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# VARIATIONAL CALCULATION OF THE TRAPPING RATE IN THERMAL BARRIERS

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Abstract

A variational calculation of the trapping rate and trapped ion density in thermal barriers is presented. The effects of diffusion in energy as well as pitch angle scattering are retained. The variational formulation uses the actual trapped-passing boundary in velocity space. The boundary condition is that the trapped ion distribution function matches the passing ion distribution function, which is taken to be a Maxwellian, on the boundary. The results compare well with two-dimensional Fokker-Planck code calculations by Futch and LoDestro.

## 1. Introduction

The thermal barrier[1] is a recently introduced concept in tandem mirrors. It is a region of depressed potential between the central cell and the plug; the thermal barrier reduces the rate of energy transfer[2] between the plug and central cell electrons and thereby allows one to heat the plug electrons without excessive heating of the central cell electrons.

A fundamental problem is the trapping of ions in thermal barriers. Trapped ions increase the electron density in the thermal barrier (through quasi-neutrality) and thereby reduce the magnitude of the potential depression; this reduces the effectiveness of the thermal barrier. In order to remove the trapped ions, a "pumping" mechanism is required. One such method for pumping the thermal barrier is to inject a neutral beam at the appropriate energy and angle.[3] Charge exchange between the trapped ions and the injected neutral atoms removes trapped ions and replaces them with "passing" (i.e., they can pass back into the central cell) ions. Alternative techniques using drift orbits,[4,5] or induced radial losses of the trapped ions by time-varying fields[6] have also been proposed. In a previous paper[7] we attempted to set up a model for the drift pumping case. In this paper we treat the neutral beam pumping case.

Numerical calculations of the trapping rate in thermal barriers have been done by Futch and LoDestro[8] using Fokker-Planck codes. Recently Carrera and Callen[9] have given an analytical calculation of the trapping rate using a pitch-angle scattering collision operator; diffusion in energy was neglected. In this paper we present an alternative analytical calculation of the trapping rate; it is based upon a variational method,[10] extended to the non-Maxwellian field particle case. It allows a better treatment of the boundary condition and includes the effect of energy diffusion. The results are generally in good agreement with the numerical results in both the high and low barrier mirror ratio cases.

## 2. Formulation of the Problem

We model the magnetic field and the potential in the thermal barrier as a square well, as shown in Fig. 1. The boundary in velocity space between ions which are trapped in the thermal barrier and ions which can pass back into the central cell (i.e., "passing" ions) is shown in Fig. 2. This boundary is given by

$$v_{\parallel}^2 + v_{\perp}^2(1 - R_b) - v_{\phi b}^2 = 0 \tag{2.1}$$

where

$$R_b = B_{mb}/B_b \quad , \quad v_{\phi b}^2 = \frac{2\phi_b}{m} \tag{2.2}$$

and $v_{\parallel}$, $v_{\perp}$ are measured at the bottom of the barrier. Here, potential and temperature are measured in units of energy, and m is the ion mass.

The kinetic equation for the trapped ion distribution function, f, is

$$-\vec{\nabla} \cdot \vec{I} - \nu f = 0 \tag{2.3}$$

where $\vec{I}$ is the diffusion current in velocity space and charge exchange pumping is modeled as an "absorption" term with a constant coefficient $\nu$, which has units of $sec^{-1}$. Note that all $\vec{\nabla}$ operators are in velocity space in this report. The diffusion current $\vec{I}$ can be written as

$$\vec{I} = \vec{A}f - \overleftrightarrow{D} \cdot \vec{\nabla}f \ . \tag{2.4}$$

