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Abstract

The moment equation approach to neoclassical transport theory has been generalized to nonaxisymmetric toroidal systems under the assumption of the existence of magnetic surfaces. In particular, the parallel plasma flows and bootstrap current are calculated in both the Pfirsch-Schlüter and banana regimes. It is found that both parallel plasma flows and the bootstrap current can be reduced as the toroidal bumpiness increases in an otherwise axisymmetric system.

I. Introduction

The moment (or fluid) equation approach developed by Hirshman and Sigmar[1] has provided a relatively physical procedure for calculating the transport properties of an axisymmetric tokamak. This approach can be summarized as follows: first, the drift kinetic equation is employed to calculate parallel viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$ directly in terms of the plasma flows ($\vec{B}$ is the magnetic field, $\overleftrightarrow{\pi}_a$ and $\overleftrightarrow{\Theta}_a$ are stress and heat flux tensors and $\langle\ \rangle$ indicates a flux surface average); second, the parallel (to the magnetic field) momentum and heat flux balance equations are used to determine the parallel plasma flows and bootstrap current;[2] and finally, the flux-friction relations are used to obtain the particle and heat fluxes in terms of the flows and hence gradients. We generalize this approach to nonaxisymmetric toroidal systems under the assumption that magnetic surfaces exist. In this paper we begin by setting up a fundamental framework for the moment equation approach to nonaxisymmetric transport theory, and then focus on the problems of the parallel plasma flows and bootstrap current in a nonaxisymmetric toroidally confined plasma.

A typical nonaxisymmetric torus is the stellarator. Since the rotational transform is provided by the external coils, a stellarator can in principle be operated in a current free regime. Most of the equilibrium and stability studies of such plasmas have been based on such an assumption. Even though the plasma is free of an externally driven current it can still have a diffusion driven "bootstrap current." The bootstrap current may have a profound impact on the equilibrium and stability of stellarator plasmas. Thus, it is interesting to investigate how large the bootstrap current is in a stellarator.

Another nonaxisymmetric torus example is the rippled tokamak. Recent experiments in ISX-B have observed that the toroidal rotation is reduced as the toroidal ripple strength increases.[3] This provided an additional motivation for us to study the plasma flows in nonaxisymmetric toroids.

Even though the particle and heat fluxes in such a system have been calculated by many authors,[4-6] a systematic study of neoclassical plasma transport in nonaxisymmetric toroids has been lacking. This has caused debate[7,8] on whether the bootstrap current exists in a nonaxisymmetric torus. One of the purposes of this paper is to clarify this point and give an explicit expression for the bootstrap current. Similarly, the parallel plasma flow velocity has only been calculated in the Pfirsch-Schlüter regime by solving the drift kinetic equation.[9] In this paper, we will calculate the parallel plasma flow velocities using the moment equation approach in both the Pfirsch-Schlüter and banana regimes and give a simple physical picture to explain the qualitative results.

This paper is organized as follows. In Sec. II, we specify the coordinate system and list the transport equations which will be used in the paper. We also employ the small gyroradius expansion scheme to discuss the first order plasma flows and the relationships among them. The flux-friction relationships are derived in Sec. III. Some properties of the parallel viscosities are discussed in Sec. IV. In Secs. V and VI, we calculate the parallel viscosities in both the Pfirsch-Schlüter and banana regimes. The plasma currents in a nonaxisymmetric torus are discussed in Sec. VII. In Sec. VIII we calculate the parallel plasma flows and bootstrap current. A simple physical picture is also given there to explain the results qualitatively. The slowing down of the toroidal rotation and determination of the radial electric field are discussed in Sec. IX. Concluding remarks are given in Sec. X.

II. Coordinates and Basic Equations

To describe a general nonaxisymmetric torus, we employ the conventional Hamada coordinates[10] V,θ,ζ with

$$\vec{B} = \vec{\nabla}V\times\vec{\nabla}\,(\psi'\theta - \chi'\zeta) = \vec{B}_t + \vec{B}_p \quad , \tag{1}$$

where V is the volume enclosed by each toroidal flux surface, θ and ζ are angle variables in roughly the poloidal and toroidal directions. Also, here $\psi' = d\psi/dV = \vec{B}\cdot\vec{\nabla}\zeta$ and $\chi' = d\chi/dV = \vec{B}\cdot\vec{\nabla}\theta$ are toroidal and poloidal flux densities, and $\vec{B}_t = \psi'\vec{\nabla}V\times\vec{\nabla}\theta$ and $\vec{B}_p = -\chi'\vec{\nabla}V\times\vec{\nabla}\zeta$ are the toroidal and poloidal magnetic field components. The Jacobian $J \equiv \vec{\nabla}V\times\vec{\nabla}\theta\cdot\vec{\nabla}\zeta$ is unity everywhere. The line integral $\oint d\ell/B$ is assumed constant on rational surfaces,[11] which implies the existence of magnetic surfaces.[12] Throughout this paper, we will assume the existence of magnetic surfaces.

It is well-known that the macroscopic plasma behavior can be described by a set of conservation equations[1,13,14]

$$\frac{\partial n_a}{\partial t} + \vec{\nabla}\cdot(n_a\vec{u}_a) = 0 \quad , \tag{2}$$

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\, n_a m_a u_a^2 + \frac{3}{2}\, P_a\right) + \nabla\cdot\left[\left(\frac{1}{2}\, n_a m_a u_a^2 + \frac{5}{2}\, P_a\right)\vec{u}_a + \overleftrightarrow{\pi}_a\cdot\vec{u}_a + \vec{q}_a\right]$$

$$= (n_a e_a\vec{E} + \vec{F}_{a1})\cdot\vec{u}_a + Q_a \quad , \tag{3}$$

$$m_a n_a \frac{d\vec{u}_a}{dt} = n_a e_a\left(\vec{E} + \frac{1}{c}\,\vec{u}_a\times\vec{B}\right) + \vec{F}_{a1} - \vec{\nabla}P_a - \vec{\nabla}\cdot\overleftrightarrow{\pi}_a \quad , \tag{4}$$

$$\frac{\partial \vec{Q}_a}{\partial t} = \frac{e_a}{m_a} [\vec{E}\cdot(\frac{5}{2} P_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a + m_a n_a \vec{u}_a \vec{u}_a) + \frac{1}{c} \vec{Q}_a \times \vec{B}] + \vec{G}_a - \vec{\nabla}\cdot\overleftrightarrow{r}_a , \qquad (5)$$

The notation used in Eqs. (2-5) are standard and are defined in Ref. 1.

It is convenient for the transport calculation to employ the gyroradius ordering with small parameter $\Delta \equiv \rho/\ell$, where ρ is gyroradius and ℓ is a typical scale length of the system (cf. Refs. 1 and 14) to solve Eqs. (2-5). With this ordering, the first order stress tensors $\overleftrightarrow{\pi}_a$ and $\overleftrightarrow{r}_a$ have diagonal forms,[15] namely $\overleftrightarrow{\pi}_a = (P_{\parallel a} - P_{\perp a})(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})$ and $\overleftrightarrow{r}_a = r_a\overleftrightarrow{I} + (r_{\parallel a} - r_{\perp a})(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})$. The subscripts "$\parallel$" and "$\perp$" in the plasma variables are used to indicate components parallel and perpendicular to the magnetic field line components, respectively, and $\hat{n} \equiv \vec{B}/B$ is the unit vector along the magnetic field. When the collision frequency ν is of the order of the drift frequency ω_d, another small parameter $\Delta r/\ell << 1$ has to be assumed to exist, where Δr is the typical radial step size induced by the drift orbits. The discussion in this paper is limited to the high collisionality regime $\nu > \omega_d$ where the radial step size Δr is limited by collisional processes; its applicability to more collisionless regimes remains to be clarified.

