

$$\frac{\partial \ell}{\partial z} = - \frac{\partial^2 \ell}{\partial x^2} + \gamma \ell .$$

This system of partial differential equations is coupled through the boundary conditions, which will be given shortly. The effects of trapped and passing particles can be superposed due to the linearity of the problem; as an example, the boundary layer solution in the central cell is $f = t_+ + p_+ + f_0$, for $\zeta > 0$.

There are three types of boundary conditions which apply to Eqs. (9):

- continuity of the solution at the barrier peaks ($B = B_{maxb}$),
- reflection of particles at the plug end of the barrier region and the barrier peaks,
- symmetry of the central cell and the entire device.

As indicated above the symmetry conditions decouple the various boundary layer problems so that only the four problems given in Eqs. (9) are independent, namely, the central cell trapped and passing boundary layers and the $v_\parallel \gtrless 0$ boundary layer problems in the right-side barrier cell (Fig. 3). Thus, the following boundary conditions can be written,

$$
\begin{aligned}
t_+(x,0) &= t_+(x,1) \,, & x < 0 \\
t_+(x,0) &= 0 \,, & x > 0 \\
p_+(x,0) &= p_+(x,1) \,, & x < 0 \\
p_+(x,0) &= \ell(x,0) - f_0 \,, & x > 0 \\
r(x,0) &= \ell(x,0) \,, & x < 0 \\
r(x,0) &= t_+(x,1) + p_+(x,1) + f_0 \,, & x > 0 \\
r(x,1) &= \ell(x,1) \,. & x \lessgtr 0
\end{aligned}
\tag{10}
$$

Moreover, the solutions should have the following asymptotic behavior,

$$
\begin{aligned}
& t_+(x) \xrightarrow[x\to\infty]{} 0 \,, \\
& t_+(x) \xrightarrow[x\to-\infty]{} \sigma_t x + c_t \,, \\
& p_+(x) \xrightarrow[x\to+\infty]{} \sigma_p x + c_p \,, \\
& p_-(x) \xrightarrow[x\to-\infty]{} 0 \,, \\
& r(x) \xrightarrow[x\to+\infty]{} \sigma_r x + c_r \,, \\
& r(x) \xrightarrow[x\to-\infty]{} 0 \,, \\
& \ell(x) \xrightarrow[x\to\infty]{} \sigma_\ell x + c_\ell \,, \\
& \ell(x) \xrightarrow[x\to-\infty]{} 0 \,,
\end{aligned}
\tag{11}
$$

where the σ and c constants are determined by the results of the bulk analysis.[2] The linear asymptotic limits derive from the z independence of the distribution function far from the boundary layer in the central cell and from the condition of a large central cell ($\eta \ll 1$). The effect of the particles pumped into the passing region of the device is an $O(\eta)$ effect which is neglected. The system of equations (9) together with the conditions (10) and (11) form a mixed boundary value problem which can be reduced to a system of functional equations in the complex plane of the Wiener-Hopf type.[6] The solution to the problem must have a continuous derivative at the boundary contours in the velocity spaces of the central cell and barrier regions.

Now we introduce the Fourier transforms

$$T(k,z) = \int_{-\infty}^{\infty} \exp(ikx)\ t_+(r,z)\ dx \ , \tag{12}$$

$$P(k,z) = \int_{-\infty}^{\infty} \exp(ikx)\ p_+(x,z)\ dx \ ,$$

$$R(k,z) = \int_{-\infty}^{\infty} \exp(ikx)\ r(x,z)\ dx \ ,$$

$$L(k,z) = \int_{-\infty}^{\infty} \exp(ikx)\ \ell(x,z)\ dx \ ,$$

where $k = \sigma + i\tau$. The transformed system of equations (9) is then

$$\frac{\partial T}{\partial z} = -k^2 T(k,z) \ , \tag{13}$$

$$\frac{\partial P}{\partial z} = -k^2 P(k,z) \ ,$$

$$\frac{\partial R}{\partial z} = -(k^2 + \gamma)\ R(k,z) \ ,$$

$$\frac{\partial L}{\partial z} = (k^2 + \gamma)\ L(k,z) \ ,$$

whose solutions are

$$T(k,z) = T(k,0)\ \exp(-k^2 z)\ , \tag{14}$$

$$P(k,z) = P(k,0)\ \exp(-k^2 z)\ ,$$

$$R(k,z) = R(k,0)\ \exp[-(k^2 + \gamma)z]\ ,$$

$$L(k,z) = L(k,1)\ \exp[(k^2 + \gamma)(z - 1)]\ .$$

Before applying the boundary conditions given in Eqs. (10) we need to define the one-sided Fourier transforms