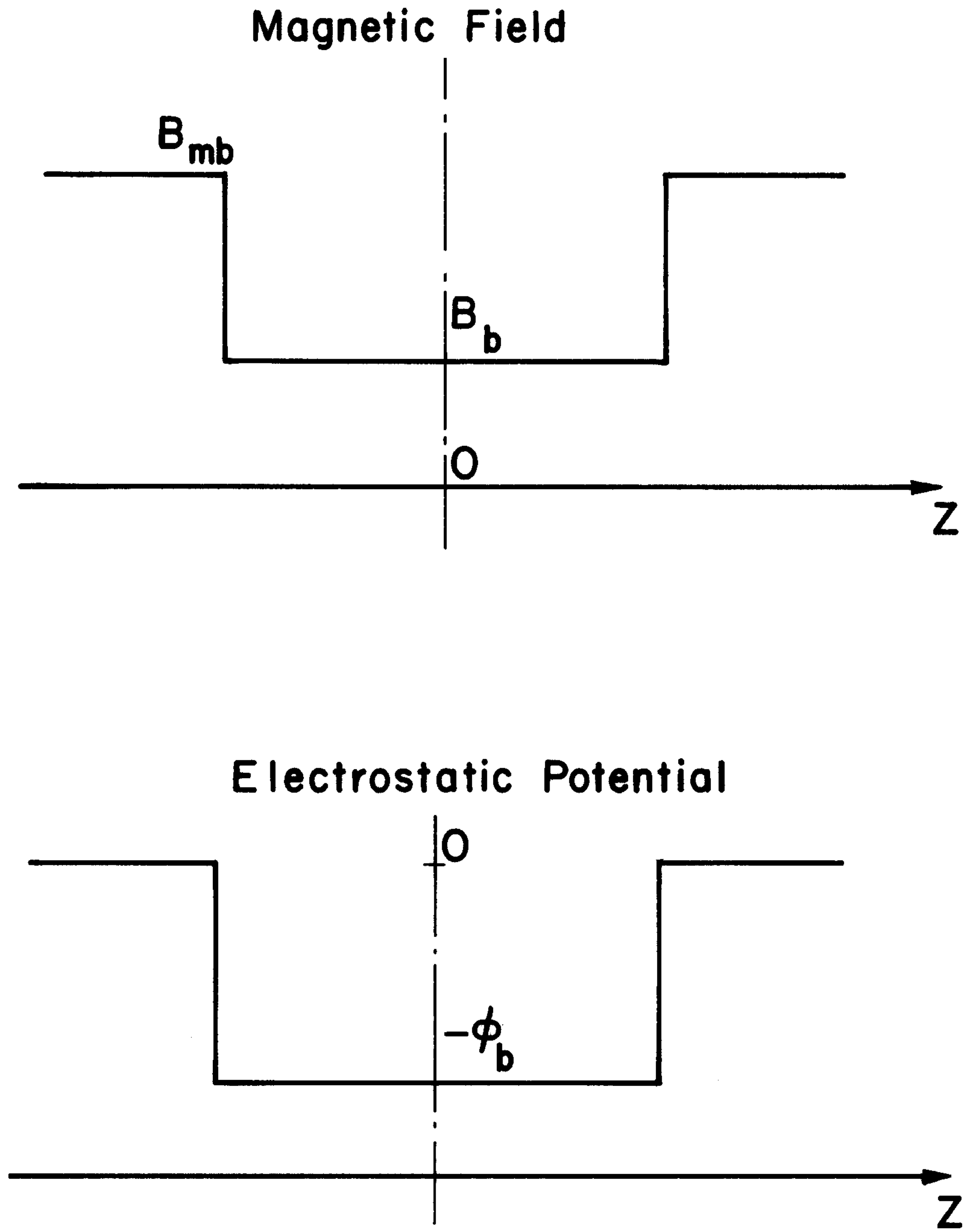


Fig. 1. Square well model for the thermal barrier.

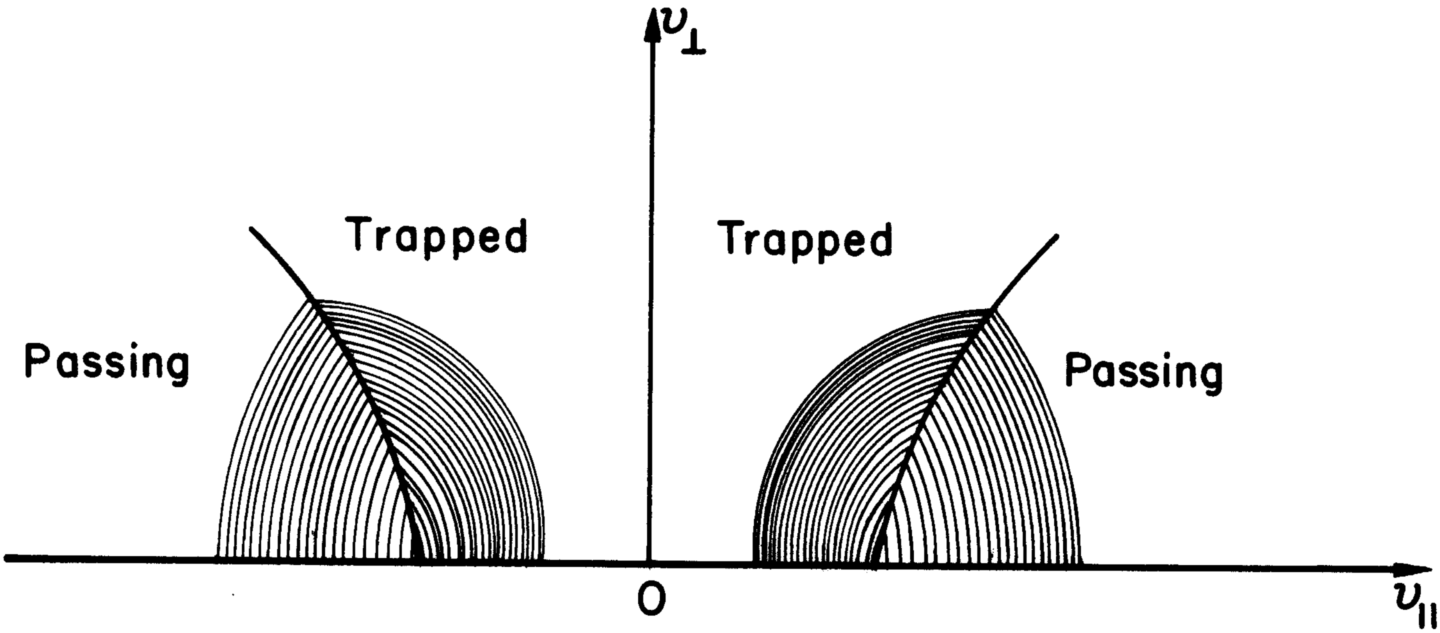


Fig. 2. Trapped and passing particles in velocity space.

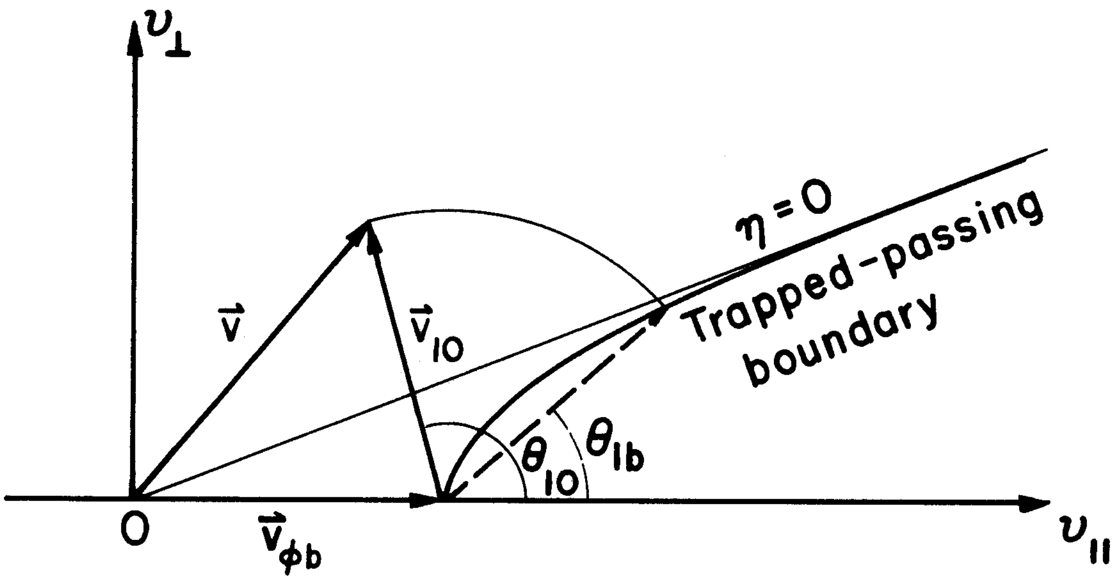


Fig. 3. The center-shifted coordinate system $(v_{10}, \theta_{10}, \psi)$ in velocity space.