As in an axisymmetric system, from the lowest order equation and the equations of motion Eqs. (2) and (4), we obtain

$$n_a e_a(-\vec{\nabla}\Phi + \frac{1}{c} \vec{u}_{a1} \times \vec{B}) = \vec{\nabla}P_a , \qquad (6)$$

$$\vec{u}_{\perp a1} = c \frac{\vec{B}\times\vec{\nabla}\Phi}{B^2} + c \frac{\vec{B}\times\vec{\nabla}P_a}{n_a e_a B^2} , \qquad (7)$$

$$\vec{\nabla}\cdot\vec{u}_{a1} = 0 , \qquad (8)$$

which means that the first order flow velocity $\vec{u}_{a1}$ is divergence free or incompressible and lies within the flux surface. With Eqs. (7) and (8) we can show that the first order flow velocity vectors $\vec{u}_{a1}$ are straight lines in the Hamada coordinates (the proof of this statement is the same as that which shows the current lines $\vec{j}$ are straight lines in Hamada coordinates and can be found in Refs. 10 and 11) and can be expressed as

$$\vec{u}_{a1} = u_{ta}\vec{\nabla}V\times\vec{\nabla}\theta + u_{pa}\vec{\nabla}\zeta\times\vec{\nabla}V \ , \tag{9}$$

where $u_{ta} = u_{ta}(V)$ and $u_{pa} = u_{pa}(V)$ are functions of the flux coordinate V only. The flow velocity can also be decomposed into the form

$$\vec{u}_{a1} = u_{\parallel a}\hat{n} + \vec{u}_{\perp a1} \ , \tag{10}$$

where $u_{\parallel a}$ is the parallel flow-speed along the magnetic field line. Taking the poloidal and toroidal components of Eqs. (9) and (10), we obtain

$$u_{pa} = \vec{u}_{a1}\cdot\vec{\nabla}\theta = u_{\parallel a}\hat{n}\cdot\vec{\nabla}\theta + \vec{u}_{\perp a1}\cdot\vec{\nabla}\theta \ , \tag{11a}$$

$$u_{ta} = \vec{u}_{a1}\cdot\vec{\nabla}\zeta = u_{\parallel a}\hat{n}\cdot\vec{\nabla}\zeta + \vec{u}_{\perp a1}\cdot\vec{\nabla}\zeta \ . \tag{11b}$$

Equation (11) is a geometric relationship between the components of the first order flow velocity $\vec{u}_{a1}$.

Similarly, employing the lowest order steady state energy and heat flux balance equations [Eqs. (3) and (5) and Eqs. (6-8)], we obtain

$$\vec{q}_{a1}\times\vec{B} = \frac{5}{2}\frac{c}{e_a} P_a \vec{\nabla} T_a \ , \qquad (12)$$

and

$$\vec{\nabla}\cdot\vec{q}_{a1} = 0 \ . \qquad (13)$$

From Eqs. (12) and (13), we can again show that the first order heat flux vector $\vec{q}_{a1}$ is also a straight line in the Hamada coordinates, and obtain the following relationships

$$q_{pa} = \vec{q}_{a1}\cdot\vec{\nabla}\theta = q_{\parallel a}\hat{n}\cdot\vec{\nabla}\theta + \vec{q}_{\perp a1}\cdot\vec{\nabla}\theta \ , \qquad (14a)$$

$$q_{ta} = \vec{q}_{a1}\cdot\vec{\nabla}\zeta = q_{\parallel a}\hat{n}\cdot\vec{\nabla}\zeta + \vec{q}_{\perp a1}\cdot\vec{\nabla}\zeta \ . \qquad (14b)$$

where $\vec{q}_{\perp a1} = 5\, cp_a \vec{\nabla} T_a / 2e_a$. The geometric relationships between the components of the first order flows $\vec{u}_{a1}$ and $\vec{q}_{a1}$ given in Eqs. (11) and (14) will be used in the calculation of the parallel viscosities.

III. Flux-Friction Relationships

As in an axisymmetric tokamak,[1] the relationship between the second order cross field particle flux and dissipative forces can be derived from the first order momentum balance equation

$$-m_a n_a \frac{\partial \vec{u}_{a1}}{\partial t} + n_a e_a(\vec{E} + \frac{1}{c}\vec{u}_a\times\vec{B}) + \vec{F}_{a1} - \vec{\nabla} P_a - \vec{\nabla}\cdot\overleftrightarrow{\pi}_a = 0 \ . \qquad (15)$$

Taking the perpendicular component of Eq. (15) ($\vec{B}$ × Eq. (15)) and averaging it over a flux surface, we obtain the flux-surface-averaged radial particle flux Γ_V^a of species a:

$$\langle\Gamma_V^a\rangle = \langle n_a\vec{u}_{\perp a}\cdot\vec{\nabla}V\rangle = \langle n_a c\,\frac{\vec{E}\cdot\hat{n}\times\vec{\nabla}V}{B}\rangle + \langle\frac{\vec{F}_{a1}\cdot\hat{n}\times\vec{\nabla}V}{m_a\Omega_a}\rangle - \langle\frac{\hat{n}\times\vec{\nabla}V\cdot\vec{\nabla}\cdot\overleftrightarrow{P}_a}{m_a\Omega_a}\rangle\ , \qquad (16)$$

where $\overleftrightarrow{P}_a = P_a\overleftrightarrow{I} + \overleftrightarrow{\pi}_a$, $\Omega_a = e_aB/m_ac$ is the gyrofrequency and the angular brackets denote the flux surface averaging $\langle A\rangle \equiv \oint d\theta d\zeta A$. The interpretation of Eq. (16) is the same as that in the axisymmetric tokamak.[1,14] Namely, the classical flux is driven by the perpendicular friction force $\vec{F}_{a1}\cdot\hat{n}\times\vec{\nabla}V$, and the neoclassical flux is driven by the viscosity (banana-plateau) and the pressure variation (Pfirsch-Schlüter) within the flux surface.