$$G^+(k,z) = \int_0^\infty \exp(ikx)\ g(x,z)\ dx\ ,$$

$$G^-(k,z) = \int_{-\infty}^0 \exp(ikx)\ g(x,z)\ dx\ , \tag{15}$$

for the functions $g = t_+,\ p_+,\ r,\ \ell$ where $G = T,\ P,\ R,\ L$. According to the asymptotic limits of Eqs. (11) $T^+(k,z)$, $P^-(k,z)$, $R^-(k,z)$ and $L^-(k,z)$ are entire functions; $T^-(k,z)$ is an analytic function for $\tau < 0$; and $P^+(k,z)$, $R^+(k,z)$ and $L^+(k,z)$ are analytic functions for $\tau > 0$. Then, substituting the transformed boundary conditions from Eqs. (10) into the system of equations (14) we obtain

$$T^+ = -WT^-\ , \tag{16}$$

$$P^+ - (L^+ - F_0^+)\ \exp(-k^2) = -WP^-\ ,$$

$$L^+ - (T^+ + P^+ + F_0^+)\ \exp[-2(k^2 + \gamma)] = -UL^-\ ,$$

$$R^+ - T^+ - P^+ - F_0^+ = -R^- + L^-\ ,$$

where $W(k) = 1 - \exp(-k^2)$ is an entire function with a denumerable infinity of zeros along the diagonals of the complex k-plane and a double zero at the origin; $F_0^+(k) = if_0/k$ is an analytic function for $\tau > 0$; and $U(k) = 1 - \exp[-2(k^2 + \gamma)]$ is an entire function with a denumerable infinity of simple zeros. The unknown functions in Eqs. (16) are $T^{\pm} = T^{\pm}(k,1)$, $P^{\pm} = P^{\pm}(k,1)$, $R^{\pm} = R^{\pm}(k,0)$ and $L^{\pm} = L^{\pm}(k,0)$. We note that the first equation is defined in the lower-half complex k-plane whereas the remaining equations of the system are defined in the upper-half plane. The boundary layer problem, therefore, has been converted into a system of coupled Wiener-Hopf equations along a boundary line in phase space whose solution yields the desired distribution function. The first equation of the system of equations (16), which corresponds to the boundary layer for the central cell trapped particles, was solved by Baldwin, Cordey and Watson[5] within the boundary layer analysis for a simple mirror.

In a tandem mirror with a "large" central cell ($\eta \ll 1$) the system of equations (16) may be expanded in terms of the small quantity η and to lowest order we find

$$\begin{aligned} L^+ - F_0^+ \exp[-2(k^2 + \gamma)] &= -UL^- \,, \\ R^+ - F_0^+ &= -R^- + L^- \,, \end{aligned} \tag{17}$$

which are defined for $\tau > 0$.

3. THERMAL BARRIER DISTRIBUTION FUNCTION

The thermal barrier particles are reflected by the plug whereupon they reverse their direction of motion. Therefore, we can consider the barrier region as a cell of length $2L_b$ with a source of particles entering from the central cell. This source is isotropic to lowest order in η ($\eta \ll 1$). One expects that for the deeply trapped particles the distribution function will remain constant along field lines, i.e. $\partial f/\partial z \; (x \to -\infty) \to 0$ whereas for the passing particles the distribution function will remain isotropic, i.e. $\partial^2 f/\partial x^2 \; (x \to +\infty) \to 0$, away from the boundary layer. Thus the asymptotic behavior in the boundary layer analysis matches the bulk analysis results.(2) Hence, the boundary layer problem for the thermal barrier of a tandem mirror (see Fig. 4) with a large central cell can be formulated as

$$\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x^2} - \gamma f(x,z) \;, \tag{18}$$

$$x \varepsilon (-\infty, \infty) \;, \qquad z \varepsilon [0,1] \;,$$

$$f(x,0) = f(x,1) \;, \qquad x < 0 \;,$$

$$f(x,0) = f_0 \;, \qquad x > 0 \;,$$

$$f(x,1) \xrightarrow[x \to -\infty]{} e^{\sqrt{\gamma}\, x} \;,$$

$$f(x,1) \xrightarrow[x \to +\infty]{} \text{constant} \;,$$

with the boundary layer variables $x = (\zeta - \zeta_b)/(D\lambda)^{1/2}$ and $z = s/2L_b$ where $D = (1 - \zeta_b^2)/2$, $\lambda = \nu_s/\omega_b$, $\nu_s = n_{b_0}\Gamma/v^3$, n_{b_0} = density of the barrier midplane (assumed symmetry-plane of the cell), $\omega_b = v\zeta_b/2L_b$ = frequency for one barrier full bounce and L_b = barrier length. The statement of the problem given in Eqs. (18) deals with both barrier boundary layers ($v_\parallel \gtrless 0$) at once.