$\vec{A}$ and $\overleftrightarrow{D}$ are the dynamical frictional and diffusion coefficients, respectively:

$$\vec{A}(\vec{v}) = - \frac{4\pi e^4 \ln \Lambda}{m^2} \int f(\vec{v}') \frac{(\vec{v} - \vec{v}')}{|\vec{v} - \vec{v}'|^3} d^3v' \tag{2.5}$$

$$\overleftrightarrow{D}(\vec{v}) = \frac{2\pi e^4 \ln \Lambda}{m^2} \vec{\nabla} \vec{\nabla} \int f(\vec{v}')|\vec{v} - \vec{v}'| d^3v' \ . \tag{2.6}$$

For simplicity, we consider only ion-ion collisions. The boundary condition to be satisfied by f is that it match the distribution function of the passing ions on the trapped-passing boundary in Fig. 2. The passing ion distribution function is taken to be a Maxwellian under the assumption that the central cell is long compared with the thermal barrier. The trapped ion distribution function is symmetric in $v_\parallel$ and has its dominant part in the neighborhood of the trapped-passing boundary.

We assume a separable form for the trapped ion distribution function $f(\vec{v})$,

$$f(\vec{v}) = R(v_{10})Z(\eta) \tag{2.7}$$

where $\vec{v}_{10}$ is the vector from the vertex of the boundary hyperbola to a field point $\vec{v}$ (see Fig. 3)

$$\vec{v}_{10} = \vec{v} - \vec{v}_{\phi b} \tag{2.8}$$

and $\eta$ is a second velocity space variable to be determined later. The surface $\eta = 0$ is defined to be the trapped-passing boundary. We define $Z(\eta = 0) = 1$. The boundary condition on f is that it matches the Maxwellian distribution of the passing particles; this becomes a constraint on the choice of R, which is taken to be only a function of $v_{10}$, the magnitude of $\vec{v}_{10}$. On the boundary, we need

$$R = Ge^{-mv^2/2T_p} e^{-\phi_b/T_p} \tag{2.9}$$

where G is a normalization constant determined by the passing ion density, $n_p$, in the thermal barrier,

$$G = n_p\left(\frac{m}{2\pi T_p}\right)^{3/2} H \tag{2.10}$$

where

$$H^{-1} = e^{\phi_b/T_p} \operatorname{erfc}\left(\sqrt{\frac{\phi_b}{T_p}}\right) - \sqrt{1 - \frac{1}{R_b}} \exp\left(\frac{\phi_b}{T_p} \frac{R_b}{(R_b - 1)}\right) \operatorname{erfc}\left(\sqrt{\frac{\phi_b}{T_p} \frac{R_b}{(R_b - 1)}}\right) . \tag{2.11}$$

For large $\phi_b$, $R_b$,

$$H \approx R_b \sqrt{\frac{\pi\phi_b}{T_p}} .$$

The desired expression for R is obtained by letting

$$v^2 = (\vec{v}_{10} + \vec{v}_{\phi b})^2 = v_{10}^2 + v_{\phi b}^2 + 2v_{10}v_{\phi b}\cos\theta_{10}$$

and replacing $\theta_{10}$ by $\theta_{1b}$, the value of $\theta_{10}$ on the boundary. One gets

$$R(v_{10}) = Ge^{-mv_{10}^2/2T_p} e^{-mv_{10}v_{\phi b}\cos\theta_{1b}/T_p} \tag{2.12}$$

where

$$\cos\theta_{1b} = \frac{1}{R_b}\left[-\frac{v_{\phi b}}{v_{10}} + \sqrt{\left(\frac{v_{\phi b}}{v_{10}}\right)^2 + R_b(R_b - 1)}\ \right] . \tag{2.13}$$

Substituting Eq. (2.7) and (2.4) into Eq. (2.3), we get

$$-\vec{\nabla} \cdot [Z(A\vec{R} - \overleftrightarrow{D}\cdot\vec{\nabla}R)] + \vec{\nabla}\cdot[R\overleftrightarrow{D}\cdot\vec{\nabla}Z] - \nu RZ = 0 \ , \tag{2.14}$$

which is an equation determining Z. In the expressions for $\vec{A}$ and $\overleftrightarrow{D}$ (Eq. (2.5) and (2.6)) we replace the distribution function f by $R(v_{10})$ and integrate only over the $v_{\parallel} > 0$ half (including the passing ions) of the phase space. The justification for this latter assumption is that the relative velocity is smaller for trapped particles having the same sign for $v_{\parallel}$ and hence this should be the dominant contribution. We renormalize the transport coefficients to the total ion density, $n_b$, in the barrier. Hence, replacing f by $R_{10}$ in $\vec{A}$ and $\overleftrightarrow{D}$ affects only the isotropy of the field particle distribution function and not the total field particle density, $n_b$. Since $\vec{A}$ and $\overleftrightarrow{D}$ no longer depend on Z, Eq. (2.14) has been linearized by this assumption. This allows us to construct a variational principle from Eq. (2.14).

We define

$$\vec{t} = \vec{A}R - \overleftrightarrow{D}\cdot\vec{\nabla}R \ . \tag{2.15}$$

The vector $\vec{t}$ is, in general, non-zero since $R(v_{10})$ is not Maxwellian, even centered around $\vec{v}_{10} = 0$. Since $\vec{A}$ and $\overleftrightarrow{D}$ are isotropic about $\vec{v}_{10} = 0$, we can write

$$\vec{t} = t\vec{v}_{10}$$

so that Eq. (2.14) becomes

$$\vec{\nabla}\cdot(R\overleftrightarrow{D}\cdot\vec{\nabla}Z) - (\nu R + \vec{\nabla}\cdot(t\vec{v}_{10}))Z - t\vec{v}_{10}\cdot\vec{\nabla}Z = 0 \; . \qquad (2.16)$$

Near the boundary, we expect $\vec{v}_{10}\cdot\vec{\nabla}z$ to be small. Consequently, we neglect the last term in Eq. (2.16) in order to put it in self-adjoint (i.e., Sturm-Liouville) form.