To derive flux-friction relationships, we take the toroidal component of Eq. (15) ($\vec{\nabla}V\times\vec{\nabla}\theta\cdot$Eq. (15)) and obtain the flux-surface-averaged particle flux

$$\langle\Gamma_V^a\rangle = \Gamma_{c\ell}^a + \Gamma_{bp}^a + \Gamma_{ps}^a + \Gamma_{na}^a + \frac{n_ac}{\chi'}\langle\frac{E_\parallel^{(A)}}{B}\left(\langle I\rangle\frac{B^2}{\langle B^2\rangle} - I\right)\rangle \qquad (17a)$$

$$- \frac{n_ac}{\chi'\psi'}\langle\vec{B}_t\cdot\vec{E}^{(A)} - \vec{B}_t\cdot\hat{n}E_\parallel^{(A)}\rangle\ ,$$

where

$$\Gamma_{c\ell}^a = -\frac{c}{e_a\chi'\psi'}\langle\vec{B}_t\cdot\vec{F}_{\perp a1}\rangle\ , \qquad (17b)$$

$$\Gamma_{bp}^a = -\frac{c}{e_a\chi'}\,\frac{\langle I\rangle\langle B(F_{\parallel a1} + n_ae_aE_\parallel^{(A)})\rangle}{\langle B^2\rangle}\ , \qquad (17c)$$

$$\Gamma_{ps}^a = -\frac{c}{e_a\chi'}\langle\frac{F_{\parallel a1}}{B}\left(I - \langle I\rangle\frac{B^2}{\langle B^2\rangle}\right)\rangle\ , \qquad (17d)$$

$$\Gamma_{na}^a = \frac{c}{e_a\chi'\psi'}\langle\vec{B}_t\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle\ , \qquad (17e)$$

and $I \equiv \vec{B}_t\cdot\vec{B}/\psi'$, $\vec{E} = -\vec{\nabla}\phi + \vec{E}^{(A)}$, and $\vec{F}_{a1} = \vec{F}_{\parallel a1}\hat{n} + \vec{F}_{\perp a1}$. The classical ($\Gamma_{c\ell}^a$), banana-plateau ($\Gamma_{bp}^a$), and Pfirsch-Schlüter (Γ_{ps}^a) fluxes have the same meaning as in an axisymmetric tokamak, and are ambipolar. The nonaxisymmetric flux Γ^a_{na} in Eq. (17e) is induced by the toroidal viscosity $\langle \vec{B}_t \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle$ and is nonambipolar. In the low collisionality regime (usually this means for the collision frequency ν less than the bounce frequency ω_b), the dominant contribution to $\langle \Gamma^a_V \rangle$ is usually from Γ^a_{na}. From the flux-surface-averaged parallel component of Eq. (15), we have

$$\langle B(F_{a1\parallel} + n_a e_a E_\parallel^{(A)}) \rangle = \langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle \ . \tag{18}$$

Hence, we can also write Eq. (17c) as

$$\Gamma^a_{bp} = - \frac{c}{e_a \chi'} \frac{\langle I \rangle \langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle}{\langle B^2 \rangle} \ , \tag{19}$$

i.e., as in the axisymmetric tokamak, the banana-plateau flux can be viewed as driven either by the parallel friction force or parallel viscosity.

To relate Eqs. (16) and (17), we need a geometric relationship between $\vec{B}_t$ and $\hat{n} \times \vec{\nabla} V$. The desired geometric identity is

$$\frac{\vec{B}_t}{\psi'} = \frac{I}{B} \hat{n} - \chi' \frac{\hat{n} \times \vec{\nabla} V}{B} \ . \tag{20}$$

With Eq. (20) we can show that Eqs. (16) and (17) are equivalent. Thus Eq. (16) can be written as

$$\langle \Gamma^a_V \rangle = \Gamma^a_{c\ell} + \Gamma^a_{bp} + \Gamma^a_{ps} + \Gamma^a_{na} + \frac{n_a c}{\chi'} \left\langle \frac{E_\parallel^{(A)}}{B} \left(\langle I \rangle \frac{B^2}{\langle B^2 \rangle} - I \right) \right\rangle + \left\langle n_a c \frac{\vec{E} \cdot \hat{n} \times \vec{\nabla} V}{B} \right\rangle \ . \tag{21}$$

Usually, the magnetic flux surface is not totally static during the transport process. From Faraday's law $\partial\vec{B}/\partial t = -c\vec{\nabla}\times\vec{E}$, we can define a toroidal flux velocity[1,16] $\langle\vec{u}_\psi\cdot\vec{\nabla}V\rangle = \partial\psi/\partial t$. Taking the $\vec{\nabla}\zeta$ component of Faraday's law we have

$$\langle\vec{u}_\psi\cdot\vec{\nabla}V\rangle = +c\left\langle\frac{\vec{E}\cdot\vec{B}_p}{\chi'\psi'}\right\rangle . \tag{22}$$

Combining Eqs. (21) and (22), we find the particle flux relative to the toroidal flux $\langle\Gamma_R^a\rangle \equiv \langle\Gamma_V^a\rangle - n_a\langle\vec{u}_\psi\cdot\vec{\nabla}V\rangle$ to be

$$\langle\Gamma_R^a\rangle = \Gamma_{c\ell}^a + \Gamma_{bp}^a + \Gamma_{ps}^a + \Gamma_{na}^a + \frac{n_a c}{\chi'}\frac{\langle E_\parallel^{(A)}B\rangle\langle I\rangle}{\langle B^2\rangle}\left(1 - \frac{1}{\psi'}\frac{\langle B^2\rangle}{\langle I\rangle}\right) . \tag{23}$$

The last term in Eq. (23) is the usual $\vec{E}\times\vec{B}_p$ radial flux. The relative particle flux $\langle\Gamma_R^a\rangle$ can be used in three-dimensional transport calculations.[16]

We have derived the flux-friction relation, Eq. (17), by taking the toroidal component of Eq. (15). However, there is no particular reason that we have to do so. We can also take the poloidal component of Eq. (15) $[\vec{\nabla}V\times\vec{\nabla}\zeta\cdot\text{Eq. (15)}]$ and obtain

$$\langle\Gamma_V^a\rangle_p = \Gamma_{c\ell}^a + \Gamma_{bp}^a + \Gamma_{ps}^a - \frac{n_a c}{\psi'}\left\langle\frac{E_\parallel^{(A)}}{B}\left(\langle I_p\rangle\frac{B^2}{\langle B^2\rangle} - I_p\right)\right\rangle , \tag{24a}$$

$$+ \frac{n_a c}{\chi'\psi'}\langle\vec{B}_p\cdot\vec{E}^{(A)} - \vec{B}_p\cdot\hat{n}E_\parallel^{(A)}\rangle ,$$

where

$$\Gamma_{c\ell}^a = \frac{c}{e_a\chi'\psi'}\vec{B}_p\cdot\vec{F}_{\perp a1} , \tag{24b}$$

$$\Gamma_{bp}^a = \frac{c}{e_a\chi'\psi'}\frac{\langle I_p\rangle\langle B(F_{\parallel a1} + n_a e_a E_\parallel^{(A)})\rangle}{\langle B^2\rangle} , \tag{24c}$$

$$\Gamma^a_{ps} = \frac{c}{e_a\psi'} \langle \frac{F_{\parallel a1}}{B} (I_p - \langle I_p\rangle \frac{B^2}{\langle B^2\rangle})\rangle \ , \tag{24d}$$

$$\Gamma^a_{na} = - \frac{c}{e_a\chi'\psi'} \langle \vec{B}_p \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle \ , \tag{24e}$$

and $I_p = \vec{B}_p \cdot \vec{B}/\chi'$. Since $(I_p - \langle I_p\rangle B^2/\langle B^2\rangle/\psi' = -(I - \langle I\rangle B^2/\langle B^2\rangle)/\chi'$, the Pfirsch-Schlüter fluxes defined in (24d) and (17d) are identical. With Eq. (18), we can show that the sum of the banana-plateau and nonaxisymmetric fluxes $\Gamma_{bp} + \Gamma_{na}$ are identical in Eqs. (17) and (24). To transform from Eq. (17c) to Eq. (24c) (or Eq. (17e) to Eq. (24e)) or vice versa, we need only add (or subtract) an ambipolar flux that is proportional to the parallel friction $\langle B[F_{a1\parallel} + n_a e_a E_\parallel^{(A)}]\rangle$ or Γ^a_{bp}. Thus in general, to transform from one representation of the flux-friction relationships to the other, we need only add (or subtract) a term in Γ^a_{bp} (or Γ^a_{na}). Since this term is proportional to the known flux Γ^a_{bp} this involves no additional computational effort and does not alter the net radial transport arising from the sum over the various pieces.