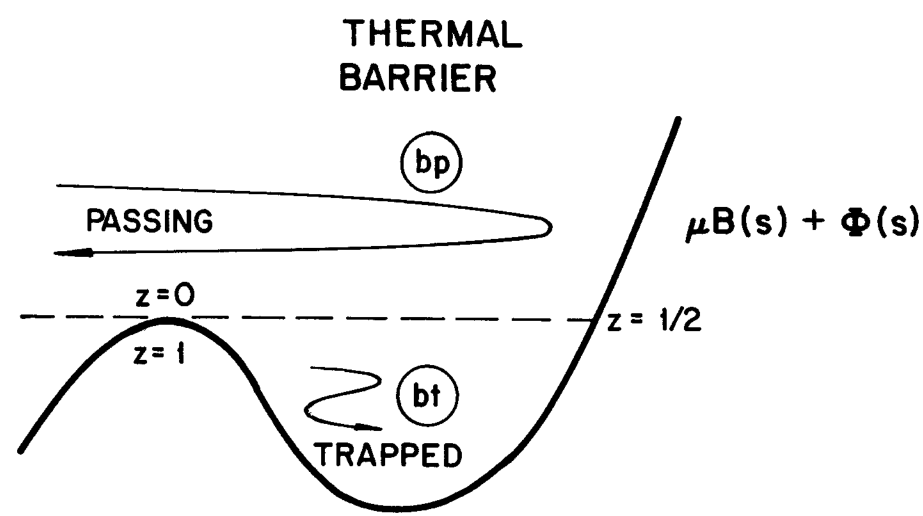

Fig. 4. Effective parallel potential $\mu B(s) + \Phi(s)$ in the thermal barrier. The regions of phase space bp and bt correspond to the barrier passing and barrier trapped particles, respectively. The variation of the axial coordinate, z, as defined in the text, is indicated.

This mixed boundary value problem can be reduced to a Wiener-Hopf equation (equivalent to Eqs. (17)) of the form

$$F^+(k) - Y^+(k) = -V(k)F^-(k) \ , \tag{19}$$

which is defined on the strip $\tau\varepsilon(0,\sqrt{\gamma}\ )$, $\sigma\varepsilon(-\infty,\infty)$ of the complex k-plane; the function

$$Y^+(k) = \frac{if_0}{k} \exp[-(k^2 + \gamma)] \ , \tag{20}$$

is analytic for $\tau > 0$; and

$$V(k) = 1 - \exp[-(k^2 + \gamma)] \ , \tag{21}$$

is an entire function with a denumerable infinity of simple zeros at

$$k^{\pm}_{mn} = [\gamma^2 + 4\pi^2 n^2]^{1/4} \exp[(\pm \frac{\pi}{2} + \phi_{mn})i] \ , \tag{22}$$

where $\phi_{mn} = \big((-1)^m/2\big) \arctan \big(2\pi n/\gamma\big)$, $m = 0,\ 1$, and $n\varepsilon[0,\infty]$. The locus of these zeros in the complex k-plane is indicated in Fig. 5 for $\tau > 0$. We can write

$$\mathrm{Im} K^{\pm}_{mn} = \tau^{\pm}_n = \pm\sqrt{\gamma}\ p_n \ , \tag{23}$$

where $(2p_n^2 - 1)^2 = 1 + 4\pi^2 n^2/\gamma^2$. We note that the change in the argument of the function V(k) when passing along some horizontal path in the k-plane is