Thus
$$\vec{\nabla}\cdot(R\overleftrightarrow{D}\cdot\vec{\nabla}Z) - (\nu R + \vec{\nabla}\cdot(t\vec{v}_{10}))Z = 0 \; . \qquad (2.17)$$

## 3. The Variational Form

We define

$$L(Z) = \int d^3v \; \{\vec{\nabla}Z\cdot R\overleftrightarrow{D}\cdot\vec{\nabla}Z + [\nu R + \vec{\nabla}\cdot(t\vec{v}_{10})]Z^2\} \; , \qquad (3.1)$$

where the integration is over the total trapped particle part of velocity space. Requiring that L(Z) be stationary with respect to variations in the function Z yields Eq. (2.17). Before discussing the physical meaning of L(Z), we first rewrite it in another form. We transform from the spherical coordinates $(v_{10},\theta_{10},\psi)$ centered about the vertex of the hyperbola to $(v_{10},\eta,\psi)$, where $\psi$ is the azimuthal angle in velocity space. Then L(Z) can be written as

$$L(Z) = \int d\eta \; \{P(\eta)(\frac{dZ}{d\eta})^2 + q(\eta)Z^2(\eta)\} \qquad (3.2)$$

where
$$P(\eta) = \int dv_{10}d\psi \; J(v_{10},\eta)\vec{\nabla}\eta\cdot R\overleftrightarrow{D}\cdot\vec{\nabla}\eta \qquad (3.3)$$

$$q(\eta) = q_0(\eta) + q_1(\eta) \qquad (3.4)$$

$$q_o(\eta) = \int dv_{10} \int d\psi \; J(v_{10},\eta) \; \nu R \tag{3.5}$$

$$q_1(\eta) = \int dv_{10} \int d\psi \; J(v_{10},\eta)\vec{\nabla}\cdot(t\vec{v}_{10}) \tag{3.6}$$

We also define
$$q_2(\eta) = \int dv_{10} \int d\psi \; J(v_{10},\eta)t\vec{v}_{10}\cdot\vec{\nabla}\eta \;, \tag{3.7}$$

which will be used later. Note that $q_2$ involves the term neglected in obtaining Eq. (2.17). Hence we expect $q_2$ to be small. Here $J(v_{10},\eta)$ is the Jacobian of the transformation and is given by

$$J(v_{10},\eta) = 2v_{10}^2 \sin\theta_{10} \left.\frac{\partial\theta_{10}}{\partial\eta}\right|_{v_{10},\psi} . \tag{3.8}$$

The factor of 2 comes from integrating over both $v_{\parallel} > 0$ and $v_{\parallel} < 0$.

The Euler-Lagrange equation which determines $Z(\eta)$ such that $L(Z)$ in Eq. (3.2) is stationary is

$$\frac{d}{d\eta}\left(P(\eta)\frac{dZ}{d\eta}\right) - q(\eta)Z(\eta) = 0 \; . \tag{3.9}$$

We have now obtained an ordinary differential equation from the partial differential equation (2.17). Substituting Eq. (3.9) into (3.2), we obtain the stationary value of $L(z)$

$$\left.L(Z)\right|_{\text{stationary}} = \int d\eta \; \left\{P(\eta)\left(\frac{dZ}{d\eta}\right)^2 + Z(\eta)\frac{d}{d\eta}\left[P(\eta)\frac{dZ}{d\eta}\right]\right\}$$

$$= \int d\eta \; \frac{d}{d\eta}\left\{P(\eta)\frac{dZ}{d\eta}Z(\eta)\right\}$$

$$= \left[P(\eta) \frac{dZ}{d\eta} Z(\eta)\right]\Big|_{\eta=0}^{\eta=\eta_1}$$

where $\eta_1$ is the value of $\eta$ on the midplane ($v_\parallel = 0$). In a well-pumped thermal barrier, there are few trapped particles at the midplane ($v_\parallel = 0$) so that we can write

$$L(Z)\Big|_{stationary} \simeq -\left[P(Z) \frac{dZ}{d\eta} Z(\eta)\right]\Big|_{\eta=0} . \tag{3.10}$$

The trapping current per unit volume can be written as

$$J_{trap} = \int d^3v \; \nu f \tag{3.11}$$

since, in equilibrium, the trapping current equals the pumping current. Using (2.3) and (2.4), this becomes

$$J_{trap} = -\int d^3v \; \vec{\nabla}\cdot(\vec{A}f - \overleftrightarrow{D}\cdot\vec{\nabla}f) .$$

We now use (2.7), (2.15), and (2.16) to write this as

$$J_{trap} = -\int d^3v \; \vec{\nabla}\cdot(Zt\vec{v}_{10} - R\overleftrightarrow{D}\cdot\vec{\nabla}Z) \tag{3.12}$$

$$= -\int dv_{10} d\psi \; \frac{\partial\vec{v}}{\partial v_{10}} \times \frac{\partial\vec{v}}{\partial\psi} \cdot (Zt\vec{v}_{10} - R\overleftrightarrow{D}\cdot\vec{\nabla}Z)\Big|_{\eta=0}^{\eta=\eta_1}$$

$$= -\int dv_{10} d\psi \; J(v_{10},\eta)\vec{\nabla}\eta\cdot(Zt\vec{v}_{10} - R\overleftrightarrow{D}\cdot\vec{\nabla}Z)\Big|_{\eta=0}^{\eta=\eta_1}$$

$$= \int dv_{10} d\psi \, J(v_{10},\eta)[\vec{\nabla}\eta \cdot R\overleftrightarrow{D} \cdot \vec{\nabla}\eta \frac{dZ}{d\eta} - Zt\vec{v}_{10} \cdot \vec{\nabla}\eta]\Big|_{\eta=0}^{\eta=\eta_1}$$

$$= [P(\eta) \frac{dZ}{d\eta} - q_2(\eta)Z(\eta)]\Big|_{\eta=0}^{\eta=\eta_1}$$

$$\cong -[P(\eta) \frac{dZ}{d\eta} - q_2(\eta)Z(\eta)]\Big|_{\eta=0} .$$

Since $Z(\eta=0) = 1$, we find that

$$J_{trap} = L(Z)\Big|_{stationary} + q_2(\eta=0) . \tag{3.13}$$

Consequently, the stationary value of L is related to the trapping current. The second term $q_2(0)$ is usually small, since in Eq. (3.7) $\vec{v}_{10} \cdot \vec{\nabla}\eta \sim 0$ on the trapped-passing boundary.