The physical reason that the representation of the flux-friction relation is not unique is that a nonaxisymmetric torus can be thought of as a perturbation of an originally axisymmetric torus. For every axisymmetric configuration, there is a representation of the flux-friction relationships corresponding to it. For example, to obtain Eq. (21), we implicitly assumed that the nonaxisymmetric torus results from the perturbation of a toroidally symmetric torus; conversely, to obtain Eq. (29), we implicitly assume that the original torus is poloidally symmetric.

So far, we have discussed the relationship between the particle flux and the dissipative forces. A similar relationship between the heat flux and the dissipative forces can be obtained from the steady state heat flux balance equation (accurate to first order in the gyroradius expansion):

$$\frac{e_a}{m_a}\left[\vec{E}\cdot\left(\frac{5}{2}\,p_a\overleftrightarrow{I} + \overleftrightarrow{\pi}_a\right) + \frac{1}{c}\,\vec{Q}_a\times\vec{B}\right] + \vec{G}_a - \nabla\cdot\overleftrightarrow{r}_a = 0 \ . \tag{25}$$

Taking the toroidal component of Eq. (25) ($\vec{\nabla}V\times\vec{\nabla}\theta\cdot$Eq. (25)), we obtain the radial component of the flux-surface-averaged heat flux $\langle\vec{q}^{\,a}_V\cdot\vec{\nabla}V\rangle \equiv \langle\vec{Q}_a\cdot\vec{\nabla}V - \frac{5}{2}\,T_a\Gamma^a_V\rangle$ as

$$\left\langle\frac{\vec{q}^{\,a}_V\cdot\vec{\nabla}V}{T_a}\right\rangle \equiv \left\langle\frac{q^a_V}{T_a}\right\rangle = \left(q^a_{c\ell}q^a_{bp} + q^a_{ps} + q^a_{na}\right)T_a^{-1} \tag{26}$$

where the classical $q^a_{c\ell}$, banana-plateau q^a_{bp}, Pfirsch-Schlüter q^a_{ps} and nonaxisymmetric q^a_{na} heat fluxes are defined by

$$\frac{q^a_{c\ell}}{T_a} = -\frac{c}{e_a\chi'\psi'}\,\langle\vec{B}_t\cdot\vec{F}_{\perp a2}\rangle \ , \tag{27a}$$

$$\frac{q^a_{bp}}{T_a} = -\frac{c\langle I\rangle}{e_a\chi'}\,\frac{\langle BF_{\parallel a2}\rangle}{\langle B^2\rangle} \ , \tag{27b}$$

$$\frac{q^a_{ps}}{T_a} = -\frac{c}{e_a\chi'}\left\langle\frac{F_{\parallel a2}}{B}\left(I - \langle I\rangle\,\frac{B^2}{\langle B^2\rangle}\right)\right\rangle \ , \tag{27c}$$

$$\frac{q^a_{na}}{T_a} = \frac{c}{e_a\chi'\psi'}\,\langle\vec{B}_t\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle \ . \tag{27d}$$

and $\vec{F}_{a2} = F_{\parallel a2}\hat{n} + \vec{F}_{\perp a2}$ and $\overleftrightarrow{\Theta}_a = [m_a(r_{\parallel a} - r_{\perp a})/T_a - 5(p_{\parallel a} - p_{\perp a})/2](\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I}) = (\Theta_{\parallel a} - \Theta_{\perp a})\ (\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})$. From the flux-surface-averaged parallel components of Eq. (25) and Eq. (18), we obtain the parallel heat flux balance equation

$$\langle BF_{\parallel a2}\rangle = \langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle \ . \tag{28}$$

Thus, like the banana-plateau particle flux Γ_{bp}^{a}, the banana-plateau heat flux q_{bp}^{a} can be viewed as driven either by parallel heat friction $F_{\parallel a2}$ or parallel heat viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$.

IV. Parallel Viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$

The plasma parallel flows and bootstrap current can be calculated from the parallel momentum and heat flux balance equations[1] Eqs. (18) and (28). Since the general expressions for $F_{\parallel a1}$ and $F_{\parallel a2}$ in terms of the relative plasma parallel flows and heat flows are known,[1,17] we need only calculate the parallel viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$. With $\overleftrightarrow{\pi}_a = (p_{\parallel a} - p_{\perp a})(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})$ and $\overleftrightarrow{\Theta}_a = (\Theta_{\parallel a} - \Theta_{\perp a})(\hat{n}\hat{n} - \frac{1}{3}\overleftrightarrow{I})$, we can simplify the parallel viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$ and obtain

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle = \langle (p_{\perp a} - p_{\parallel a})\hat{n}\cdot\vec{\nabla}B \rangle \ , \tag{29a}$$

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle = \langle (\Theta_{\perp a} - \Theta_{\parallel a})\hat{n}\cdot\vec{\nabla}B \rangle \ . \tag{29b}$$

In principle, we can calculate the parallel viscosities from Eq. (29) as long as we know the particle distribution function f_a of particle species a.

It is well-known that in the low collisionality regime, $\nu \ll \omega_b$, the particle flux is dominated by the nonaxisymmetric flux Γ_{na}^{a}. Since the parallel viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ is proportional to the banana-plateau flux Γ_{bp}^{a}, we do not expect the parallel viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ in a nonaxisymmetric system to have a different collision frequency dependence than that in an axisymmetric system. Indeed, we can show that the solution of the bounce averaged drift kinetic equation f_b which can give rise to the nonaxisymmetric flux Γ_{na}^{a} in the low collisionality regime $\nu < \omega_b$ will not contribute to the parallel viscosity. To show this, we note that since f_b is a solution of the bounce averaged drift kinetic equation, f_b will not depend on the variable that measures the distance along the field line. For convenience we define $\vec{B} = \vec{\nabla}V \times \vec{\nabla}\beta$ with $\beta = \psi'\theta - \chi'\zeta$ and choose ζ as the variable that measures the distance along the field line. Then, f_b will have the functional form $f_b = f_b(E,\mu,V,\beta)$ where $E = 1/2\ m_a v^2$ and $\mu = 1/2\ m_a v_\perp^2/B$ are total energy and magnetic moment of particle species a. From Eq. (29) and neglecting the flow velocity $\vec{u}_a$ compared with the thermal velocity $\vec{v}$, we have