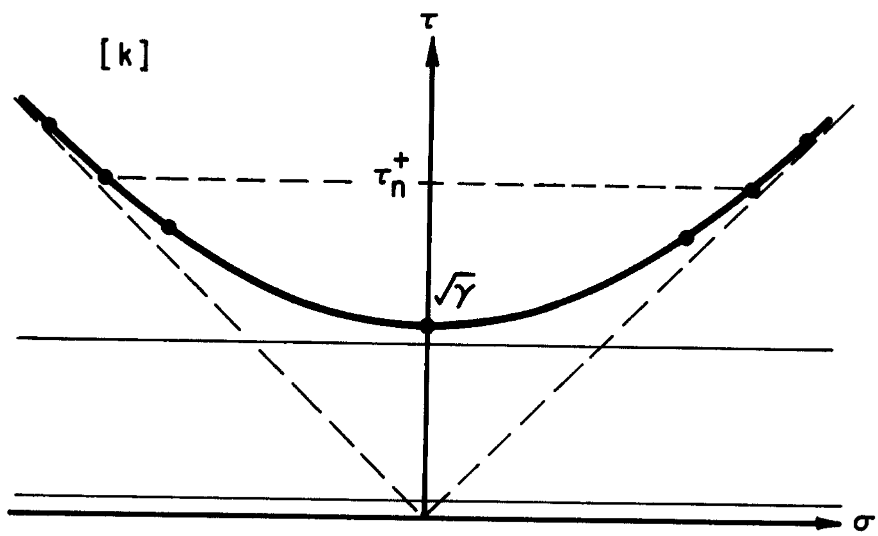

Fig. 5. Sketch of the complex k-plane ($k = \sigma + i\tau$) representing the locus of the zeros of $V(k) = 1 - \exp[-(k^2 + \gamma)]$ with $\tau > 0$. Indicated are also the strip of definition of Eq. (19), $\tau \varepsilon (0,\sqrt{\gamma})$, $\sigma \varepsilon (-\infty,\infty)$ and the contours of integration for the functions $\psi^{\pm}(k)$ given in Eq. (31) of the text.

$$\Delta \arg V(k) \Big|_{-\infty + i\tau}^{\infty + i\tau} = \mp 2\pi(n + 1) , \tag{24}$$

when $\tau \varepsilon (\tau_n^{\pm}, \tau_{n+1}^{\pm})$ with $n \varepsilon [o,\infty]$ and

$$\Delta \arg V(k) \Big|_{-\infty + i\tau}^{\infty + i\tau} = 0 , \tag{25}$$

when $0 < |\tau| < |\tau_0|$. However, when $\gamma \to 0$, we obtain

$$\lim_{\gamma \to 0} \tau_n^{\pm} = \pm\sqrt{\pi n} , \tag{26}$$

and Eq. (24) applies for the whole k-plane. Therefore when $\gamma \to 0$, Eq. (25) is never satisfied. Then, the function V(k) is analytic and non-zero in the strip $\tau \varepsilon (0,\sqrt{\gamma}\,)$ and $V(k) \to +1$ as $\sigma \to \pm\infty$ in the strip. In conclusion, Eq. (21) defines a function such that it is possible to select a branch of ln V(k) which does not vary along the strip $\tau \varepsilon (0,\sqrt{\gamma}\,)$. The nature of V(k) as well as the fact that the periodicity conditions ($x < 0$) are imposed on the decaying part of the boundary layer distribution function are the main differences of this problem from that of the boundary layer analysis in a simple mirror[5] or equivalent situations.[7,8]

The functions

$$\begin{aligned} F^+ &= F^+(k,1) = \int_0^{\infty} \exp(ikx)\; f(x,1)\; dx , \\ F^- &= F^-(k,1) = \int_{-\infty}^{0} \exp(ikx)\; f(x,1)\; dx , \end{aligned} \tag{27}$$

are analytic for $\tau > 0$ and $\tau < \sqrt{\gamma}$, respectively. In the strip $\tau \varepsilon (0, \sqrt{\gamma})$

$$F(k) = F^{+}(k) + F^{-}(k) = \int_{-\infty}^{\infty} \exp(ikx)\, f(x)\, dx \, . \tag{28}$$