## 4. The Trial Function

The property of L(Z) being stationary with respect to changes in Z allows us to obtain an estimate of the trapping rate which is second order in the error in the trial function. It is necessary, however, that the trial function satisfy the boundary condition Z = 1 on the trapped-passing boundary. In order to choose the appropriate trial function Z, we first consider the case obtained by letting $R(v_{10}) = Ge^{-mv_{10}^2/2T_p}$ in Eq. (2.17). Then from Eq. (2.15) t = 0 and Eq. (2.17) becomes

$$\frac{1}{v_{10}\sin\theta_{10}} \frac{\partial}{\partial\theta_{10}} \left(\frac{RD_\perp \sin\theta_{10}}{v_{10}} \frac{\partial Z}{\partial\theta_{10}}\right) - \nu RZ = 0 \tag{4.1}$$

where $D_\perp$ is the diffusion coefficient in the direction $\perp \vec{v}_{10}$. Equation (4.1) has the approximate solution

$$Z(\theta_{10}) = e^{-\eta} \tag{4.2}$$

where
$$\eta = \sqrt{\frac{\nu v_{10}^2}{D_\perp}} \left(1 - \frac{1}{2} (\cot \theta_{1b})(\theta_{10} - \theta_{1b})\right)(\theta_{10} - \theta_{1b}) \ . \tag{4.3}$$

This solution becomes exact as $\theta_{10} \to \theta_{1b}$.

This solution, obtained for a choice of R corresponding to a Maxwellian distribution centered about $\vec{v}_{10} = 0$, has the property of $\eta = 0$ and $Z = 1$ for $\theta_{10} = \theta_{1b}$, as required by the boundary condition. Hence, we choose the form in Eqs. (4.2) and (4.3) as our trial function to use in L(Z), but retain now the more exact expression for R in Eq. (2.12) and keep the t term in Eq. (2.17). The estimate for $J_{trap}$ is then obtained from the stationary value of L (see Eq. (3.13)). Because of the stationary property of L, we can estimate the trapping rate with better accuracy than that possessed by the trial function Z.

The diffusion coefficient $D_\perp$ in Eq. (4.3) is simplified further by taking the value for a Maxwellian distribution with effective temperature, $T_{eff}$.

$$D_\perp = \frac{2\pi e^4 \ \ln \Lambda}{m^2} \left(\frac{n_b}{2}\right) \frac{1}{v_{10}} \left[\phi(x) - \frac{1}{2x^2} \left(\phi(x) - x \frac{d\phi(x)}{dx}\right)\right] \tag{4.4}$$

where
$$\phi(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \ dt$$

and
$$x = \frac{v_{10}}{\sqrt{2T_{eff}/m}} \ . \tag{4.5}$$

In Eq. (4.4) $n_b$ is the total ion density in the barrier. We treat $T_{eff}$ as a variational parameter determined by requiring that L be stationary using the trial function Z given in Eqs. (4.2) and (4.3). The parameter $n_b$ in Eq. (4.4) is the total ion density (trapped plus passing) in the thermal barrier and is discussed in the next section.

The Euler-Lagrange equation (Eq. (3.9)) evaluated at $\eta = 0$ with

$$Z(\eta{=}0) = 1$$

$$\left.\frac{dZ}{d\eta}\right|_{\eta=0} = -1$$

$$\left.\frac{d^2Z}{d\eta^2}\right|_{\eta=0} = 1$$

gives

$$p(0) - \left.\frac{dp}{d\eta}\right|_{\eta=0} - q(0) = 0 \; . \tag{4.6}$$

The quantities $p(\eta)$ and $q(\eta)$ are given in Eqs. (3.3)-(3.7) and can be evaluated now that $\eta$ has been chosen. We choose $T_{eff}$ in Eq. (4.5) by requiring that the results $p(\eta)$ and $q(\eta)$ satisfy (4.6). This is equivalent to requiring that our trial function satisfy the Euler-Lagrange equation on the trapped-passing boundary.

In the evaluation of $p(\eta)$ and $q(\eta)$ the dynamical friction $\vec{A}$ and the diffusion coefficient $\vec{\vec{D}}$ are needed as well. For $\vec{A}$ and $\vec{\vec{D}}$ in these expressions, Eq. (2.5) and (2.6) with f replaced by $R(v_{10})$, as discussed earlier, are used. These can be rewritten as

$$D_{\parallel}^{o} = \frac{2\pi e^4}{m^2} \ln \Lambda \left(\frac{n_b}{2C_o v_{10}}\right) \frac{2}{3} \left[\frac{1}{v_{10}^2} \int_0^{v_{10}} v^2 R(v) 4\pi v^2 \, dv + v_{10} \int_{v_{10}}^{\infty} \frac{1}{v} R(v) 4\pi v^2 \, dv\right] \tag{4.7}$$

$$D_\perp^o = \frac{2\pi e^4 \ln \Lambda}{m^2} \left(\frac{n_b}{2C_o v_{10}}\right)\left[\int_0^{v_{10}} R(v)4\pi v^2 \, dv - \frac{1}{3v_{10}^2} \int_0^{v_{10}} v^2 R(v)4\pi v^2 \, dv \right.$$

$$\left. + \frac{2}{3} v_{10} \int_{v_{10}}^{\infty} \frac{1}{v} R(v)4\pi v^2 \, dv\right] . \tag{4.8}$$

Here $C_0$ is a normalization constant

$$C_o = \int_0^{\infty} R(v)4\pi v^2 \, dv .$$

Note that the total ion density, $n_b$, has been introduced explicitly in Eq. (4.7) and (4.8).