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle = \langle \hat{n}\cdot\vec{\nabla}B \int d\vec{v}\ m_a \left(\frac{v_\perp^2}{2} - v_\parallel^2\right) f_b \rangle \ . \tag{30a}$$

Changing variables from $d\vec{v}$ to $4\pi dE d\mu B/m_a^2|v_\parallel|$ and expressing the flux surface average explicitly, we can write Eq. (30a) as

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle = \oint d\beta d\zeta \int \frac{4\pi}{m_a^2}\ dE d\mu m_a \left[\frac{1}{|v_\parallel|}\frac{\partial B}{\partial \zeta}\left(\frac{v_\perp^2}{2} - v_\parallel^2\right)\right] f_b \ . \tag{30b}$$

The term inside the square brackets can be written as a total derivative with respect to ζ, i.e.,

$$\frac{1}{|v_\parallel|}\frac{\partial B}{\partial \zeta}\left(\frac{v_\perp^2}{2} - v_\parallel^2\right) = -\frac{\partial}{\partial \zeta}\left(|v_\parallel| B\right) \ . \tag{30c}$$

Since f_b is independent of ζ, we can carry out the ζ averaging first and obtain $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle = 0$. This result is physically related to the fact that the bounce averaged drift kinetic equation is not sensitive to the Coulomb collisional contributions in the parallel momentum balance, and thus will make no contribution to the parallel friction force and Γ^a_{bp}. Similarly, one can show that f_b will not contribute to the parallel heat viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$.

V. Calculation of Parallel Viscosities in the Pfirsch-Schlüter Regime

Before we start calculating the parallel viscosities with the drift kinetic equation, we first calculate the parallel viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ from the fluid theory. For collision frequency ν less than the gyrofrequency Ω, the pressure anisotropy is[18]

$$p_\parallel - p_\perp = -3p\tau[\hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}) - (\hat{n}\cdot\vec{\nabla}\hat{n})\cdot\vec{u}] \,, \tag{31}$$

where p is the plasma pressure and τ the ion-ion collision time. To obtain Eq. (31), we have used the fact that $\vec{\nabla}\cdot\vec{u} = 0$, and neglected the heat flow. The second term inside the square brackets of Eq. (31) can be simplified with

$$(\hat{n}\cdot\vec{\nabla}\hat{n})\cdot\vec{u} = \frac{\vec{u}\cdot\vec{\nabla}B}{B} \,. \tag{32a}$$

In Hamada coordinates, the first term inside the square brackets of Eq. (31) can be written as

$$\hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}) = \frac{1}{B}\left[\frac{\partial}{\partial\theta}(u_\parallel\chi') + \frac{\partial}{\partial\zeta}(u_\parallel\chi')\right] \,. \tag{32b}$$

Using the flow relationships described in Eq. (11), we obtain

$$\hat{n}\cdot\vec{\nabla}(\hat{n}\cdot\vec{u}) = \frac{1}{B}\frac{\partial B}{\partial\theta}(u_p - \vec{u}_\perp\cdot\vec{\nabla}\theta) + \frac{1}{B}\frac{\partial B}{\partial\zeta}(u_t - \vec{u}_\perp\cdot\vec{\nabla}\zeta) - \left[\frac{\partial}{\partial\theta}(\vec{u}_\perp\cdot\vec{\nabla}\theta) + \frac{\partial}{\partial\zeta}(\vec{u}_\perp\cdot\vec{\nabla}\zeta)\right] \,. \tag{32c}$$

Combining Eqs. (31) and (32), we have for the pressure anisotropy

$$p_{\parallel} - p_{\perp} = -3p\tau \left[\frac{1}{B}\frac{\partial B}{\partial \theta} u_p + \frac{1}{B}\frac{\partial B}{\partial \zeta} u_t - \left(\vec{\nabla}\cdot\vec{u}_{\perp} + 2\frac{\vec{u}_{\perp}\cdot\vec{\nabla}B}{B}\right)\right] . \qquad (33)$$

Recalling that $\vec{u}_{\perp}$ is just the diamagnetic flow and $\vec{\nabla}\times\vec{B}\cdot\vec{\nabla}V = 0$, we can show that the term in parentheses in Eq. (33) is

$$\vec{\nabla}\cdot\vec{u}_{\perp} + 2\frac{\vec{u}_{\perp}\cdot\vec{\nabla}B}{B} = \frac{1}{B^2}\vec{\nabla}\cdot(\vec{u}_{\perp}B^2) = 0 .$$

Thus, the parallel viscosity becomes

$$\langle\vec{B}\cdot\vec{\nabla}\cdot\vec{\vec{\pi}}\rangle = 3\ p\tau\langle\hat{n}\cdot\vec{\nabla}B\left(\frac{1}{B}\frac{\partial B}{\partial \sigma} u_p + \frac{1}{B}\frac{\partial B}{\partial \zeta} u_t\right)\rangle . \qquad (34)$$

Equation (34) reduces to the axisymmetric tokamak result[1] by neglecting the $\partial B/\partial\zeta$ term.

Anticipating that the result should have the same form as Eq. (34), we can also calculate the parallel viscosities from the linearized drift kinetic equation

$$v_{\parallel}\hat{n}\cdot\vec{\nabla}f_{a1} + \vec{v}_{da}\cdot\vec{\nabla}V\frac{\partial f_{ao}}{\partial V} + e_{\parallel}E_{\parallel}v_{\parallel}\frac{\partial f_{ao}}{\partial E} = C_a(F_{a1}) , \qquad (35)$$

where $C_a(f_{a1})$ is the linearized collision operator, and

$$\vec{v}_{da}\cdot\vec{\nabla}V = \frac{v_{\parallel}}{B}\vec{\nabla}\cdot\left(\frac{v_{\parallel}}{\Omega_a}\vec{B}\times\vec{\nabla}V\right) = \frac{v_{\parallel}}{B}\left[\frac{\partial}{\partial\theta}\left(\frac{v_{\parallel}}{\Omega_a}\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta\right) + \frac{\partial}{\partial\zeta}\left(\frac{v_{\parallel}}{\Omega_a}\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta\right)\right] , \qquad (36)$$

is the radial drift velocity. We will use the same method to calculate the parallel viscosities as that developed in Ref. 1, except that we now remove the axisymmetric assumption. Equation (35) can be solved in the Pfirsch-