Now, we need to find $V^{+}(k)$ analytic and non-zero for $\tau > 0$ and $V^{-}(k)$ analytic and non-zero for $\tau < \sqrt{\gamma}$, such that $V(k) = V^{-}(k)/V^{+}(k)$ [9,10] and for that we consider the branch of

$$\psi(k) = \ln \{1 - \exp[-(k^2 + \gamma)]\} \, , \tag{29}$$

which is analytic and satisfies

$$\psi(k) \xrightarrow[\sigma \to \pm\infty]{} O(e^{-k^2}) \, , \tag{30}$$

in the strip $\tau \varepsilon (0, \sqrt{\gamma}\,)$. Then, the function $\psi(k)$ can be separated into the difference form

$$\psi(k) = \psi^{+}(k) - \psi^{-}(k) = \frac{1}{2\pi i} \int_{-\infty+0^{+}}^{\infty+0^{+}} \frac{\psi(z)\, dz}{z - k}$$

$$- \frac{1}{2\pi i} \int_{-\infty+\sqrt{\gamma}^{\,-}}^{\infty+\sqrt{\gamma}^{\,-}} \frac{\psi(z)\, dz}{z - k} \, , \tag{31}$$

where $0^{+} = +\varepsilon$ (ε = positive infinitesimal) and $\sqrt{\gamma}^{\,-} = \sqrt{\gamma} - \varepsilon$. From the convergence of the integrals for $\psi^{+}(k)$ and $\psi^{-}(k)$ it follows that both are bounded for large k in their respective regions of analyticity. Hence, using Eqs. (29) and (31) we obtain

$$V^{\pm}(k) = \exp[-\psi^{\pm}(k)] \ , \tag{32}$$

so that Eq. (19) takes the form

$$V^{+}(F^{+} - Y^{+}) = -V^{-}F^{-} \ . \tag{33}$$

Next, we define a function J(k) in the strip $\tau \varepsilon (o, \sqrt{\gamma}\,)$ as

$$J(k) = V^{+}(F^{+} - Y^{+}) = -V^{-}F^{-} \ , \tag{34}$$

where the second part of the equation is defined and is analytic for $\tau > 0$ and the third part is defined and is analytic for $\tau < \sqrt{\gamma}$. Therefore, by analytic continuation we can define J(k) over the whole k-plane and J(k) will be an entire function whose order may be determined by the asymptotic form of Eq. (34). We can show by integration by parts that $F^{-}(k) \sim \mathcal{O}(k^{-1})$ as $|k| \to \infty$, $\tau < 0$ and also

$$\psi^{-}(k) = \frac{1}{2\pi i} \int_{-\infty+\sqrt{\gamma}^{-}}^{\infty+\sqrt{\gamma}^{-}} \frac{\ln\{1 - \exp[-(z^2 + \gamma)]\}}{z - k}\, dz \xrightarrow[\tau<0]{|k|\to\infty} - \frac{\phi[\exp(-\gamma), \frac{3}{2}]}{2\sqrt{\pi}\, k}\, i + \mathcal{O}(k^{-3}) \ , \tag{35}$$

with $\phi(z,s) = z\zeta(s,1,z)$ where $\zeta(s,\alpha,z)$ is the Lerch zeta function.[11,12] Since $|V^{-}(k)| \sim |k|^{0}$ as $|k| \to \infty$ in $\tau < \sqrt{\gamma}$, and $V^{-}(k)F^{-}(k) \to 0$ for $|k| \to \infty$ Liouville's theorem gives $J(k) \equiv 0$.

Using the infinite product representation of the entire function $V(k)$[13,14] we see that $V^+(k)$ has no zeros in the upper-half plane whereas $V^-(k)$ has a denumerable infinity of zeros at $k = k^+_{mn}$ (Eq. 22). Then, we obtain

$$F^+(k) = Y^+(k) \ ,$$

$$F^-(k) = \sum_{m,n} A_{mn}\delta(k - k^+_{mn}) \ , \tag{36}$$

where $F^-(k)$ can be different from zero only at the simple zeros $k = k^+_{mn}$. The arbitrary constants $\{A_{mn}\}$ might be determined by continuity of the solution (and its derivatives) at the origin of the boundary layer. The solution to the problem is obtained by doing the Fourier inversion of the results given in Eq. (36)

$$f(x) = \frac{if_0}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} dk\, \frac{\exp[-(ikx + k^2 + \gamma)]}{k} \ , \qquad x \geq 0 \ ,$$

$$f(x) = \sum_n \frac{A_{mn}}{2\pi}\int_{-\infty+i\tau}^{\infty+i\tau} dk\, \exp(-ikx)\ \delta(k - k^+_{mn}) \ , \qquad x \leq 0 \ , \tag{37}$$

where $\tau\varepsilon(0,\sqrt{\gamma}\ )$. The expression given for the solution in the range $x \leq 0$ is purely formal and it joins the $x \geq 0$ result with the tail of the bulk distribution function as $x \to -\infty$.