## 5. The Total Ion Density

There is one remaining complication in the total ion density, which we now consider. We integrate Eq. (2.3) over the trapped particle part of velocity space to get the trapping current,

$$J_{trap} = \nu n_t = \nu(n_b - n_p) \tag{5.1}$$

where $n_t$, $n_p$, and $n_b$ are the trapped, passing, and total ion density in the barrier. Generally speaking, $n_p$ is determined by the central cell parameters, and $\nu$ by the barrier pump neutral beams. Consequently Eq. (5.1) is a self-consistent relationship between the trapping current per unit volume, $J_{trap}$, and the ion density, $n_b$, both of which are determined by the collisional relaxation processes in the barrier. Upon combining Eq. (5.1) with (3.13), we have

$$\nu(n_b - n_p) = p(0) + q_2(0) . \tag{5.2}$$

In the expressions for $p(0)$ and $q_2(0)$, the passing ion density enters $R(v_{10})$, and the total ion density, $n_b$, enters through $\vec{A}$ and $\vec{D}$, and also through the $D_\perp$ in the expression for $\eta$. It is convenient to define

$$g_b = \frac{n_b}{n_p} \tag{5.3}$$

and renormalize $p(0)$ and $q_2(0)$ to be independent of $g_b$, i.e.

$$p(0) \rightarrow \hat{p}(0) = \frac{1}{n_p\sqrt{g_b n_p}}\, p(0)$$

$$q_2(0) \rightarrow \hat{q}_2(0) = \frac{1}{n_p(g_b n_p)}\, q_2(0) \ .$$

Then Eq. (5.2) becomes

$$\nu(g_b - 1) = \sqrt{g_b n_p}\ \hat{p}(0) + g_b n_p \hat{q}_2(0) \ , \tag{5.4}$$

which is a quadratic equation for $\sqrt{g_b}$.

We treat Eq. (4.6) in a similar way. We define

$$\hat{q}_o(0) = \frac{1}{n_p\sqrt{g_b n_p}}\, q_o(0)$$

$$\hat{q}_1 = \frac{1}{n_p(g_b n_p)^{3/2}}\, q_1(0)$$

$$\left.\frac{d\hat{p}}{d\eta}\right|_{\eta=0} = \frac{1}{n_p(g_b n_p)} \left.\frac{d\hat{p}}{d\eta}\right|_{\eta=0}$$

and rewrite (4.6) as

$$-g_b(n_p\hat{q}_1(0)) - \sqrt{g_b n_p}\left.\frac{d\hat{p}}{d\eta}\right|_{\eta=0} + (\hat{p}(0) - \hat{q}_0(0)) = 0 \ . \qquad (5.5)$$

Equations (5.4) and (5.5) are two simultaneous equations which determine the barrier parameter, $g_b$, and implicitly through the coefficients, the effective temperature, $T_{eff}$. The coefficients involve integrals over velocity space. They are complicated, but can be done numerically. The procedure for calculating $g_b$, and hence the trapping rate through Eq. (5.1) and (5.3) is completely formulated. In the next section we present some results and compare them with the numerical Fokker-Planck results of Futch and LoDestro, and with the analytical results of Carrera and Callen.

## 6. Results from the Variational Calculation

A numerical solution of Eq. (5.4) and (5.5) for the barrier parameter, $g_b$, the trapping rate and $T_{eff}$ has been obtained for the same input data used in the two-dimensional Fokker-Planck calculations of Futch and LoDestro.[8] The input data are the passing ion density, $n_p$, temperature of the passing ions, $T_p$, barrier mirror ratio, $R_b$, barrier potential, $\phi_b$, and the neutral beam pumping rate, $\nu$. For the Coulomb logarithm, the expression[8]

$$\ln \Lambda = 34.9 - \ln \sqrt{\frac{n_b}{E_{av}}}\ T_e$$

was used. The units are $cm^{-3}$ for $n_b$, and keV for the mean ion energy, $E_{av}$, and for the electron temperature, $T_e$. The mean ion energy is approximated by $T_p + \phi_b$.

For a preliminary calculation, the $\hat{q}_1$ and $\hat{q}_2$ terms in Eq. (5.4) and (5.5) were neglected since they are expected to yield only small corrections. Table I shows the results for the low barrier mirror ratio ($R_b$ = 2) case for various values of $n_p$, $T_p$, and $\nu$. Table II gives the corresponding values for the high barrier mirror ratio (R = 20) case. We have included for comparison the analytical results of Carrera and Callen. The variational results are in rather good agreement with the Fokker-Planck code results; the error in $g_b$ is generally less than 10%, except in the 8th case in Table II, where it is 16%.

In Tables I and II, we have chosen to make our comparison for a specified passing ion density, $n_p$, rather than for a specified total ion density $n_b$, which was the procedure used in Ref. 9. The former is a more sensitive test. To see this, write Eq. (5.1) in the form

$$J_{trap} = \nu n_b (1 - \frac{1}{g_b}) .$$

In the limit of large $g_b$, $J_{trap}$ is only weakly dependent on $g_b$ and approaches $\nu n_b$ (which are simply input data) as $g_b \to \infty$. Specifying $n_p$ as input data puts Eq. (5.1) in the form

$$J_{trap} = \nu n_p (g_b - 1)$$

which is more sensitive to the calculated value of $g_b$. One hopes to obtain values for $g_b$ in the range 2 to 4 in thermal barrier tandem mirror experiments.

A graphical representation of the results and their comparison is given in Fig. 4 and Fig. 5 for both the low and high mirror ratio cases.