Schlüter regime by an auxiliary expansion with the small parameter ω_t/ν where $\omega_t = v_\parallel/L_\parallel$ is the typical transit frequency with $L_\parallel$ being the typical parallel scale length. The zeroth order equation is then

$$C_a(f_{a1}^{(o)}) = 0 ,$$

which has the solution that $f_{a1}^{(o)} = n_a(m_a/2\pi T_a)^{3/2} \exp(-E/T_a)$, where $E = (1/2)m_a v^2$ is the kinetic energy of the particle. Note that $f_{a1}^{(o)}$ is essentially a Maxwellian distribution but both density n_a and temperature T_a are allowed to have a gradient along the field line. The first order equation is

$$v_\parallel(A_{1a}L_o^{(3/2)} + A_{2a}L_1^{(3/2)})f_{ao} = C_a(f_{a1}^{(1)}) , \tag{37}$$

where $A_{1a} = \hat{n}\cdot\vec{\nabla} \ln p_a - e_a E_\parallel/T_a$, $A_{2a} = -\hat{n}\cdot\vec{\nabla}T_a/T_a$, $L_j^{(3/2)}$ for $j = 0,1,2,...$ are Laguerre polynomials[19] of order (3/2) with $L_o^{(3/2)} = 1$, $L_1^{(3/2)} = 5/2 - x_a^2,...$ and $x_a^2 = E/T_a$. The solution $f_{a1}^{(1)}$ to Eq. (37) can be expanded in terms of Laguerre polynomials and is

$$f_{a1}^{(1)} = 2\,\frac{v_\parallel}{v_{ta}^2}\,(u_{\parallel a} - \frac{2}{5}\frac{q_{\parallel a}}{p_a}L_1^{(3/2)} + ...)f_{ao} , \tag{38}$$

where $v_{ta}^2 = 2T_a/m_a$. The second order equation from Eq. (35) is

$$v_\parallel\hat{n}\cdot\vec{\nabla}f_{a1}^{(1)} + \vec{v}_{da}\cdot\vec{\nabla}V\,\frac{\partial f_{ao}}{\partial V} = C_a(f_{a1}^{(2)}) . \tag{39}$$

Using the Hamada coordinates and Eq. (36), we can write Eq. (39) explicitly as

$$\frac{v_\parallel}{B}\chi'\frac{\partial f_{a1}^{(1)}}{\partial\theta}+\frac{v_\parallel}{B}\frac{\partial}{\partial\theta}\left(\frac{v_\parallel}{\Omega_a}\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta\right)\frac{\partial f_{ao}}{\partial V}+\frac{v_\parallel}{B}\psi'\frac{\partial f_{a1}^{(1)}}{\partial\zeta}$$

$$+\frac{v_\parallel}{B}\frac{\partial}{\partial\zeta}\left(\frac{v_\parallel}{\Omega_a}\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta\right)\frac{\partial f_{ao}}{\partial V}=C_a(f_{a1}^{(2)}) \ . \tag{40}$$

Substituting Eq. (38) into (40) and using Eqs. (11) and (14), we obtain

$$2x_a^2\frac{1}{B}\frac{\partial B}{\partial\theta}f_{ao}p_2(\xi)\left(u_{pa}-\frac{2}{5}\frac{L_1^{(3/2)}}{p_a}q_{pa}\right)$$

$$+2x_a^2\frac{1}{B}\frac{\partial B}{\partial\zeta}f_{ao}p_2(\xi)\left(u_{ta}-\frac{2}{5}\frac{L_1^{(3/2)}}{p_a}q_{ta}\right)=C_a(f_{a1}^{(2)}) \ , \tag{41}$$

where $p_2(\xi) = 3\xi^2/2 - 1/2$ is the Legendre polynomial and $\xi = v_\parallel/v$. The solution $f_{a1}^{(2)}$ to Eq. (41) can be found again by expanding $f_{a1}^{(2)}$ in terms of Laguerre polynomials $L_j^{(5/2)}$ as

$$f_{a1}^{(2)}=\frac{2}{3}x_a^2\,p_2(\xi)\sum_j p_{aj}L_j^{(5/2)}(x_a^2)f_{ao} \ , \tag{42a}$$

and

$$p_{aj}=\frac{5\int d\vec{v}\;x_a^2L_j^{(5/2)}f_{a1}^{(2)}p_2(\lambda)}{n_a\{x_a^2[L_j^{(5/2)}]^2\}} \tag{42b}$$

where $\{A(v)\} = (8/3\sqrt{\pi})\int_0^\infty dx\,\exp(-x^2)x^4A(xv_{ta})$. The first few terms of $L_j^{(5/2)}$ are $L_0^{(5/2)} = 1$, $L_1^{(5/2)} = 7/2 - x^2$, ... From Eq. (42b) we have $(p_{\parallel a} - p_{\perp a}) = p_ap_{ao}$ and $\Theta_{\parallel a} - \Theta_{\perp a} = (p_{ao} - 7p_{a1}/2)p_a$. Thus to calculate parallel viscosities we need only know p_{ao} and p_{a1}. Substituting Eq. (42a) into Eq. (41) and utilizing the results in Eqs. (30) and a similar one for $\langle\vec{B}\cdot\vec{\nabla}\cdot\vec{\Theta}_a\rangle$, we obtain

$$\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle = 3p_a\tau_a\left[\langle\hat{n}\cdot\vec{\nabla}B\,\frac{1}{B}\,\frac{\partial B}{\partial\theta}\rangle\left(\mu_{a1}u_{pa} + \frac{2}{5}\,\mu_{a2}\,\frac{q_{pa}}{p_a}\right)\right.$$
$$\left. + \langle\hat{n}\cdot\vec{\nabla}B\,\frac{1}{B}\,\frac{\partial B}{\partial\zeta}\rangle\left(\mu_{a1}u_{ta} + \frac{2}{5}\,\mu_{a2}\,\frac{q_{ta}}{p_a}\right)\right] , \tag{43a}$$

$$\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle = 3p_a\tau_a\left[\langle\hat{n}\cdot\vec{\nabla}B\,\frac{1}{B}\,\frac{\partial B}{\partial\theta}\rangle\left(\mu_{a2}u_{pa} + \frac{2}{5}\,\mu_{a3}\,\frac{q_{pa}}{p_a}\right)\right.$$
$$\left. + \langle\hat{n}\cdot\vec{\nabla}B\,\frac{1}{B}\,\frac{\partial B}{\partial\zeta}\rangle\left(\mu_{a2}u_{ta} + \frac{2}{5}\,\mu_{a3}\,\frac{q_{ta}}{p_a}\right)\right] , \tag{43b}$$

where τ_a is the self-collision time. The definitions of μ_{aj} for j = 1,2,3 are given in Ref. 1. For a simple electron-ion plasma, $\mu_{i1} = 1.365$, $\mu_{i2} = 2.31$, $\mu_{i3} = 8.78$, and $\mu_{e1} = 0.733$, $\mu_{e2} = 1.51$, $\mu_{e3} = 6.06$. Note that if we neglect the heat fluxes q_{pa} and q_{ta}, Eq. (43a) is the same as Eq. (34) to within a factor of order unity.

VI. Calculation of Parallel Viscosities in the Banana Regime

Since there are in general many different kinds of trapped particles in a nonaxisymmetric torus, the banana regime for a species a is defined as the range of collision frequency ν such that the effective collision frequency ν_{eff} is less than the bounce frequency ω_{bt} for all kinds of trapped particles. The linearized drift kinetic equation, Eq. (35), can then be solved by an auxiliary expansion with small parameter ν_{eff}/ω_{bt}. Note that Eq. (35) is valid only when the geodesic drifts (i.e., those within the flux surface) are negligible since these contributions are neglected in Eq. (35). Generally speaking this limits the analysis to collision frequencies greater than a typical poloidal drift frequency -- the v_d^2/ν regime of nonaxisymmetric toroidal transport. Thus by using Eq. (35) we neglect the possible existence of superbanana drift orbits.[5] We will not discuss the effect of the superbananas on parallel viscosity in this paper.