In the origin of the boundary layer we find

$$f(x=0) = \frac{if_0 \exp(-\gamma)}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{\exp(-k^2)}{k} ,$$

(38)

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \frac{f_0 \exp(-\gamma)}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} dk \exp(-k^2) ,$$

where $\tau \varepsilon (0,\sqrt{\gamma})$. Integrating Eqs. (38) we obtain

$$f(x=0) = \frac{f_0 \exp(-\gamma)}{2} ,$$

(39)

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \frac{f_0 \exp(-\gamma)}{2\sqrt{\pi}} .$$

Using Eqs. (39) and integrating Eq. (37) for $x \geqslant 0$ we obtain

$$f(x) = f_0 \exp(-\gamma)\left[1 - \frac{1}{2} \operatorname{erfc} \left(\frac{x}{2}\right)\right] , \qquad x \geqslant 0 . \tag{40}$$

We can recover the z dependence from the solution to the Fourier transform of Eq. (18):

$$F(k,z) = F(k,1) \exp\left[-(k^2 + \gamma)(z - 1)\right] . \tag{41}$$

Hence, we obtain

(42)

$$f(x \geqslant 0,z) = \frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} dk\left[F^+(k) + F^-(k)\right]\exp\left\{-\left[ikx + (k^2 + \gamma)(z - 1)\right]\right\} ,$$

where $\tau \varepsilon (0,\sqrt{\gamma})$. The right side of Eq. (42) can be evaluated by deforming the integration path in the lower-half of the complex k-plane. There is no contribution from $F^{-}(k)$ which is analytic in the lower-half plane. Therefore, at $x = 0$

$$f(x=0,z) = \frac{if_0 \exp(-\gamma z)}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} dk \frac{\exp(-zk^2)}{k} ,$$

$$\left.\frac{\partial f(x,z)}{\partial z}\right|_{x=0} = \frac{f_0 \exp(-\gamma z)}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} dk \exp(-zk^2) , \tag{43}$$

with $\tau\varepsilon(0,\sqrt{\gamma})$; and integrating we obtain,

$$f(x=0,z) = \frac{f_0 \exp(-\gamma z)}{2} ,$$

$$\left.\frac{\partial f(x,z)}{\partial x}\right|_{x=0} = \frac{f_0 \exp(-\gamma z)}{2\sqrt{\pi z}} . \tag{44}$$

Hence, after integration of Eq. (42) and using Eqs. (44) we obtain

$$f(x,z) = f_0 \exp(-\gamma z) \left[1 - \frac{1}{2} \operatorname{erfc} \left(\frac{x}{2\sqrt{z}}\right)\right] , \qquad x \geqslant 0 . \tag{45}$$

This solution satisfies the equation and conditions given in Eqs. (18). Near the origin the solution (45) behaves like

$$f(x,z) \xrightarrow[x\to 0]{} \frac{f_0 \exp(-\gamma z)}{2\sqrt{\pi z}} \left(x + \sqrt{\pi z}\right) . \tag{46}$$

A sketch of how the distribution function evolves along field lines is given in Fig. 6. Small angle collisions smooth out the distribution function