TABLE I. Low Mirror Ratio Case

$R_b = 2$ , $\phi_b = 40$ keV , $T_e = 15$ keV

| Input Data | | | $g_b$ | | | $J_t$ ($10^{11}$ cm$^{-3}$-sec$^{-1}$) | | |
|---|---|---|---|---|---|---|---|---|
| $n_p$ ($10^{11}$cm$^{-3}$) | $T_p$ (keV) | $\nu$ (sec$^{-1}$) | Numerical* | Variational | Analytical** | Numerical* | Variational | Analytical** |
| 1.1 | 15.0 | 0.25 | 3.45 | 3.14 | 2.54 | 0.675 | 0.589 | 0.424 |
| 1.1 | 15.0 | 0.5 | 2.31 | 2.30 | 1.96 | 0.721 | 0.715 | 0.528 |
| 1.1 | 15.0 | 1.0 | 1.80 | 1.82 | 1.62 | 0.880 | 0.902 | 0.682 |
| 1.1 | 15.0 | 2.0 | 1.48 | 1.53 | 1.41 | 1.06 | 1.17 | 0.902 |
| 0.55 | 15.0 | 0.25 | 2.33 | 2.31 | 1.97 | 0.183 | 0.180 | 0.133 |
| 1.1 | 10.0 | 0.5 | 3.62 | 3.48 | 2.25 | 1.44 | 1.36 | 0.688 |
| 1.1 | 10.0 | 1.0 | 2.46 | 2.49 | 1.79 | 1.61 | 1.64 | 0.869 |
| 1.1 | 10.0 | 2.0 | 1.90 | 1.93 | 1.52 | 1.98 | 2.05 | 1.14 |
| 0.55 | 10.0 | 0.25 | 3.66 | 3.50 | 2.27 | 0.366 | 0.344 | 0.175 |
| 2.2 | 10.0 | 2.0 | 2.45 | 2.47 | 1.79 | 6.38 | 6.47 | 3.48 |

* Ref. 8.

** Ref. 9.

TABLE II. High Mirror Ratio Case

$R_b = 20$ , $\phi_b = 1.0$ keV , $T_e = 0.4$ keV

| Input Data | | | $g_b$ | | | $J_t$ ($10^{15}$ cm$^{-3}$-sec$^{-1}$) | | |
|---|---|---|---|---|---|---|---|---|
| $n_p$ ($10^{11}$cm$^{-3}$) | $T_p$ (keV) | $\nu$ ($10^3$sec$^{-1}$) | Numerical* | Variational | Analytical** | Numerical* | Variational | Analytical** |
| 8.9 | 1/3 | 2.0 | 4.76 | 4.30 | 3.90 | 6.69 | 5.87 | 5.16 |
| 8.9 | 1/3 | 4.0 | 2.90 | 2.94 | 2.73 | 6.79 | 6.91 | 6.16 |
| 8.9 | 1/3 | 8.0 | 2.12 | 2.19 | 2.07 | 7.97 | 8.47 | 7.62 |
| 8.9 | 0.2 | 4.0 | 5.0 | 5.33 | 3.51 | 14.2 | 14.9 | 8.94 |
| 8.9 | 0.4 | 4.0 | 2.54 | 2.52 | 2.52 | 5.48 | 5.41 | 5.41 |
| 8.9 | 0.5 | 4.0 | 2.22 | 2.15 | 2.30 | 4.34 | 4.09 | 4.63 |
| 17.8 | 1/3 | 4.0 | 4.68 | 4.25 | 3.86 | 26.2 | 23.1 | 20.4 |
| 4.45 | 0.2 | 4.0 | 3.06 | 3.55 | 2.53 | 3.66 | 4.54 | 2.72 |
| 17.8 | 0.4 | 4.0 | 3.79 | 3.48 | 3.48 | 19.8 | 17.7 | 17.7 |
| 4.45 | 0.4 | 4.0 | 1.94 | 1.96 | 1.96 | 1.67 | 1.71 | 1.71 |
| 17.8 | 0.5 | 4.0 | 3.10 | 2.85 | 3.11 | 15.0 | 13.2 | 15.0 |

* Ref. 8.
** Ref. 9.