The zeroth order equation from Eq. (35) for the perturbed circulating particle distribution $f_{alc}^{(o)}$ is

$$v_{\parallel}\hat{n}\cdot\vec{\nabla}f_{alc}^{(o)} + \vec{v}_{da}\cdot\vec{\nabla}V \frac{\partial f_{ao}}{\partial V} = 0 , \quad (44a)$$

or

$$\vec{B}\cdot\vec{\nabla}f_{alc}^{(o)} + \vec{\nabla}\cdot(\frac{v_{\parallel}}{\Omega_a}\vec{B}\times\vec{\nabla}V) \frac{\partial f_{ao}}{\partial V} = 0 . \quad (44b)$$

Equation (44) can be solved if $\oint (d\ell/v_{\parallel})\ \vec{v}_{da}\cdot\vec{\nabla}V = 0$ on the rational surface.[20] The physical meaning of the solubility criterion $\oint (d\ell/v_{\parallel})\ \vec{v}_{da}\cdot\vec{\nabla}V = 0$ is that $\oint v_{\parallel} d\ell$ = constant on the rational surface, or equivalently that the circulating particles do not drift off a flux surface.[21] We will assume that the solubility criterion is satisfied, so that we can solve Eq. (44). If the solubility criterion is not satisfied, Eq. (44) is not valid for describing the circulating particles and circulating particles should be treated in a manner similar to the trapped particles. We will not treat this case in this paper.

The solution to Eq. (44) is

$$f_{alc}^{(o)} = -\int_{\ell_{max}}^{\ell} \frac{d\ell'}{B}\ \vec{\nabla}\cdot(\frac{v_{\parallel}}{\Omega_a}\vec{B}\times\vec{\nabla}V) \frac{\partial f_{ao}}{\partial V} + g(V) = \overline{F}_{alc} + g(V) , \quad (45)$$

where at $\ell = \ell_{max}$, B is the maximum B_{max} along the field line. The unknown function g(V) can be determined from the next higher order (or first order) equation from Eq. (35)

$$v_{\parallel}\hat{n}\cdot\vec{\nabla}f_{alc}^{(1)} + e_a E_{\parallel} v_{\parallel} \frac{\partial f_{ao}}{\partial E} = C_a(f_{alc}^{(o)}) \ . \tag{46}$$

Flux surface averaging Eq. (46), namely $\langle Bv_{\parallel}^{-1}$ Eq. (46)$\rangle$, we obtain a constraint condition determining g:

$$e_a \langle E_{\parallel} B\rangle \frac{\partial f_{ao}}{\partial E} = \langle \frac{B}{v_{\parallel}} C_a(F_{alc})\rangle + \langle \frac{B}{v_{\parallel}} C_a(g)\rangle \ . \tag{47}$$

The general form for the function g can be written as

$$g(V) = \frac{2H(1-\lambda)V_{\parallel}G(V,v^2)}{v_{Ta}^2} f_{ao} - \sigma \frac{2v}{v_{Ta}^2} H(1-\lambda) \cdot \left[\int_{\lambda}^{1} d\lambda \frac{\langle (1-\lambda B/B_{max})^{1/2} \frac{cT_a}{e_a}\frac{\partial}{\partial\lambda}\int_{\ell_{max}}^{\ell} \frac{d\ell'}{B} \nabla\cdot[(1-\lambda B/B_{max})^{1/2}\, \hat{n}\times\vec{\nabla}V]\rangle}{\langle (1-\lambda B/B_{max})^{1/2}\rangle} \frac{\partial f_{ao}}{\partial V}\right] , \tag{48}$$

where

$$V_{\parallel} = \frac{\sigma v}{2}\int_{\lambda}^{1} d\lambda \frac{\langle B^2\rangle^{1/2}/B_{max}}{\langle (1-\lambda B/B_{max})^{1/2}\rangle} ,$$

$\sigma = \pm$ denotes the sign of the parallel velocity $\vec{v}_{\parallel}$, λ is the pitch angle variable $\lambda \equiv \mu B_{max}/E$, and H is the Heaviside step function. Note that the general form for the function g defined in Eq. (48) is different from that defined in Ref. 1. This is because in the axisymmetric tokamak the function g can only have one kind of pitch angle dependence, namely $V_{\parallel}$. However, in a nonaxisymmetric torus, the pitch angle dependence introduced by the pitch angle scattering operator operating on F_{1ac} in Eq. (47) cannot be put into a simple form such as $V_{\parallel}$. Thus we have to separate the g function into two parts: one has the simple pitch angle dependence $V_{\parallel}$, the other has the complicated pitch angle dependence introduced by $\overline{F}_{a1c}$ as in Eq. (48). The function $G(V,v^2)$ in Eq. (48) can be expanded in terms of the Laguerre polynomials $L_j^{(3/2)}$ as

$$G(V,v^2) = A_o(V)L_o^{(3/2)} + A_1(V)L_1^{(3/2)} + \dots \quad (49)$$

The coefficients $A_j(V)$ will be determined after we know the trapped particle distribution.

For the trapped particles, we separate Eq. (35) into axisymmetric and nonaxisymmetric parts[22]

$$v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1s} + (\vec{v}_{da}\cdot\vec{\nabla}V - \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b)\,\frac{\partial f_{ao}}{\partial V} = C_a(f_{a1s}) \ , \quad (50a)$$

$$v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1a} + \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b\,\frac{\partial f_{ao}}{\partial V} = C_a(f_{a1a}) \ , \quad (50b)$$

where $\langle A\rangle_b \equiv (\oint d\ell\ A/v_\parallel)/(\oint d\ell/v_\parallel)$ is the bounce average operator, and the integral $\oint$ is carried out between the turning points of the trapped particles. Note that the parallel electric $E_\parallel$ term is neglected in Eq. (50) since it will not affect the lowest order solution of Eq. (50). The separation technique performed to obtain Eq. (50) is slightly different from that originally proposed by Boozer.[22] The separation carried out in Ref. (22) is valid for a perturbation on a known axisymmetric system. However, the bounce averaging separation in Eq. (50) is valid for an arbitrary nonaxisymmetric system.

The solution to Eq. (50a) can be found by the standard technique used in the axisymmetric tokamak transport calculation, since upon bounce averaging the driving term it vanishes, i.e. $\langle\vec{v}_d\cdot\vec{\nabla}V - \langle\vec{v}_d\cdot\vec{\nabla}V\rangle_b\rangle_b = 0$. Neglecting the collisional term and integrating along the field line, we obtain

$$f_{als} = - \int_{\ell_t}^{\ell} \frac{d\ell'}{v_\parallel} (\vec{v}_{da}\cdot\vec{\nabla}V - \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b) \frac{\partial f_{ao}}{\partial V} + g_t , \tag{51}$$

where ℓ_t is one of the turning points of the trapped particles. The integration constant g_t can be determined from the higher order equation of Eq. (50a), and as in the axisymmetric tokamak $g_t = 0$.