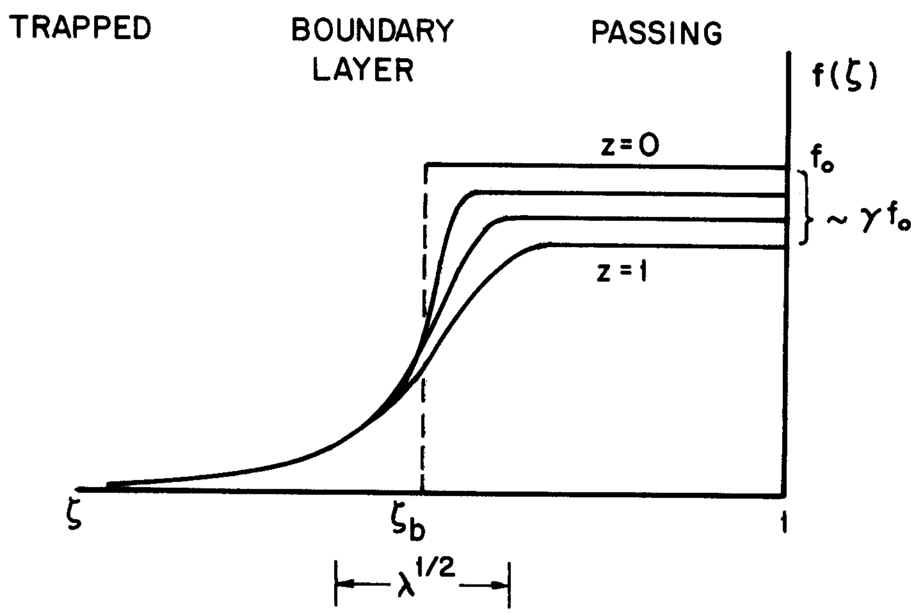

Fig. 6. Sketch of the variation of the barrier distribution function along field lines for strong pumping ($g \sim 1$). $z = 0$ corresponds to the barrier peak for particles entering the barrier from the central cell and $z = 1$ corresponds to the barrier peak for particles leaving the barrier to the central cell. The variation in the isotropic level, $\Delta f_0/f_0 \sim \gamma$, is indicated. ($\gamma \simeq \nu_{cx}/\omega_b$ and $\lambda = \nu_s/\omega_b$).

near the boundary contour in velocity space. In a bulk analysis the assumed boundary condition is $f(x=0,z) \equiv f_0$ and this maintains a higher distribution function level at the passing region of the thermal barrier than there should be. As a result of the boundary layer analysis the distribution function at the boundary contour is lower than f_0 and the asymptotic level of the passing distribution function is lowered because of the proper matching at the boundary layer. We note that this effect is different from what happens in a simple mirror where a higher asymptotic level is obtained as a consequence of boundary layer effects.(5,7,8)

4. ION TRAPPING RATE

The time rate of change of the barrier particle density due to the net flow across the boundary contour in velocity space is

$$j_t = - \frac{\int d\psi \int \frac{ds}{B} \int d^3v\ C(f)}{\int d\psi \int \frac{ds}{B}} , \tag{47}$$

where $d\psi$ is the differential magnetic flux; ds is the field line differential length; $\int d\psi \int ds/B$ = barrier cell volume enclosed by the last flux surface containing plasma; and the velocity space integration extends from $x = 0$ to ∞ and $v \geqslant v_0 = (2\Phi_b/M)^{1/2}$. Assuming that the conditions are sufficiently uniform over the cross-section and length of the barrier so that we can refer the calculation to the cell as a whole, we have

$$j_t = - \int_0^1 dz \int d^3v\ C[f(v,\zeta,z)] , \tag{48}$$

where $d^3v = 2\pi v^2\ dv d\zeta$. Since z passes over a full bounce of the barrier, for each z-point in real space the velocity space integral extends over $v_\parallel > 0$ for

$z\varepsilon[0,1/2]$ and $v_\parallel < 0$ for $z\varepsilon[1/2,1]$. Then, substituting the collision operator from Eq. (18) into Eq. (48) we obtain

$$j_t = -\int_0^1 dz \int_{v_0}^{\infty} 2\pi v^2 \, dv \, \omega_b \sqrt{D\lambda} \left. \frac{\partial f(v,x,z)}{\partial x} \right|_{x=0}^{x=\infty} . \tag{49}$$

Before proceeding with the integration we will express the isotropic part of the passing distribution function

$$f_o \simeq n_c \frac{\exp(-v^2/v_t^2)}{\pi^{3/2} v_t^3} , \tag{50}$$

(where the central cell potential is taken as zero and $v_t = (2T_{ic}/M)^{1/2}$) in terms of the barrier density at the midplane. By the continuity of the passing distribution function for particles with $v_\parallel > 0$ between the central cell midplane and the barrier at the barrier peak we can write

$$f_o \simeq \tilde{n}A \frac{e^{-v^2/v_t^2}}{\pi^{3/2} v_t^3} , \tag{51}$$

where $\tilde{n}$ = density of particles at the barrier peak moving with $v_\parallel > 0$ into the barrier and

$$A \simeq 2R_b \left(\pi \frac{\Phi_b}{T_{ic}}\right)^{1/2} \exp\left(\frac{\Phi_b}{T_{ic}}\right) . \tag{52}$$