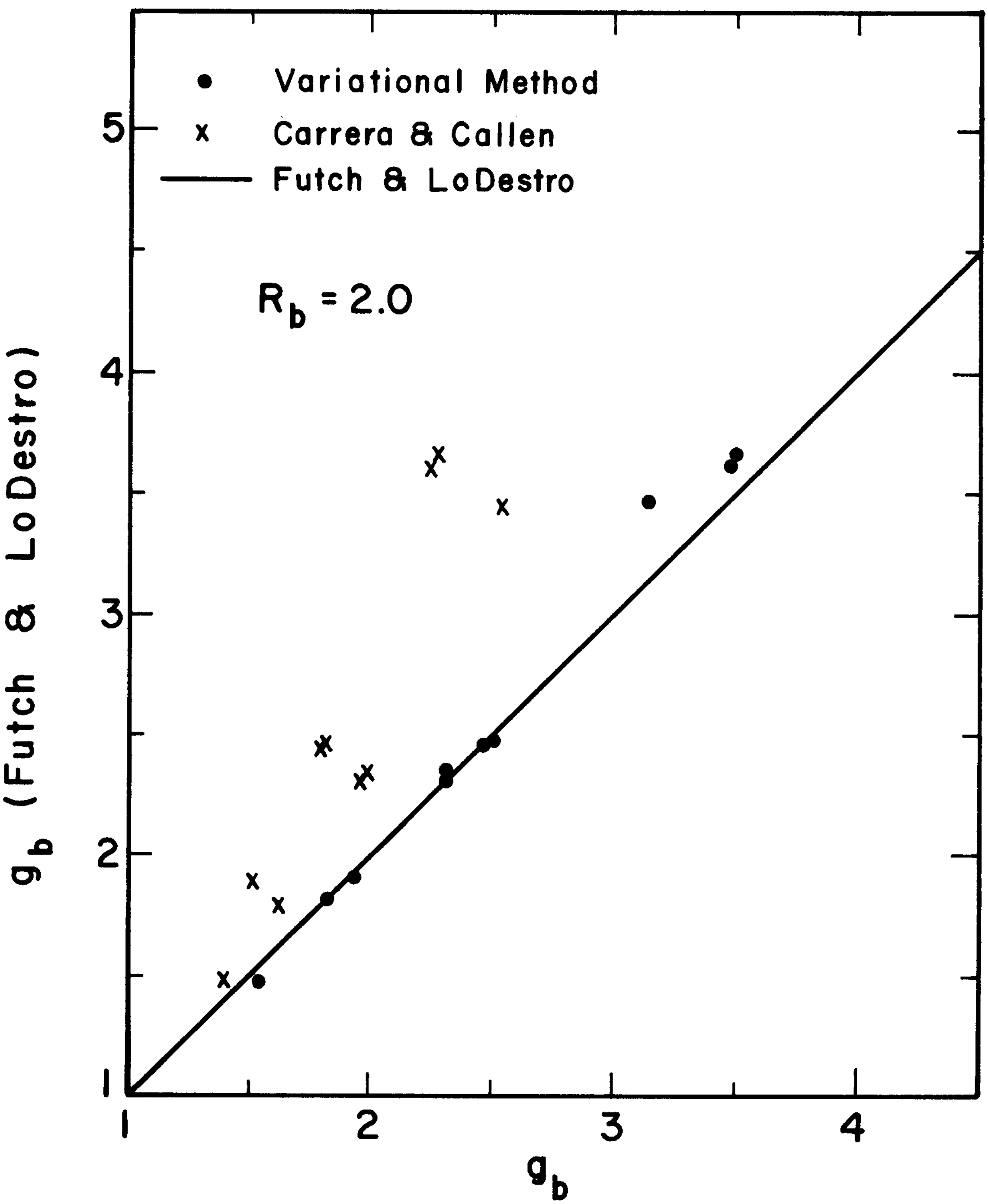


Fig. 4. Comparison of the trapping ratio, $g_b$, calculated by the three different methods (Low Mirror Ratio Cases; $R_b = 2$, $\phi_b = 40$ keV, $T_e = 15$ keV).

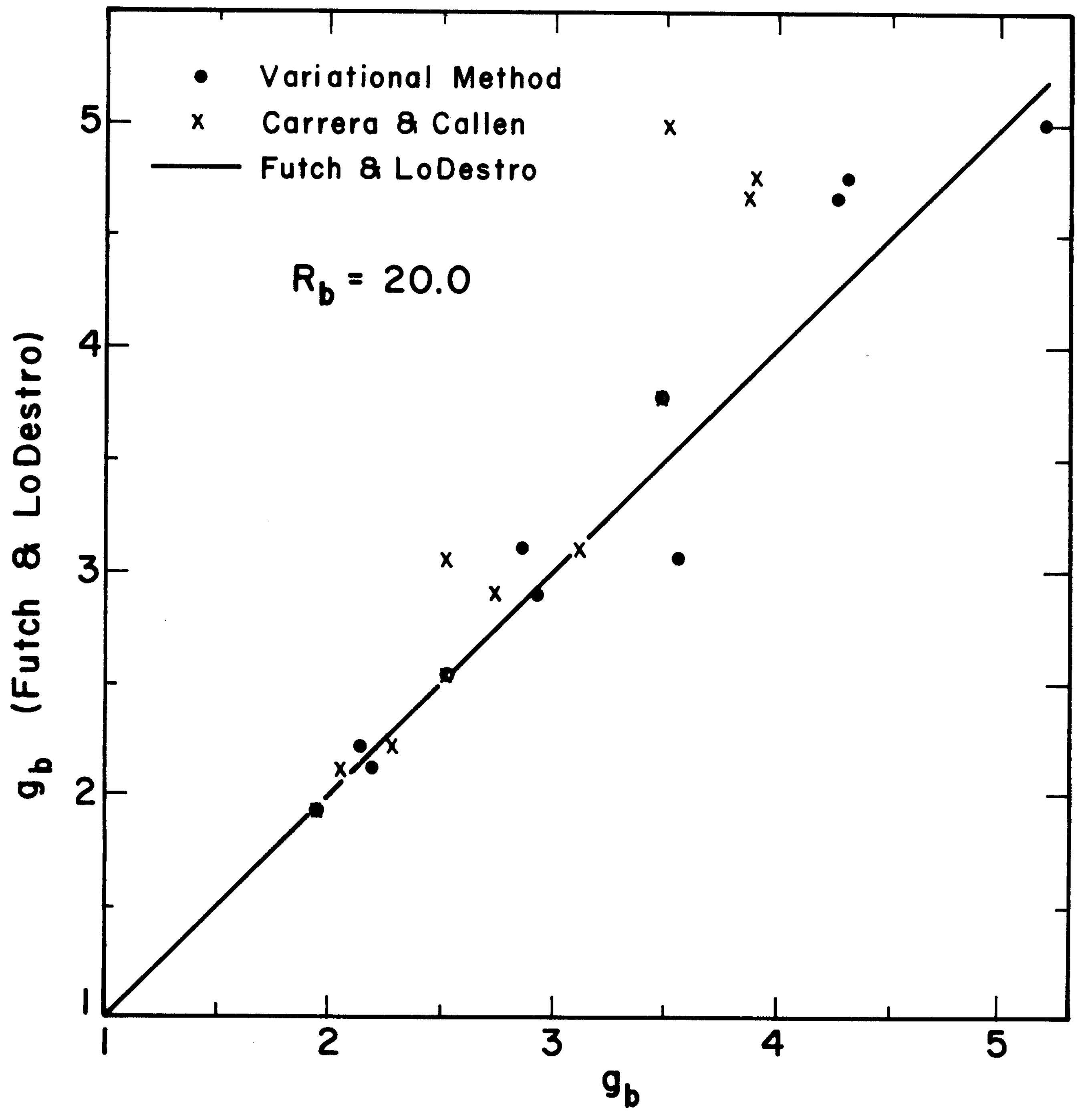


Fig. 5. Comparison of the trapping ratio, $g_b$, calculated by the three different methods (High Mirror Ratio Cases; $R_b = 20$, $\phi_b = 1$ keV, $T_e = 0.4$ keV).

## 7. Summary

A variational calculation of the trapping rate and trapped ion density in thermal barriers is presented. We retain in this formulation dynamical friction as well as pitch angle scattering. The distribution function is chosen to be a product RZ, where R is spherically symmetric about the vertex of the trapped-passing hyperbolic boundary and matches to the Maxwellian distribution of the passing ions on that boundary. We then get a kinetic equation determining Z which is linearized by appropriate choice for the field particle distribution function in the dynamical friction and diffusion coefficients. A mapping $\eta$ is introduced and a small term in the kinetic equation is neglected. We then get a variational functional whose stationary value is related to the trapping rate in the barrier. With a suitably chosen trial function, which matches exactly the desired passing ion distribution function along the actual trapped-passing boundary, estimates for the trapped ion density and trapping current are obtained. The results compare well with two-dimensional Fokker-Planck code calculations and are generally better than an analytical calculation using only pitch angle scattering.

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