Now we can obtain the barrier density profile along field lines from

$$n_b(z) = \int d^3v \, f(v,\zeta,z) , \tag{53}$$

where the integral extends over the part of velocity space with $v_\parallel \geq 0$ for $z\varepsilon[0,1/2]$ and $v_\parallel \leq 0$ for $z\varepsilon[1/2,1]$. Using Eq. (45) we may write Eq. (53) approximately as

$$n_b(z) \simeq \int d^3v\ f_o(v)\ \exp[-\gamma(v)z]\ . \tag{54}$$

The density along field lines is

$$n_b(s) = [n_b(z) + n_b(1 - z)]_{z=s/2L_b}\ , \tag{55}$$

and for $\gamma z < 1 << \Phi_b/T_{ic}$ we obtain

$$n_b(s) \simeq \tilde{n}\{\exp(\frac{-\gamma_o s}{2L_b}) + \exp[-\gamma_o(1 - \frac{s}{2L_b})]\}\ , \tag{56}$$

with $\gamma_o = \gamma(v_o)$. The expression (56) is the result of integrating (54) using Eq. (51). From Eq. (56) we find

$$\tilde{n} = \frac{n_{bo}}{2\ \exp(-\frac{\gamma_o}{2})\ \cosh(\frac{\gamma_o}{4})}\ , \tag{57}$$

where $n_{bo} = n_b(s=L_b/2)$ = density at the bottom of the barrier well. The barrier density profile along field lines may then be written as

$$n_b(s) \simeq n_{bo}\ \{\frac{\exp(-\gamma_o s/2L_b) + \exp[-\gamma_o(1 - s/2L_b)]}{2\ \exp(-\gamma_o/2)\ \cosh(\gamma_o/4)}\}\ . \tag{58}$$

Next we use Eqs. (18), (44) and (51) to evaluate Eq. (49) obtaining (for $\gamma z < 1 << \Phi_b/T_{ic}$),

$$j_t = \psi(\nu_{s\ eff}\omega_o)^{1/2}\ n_{bo}\ , \tag{59}$$

where j_t = ion trapping rate in the thermal barrier for strong pumping $(g \sim 1)$;

$$\psi = \frac{\exp(\gamma_o/2)\ \text{erf}\ \sqrt{\gamma_o}}{4\sqrt{\gamma_o}\ \cosh(\gamma_o/4)}\ , \tag{60}$$

is a factor that takes into account the variation of the barrier distribution function along field lines; $\gamma_o = \nu_{cx}/\omega_o$; $\omega_o = \omega_b(v_o) = v_o\zeta_b/2L_b$ is the bounce frequency for boundary layer particles; $v_o = (2\Phi_b/M)^{1/2}$; L_b = barrier length;

$$\nu_{s\ eff} = \nu_s(v_o)\ 2R_b\left(\frac{\Phi_b}{T_p}\right) \simeq \frac{\nu_s(v_o)}{(2\theta_b/\pi)^2}\ , \tag{61}$$

$(\theta_b = \arccos \zeta_b)$ is an effective collision frequency for particles to pitch-angle scatter out of the passing barrier region; $\nu_s(v_o) = \sqrt{2}\ \pi e^4\ \ln \Lambda\ n_{bo}/M^{1/2}\ \Phi_b^{3/2}$; $\theta_b \simeq (T_{ic}/R_b\Phi_b)^{1/2}$; and n_{bo} is the density at the bottom of the barrier. Therefore, in the case of strong pumping in the barrier $(g \sim 1)$, the barrier particles are being trapped at a rate

$$\nu_t = \psi(\nu_{s\ eff}\ \omega_o)^{1/2}\ . \tag{62}$$

This frequency takes into account the frequency for particles to scatter over the passing region of velocity space, the trapping due to the finite steepening of the distribution function at the boundary contour of velocity space, and the corrections due to the variation of the distribution function along field lines. In cases of interest $(\gamma_o < 1)$,

$$\psi \simeq \frac{1}{2\sqrt{\pi}} \left(1 + \frac{\gamma_0}{6}\right) . \tag{63}$$

For strong pumping situations ($g \sim 1$) the ion trapping rate is given by Eq. (59) and this expression sets a limit for the result given in Eq. (1) which is valid for a less strong pumping $g \gtrsim 2$.[2]

5. CONCLUSION

A boundary layer analysis of the particle distribution function in a tandem mirror with thermal barriers near the boundary contours on velocity space has been presented in order to obtain an expression for the ion trapping rate in a thermal barrier in the limit of strong pumping (filling ratios $g \sim 1$). This result sets an upper bound to the value that the barrier ion trapping rate as given by a classical analysis[2] can attain when the ion pumping rate is high enough to almost totally remove trapped particles from the thermal barrier. The expression obtained in Eq. (59) complements the result of Ref. 2 (as given in Eqs. (1) and (2) of this work) which is valid for filling ratios $g \gtrsim 2$.

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