The solution to Eq. (50b) is a little complicated since for different kinds of trapped particles, the solution could be different. For example in a rippled tokamak, banana trapped particles may be in the collisionless ripple plateau regime;[22] while ripple trapped particles may be in the $1/\nu$ ("collisionless") regime.[4] For trapped particles in the collisionless ripple plateau regime, the solution f_{alar} to Eq. (50b) is[22]

$$f_{alar} = -\int_{\ell_t}^{\ell} \frac{d\ell'}{v_\parallel} \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} + \oint \frac{d\ell'}{B} \vec{\nabla}\cdot(\frac{|v_\parallel|}{\Omega_a} \vec{B}\times\vec{\nabla}V) \frac{\partial f_{ao}}{\partial V} . \tag{52a}$$

Note that the second term in Eq. (52a) is even in $v_\parallel$ and is responsible for the collisionless ripple plateau transport. However, since this term is independent of the field line variable ℓ, it will not contribute to the parallel viscosity (cf., Section IV). For trapped particles in the $1/\nu$ regime, the solution to Eq. (50b) is[4,23]

$$f_{al a\nu} = f_{ala\nu}^{(-1)} - \int_{\ell_t}^{\ell} \frac{d\ell'}{v_\parallel} \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} + \int_{\ell_t}^{\ell} \frac{d\ell'}{v_\parallel} C_a(f_{ala\nu}^{(-1)}) = f_{ala\nu}^{(-1)} + f_{ala\nu}^{(o)} \tag{52b}$$

where $f_{ala\nu}^{(-1)}$ is the solution of the equation

$$\langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} = \langle C_a(f_{ala\nu}^{(-1)})\rangle_b \tag{53a}$$

and $f^{(o)}_{al a\nu}$ is the solution of the equation

$$v_{\parallel}\hat{n}\cdot\vec{\nabla}f^{(o)}_{ala\nu} + \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} = C_a(f^{(-1)}_{ala\nu}) \ . \tag{53b}$$

The explicit form of $f^{(-1)}_{ala\nu}$ is not needed in this calculation. Rather, we need only know that $\partial f^{(-1)}_{ala\nu}/\partial\ell = 0$, that $f^{(-1)}_{ala\nu}$ and $\partial f^{(-1)}_{ala\nu}/\partial\mu$ vanish for well-circulating particles, and that $f^{(-1)}_{ala\nu}$ is even in $v_{\parallel}$. Since $\partial f^{(-1)}_{ala\nu}/\partial\ell = 0$, $f^{(-1)}_{ala\nu}$ will not contribute to the parallel viscosity. The explicit form of $f^{(-1)}_{ala\nu}$ is obtained by many authors[17-19] and will not be presented here. The solution to Eq. (53b) was first obtained in Ref. (23) and is shown in Eq. (52b).

In summary, the trapped particle distribution can be written as

$$f_{alt} = f_{als} + f_{alar} + f_{ala\nu}$$

$$= -\int_{\ell_t}^{\ell} \frac{d\ell'}{v_{\parallel}} (\vec{v}_{da}\cdot\vec{\nabla}V - \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b) \frac{\partial f_{ao}}{\partial V} \cdots f_{als}$$

$$- \int_{\ell_t}^{\ell} \frac{d\ell'}{v_{\parallel}} \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} + \oint \frac{d\ell'}{B}\, \vec{\nabla}\cdot(\frac{|v_{\parallel}|}{\Omega_a} \vec{B}\times\vec{\nabla}V) \frac{\partial f_{ao}}{\partial V} \cdots f_{alar} \tag{54}$$

$$+ f^{(-1)}_{ala\nu} - \int_{\ell_t}^{\ell} \frac{d\ell'}{v_{\parallel}} \langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b \frac{\partial f_{ao}}{\partial V} + \int_{\ell_t}^{\ell} \frac{d\ell'}{v_{\parallel}} C_a(f^{(-1)}_{ala\nu}) \cdots f_{ala\nu} \ .$$

Note that the $\langle\vec{v}_{da}\cdot\vec{\nabla}V\rangle_b(\partial f_{ao}/\partial V)$ term in the axisymmetric distribution f_{als} is canceled out by the same terms in the distributions f_{alar} and $f_{ala\nu}$, as shown in Eq. (54).

Now we are in a position to calculate the function $G(V,v^2)$. Taking the $\int d\vec{v}(v_\parallel/B)L_j^{(3/2)}$ moments of Eqs. (45) and (54) we obtain

$$A_o = f_c^{-1} \frac{\langle B^2\rangle^{1/2}}{B_{max}} \{\frac{u_{\parallel a}}{B} + \int_{\ell_{max}}^{\ell} \frac{d\ell'}{B} \vec{\nabla}\cdot\vec{u}_{\perp a1} + \frac{2}{3}\frac{B}{B_{max}}\frac{cT_a}{e_aB}(\frac{P'}{P} + \frac{e_a\Phi'}{T_a}) \tag{55a}$$

$$\cdot\int_0^1 d\lambda \frac{\lambda\langle(1-\lambda B/B_{max})^{1/2}\frac{\partial}{\partial\lambda}\int_{\ell_{max}}^{\ell}\frac{d\ell'}{B}\vec{\nabla}\cdot[(1-\lambda B/B_{max})^{1/2}\hat{n}\times\vec{\nabla}V]\rangle}{\langle(1-\lambda B/B_{max})^{1/2}\rangle}\},$$

and

$$A_1 = -\frac{2}{5}P_a^{-1}f_c^{-1}\frac{\langle B^2\rangle^{1/2}}{B_{max}}\{\frac{q_{\parallel a}}{B} + \frac{5}{2}\int_{\ell_{max}}^{\ell}\frac{d\ell'}{B}P_a\vec{\nabla}\cdot(\frac{T_a'}{m_a\Omega_a}\hat{n}\times\vec{\nabla}V) + \frac{15}{4}P_a\frac{B}{B_{max}}\frac{cT_a'}{e_aB} \tag{55b}$$

$$\cdot\int_0^1 d\lambda \frac{\lambda\langle(1-\lambda B/B_{max})^{1/2}\frac{\partial}{\partial\lambda}\int_{\ell_{max}}^{\ell}\frac{d\ell'}{B}\vec{\nabla}\cdot[(1-\lambda B/B_{max})^{1/2}\hat{n}\times\vec{\nabla}V]\rangle}{\langle(1-\lambda B/B_{max})^{1/2}\rangle}$$

where the fraction of circulating particle f_c is defined as

$$f_c = \frac{3}{4}\frac{\langle B^2\rangle}{B_{max}^2}\int_0^1 \frac{\lambda\, d\lambda}{\langle(1-\lambda B/B_{max})^{1/2}\rangle}. \tag{56}$$

To obtain Eq. (55), we have used the definitions that $u_{\parallel a}/B = \int d\vec{v}(v_\parallel/B)(F_{alc}^{(c)} + f_{alt})$ and $q_{\parallel a}/B = -\int d\vec{v}(v_\parallel/B)L_1^{(3/2)}(f_{alc}^{(c)} + f_{alt})$. Note also that the term $\int_{\ell_t}^{\ell} d\ell' v_\parallel^{-1} C_a(f_{ala\nu}^{(-1)})$ will not contribute to the current. This can be shown by directly evaluating the integrals $\int d\vec{v}(v_\parallel/B)\int_{\ell_t}^{\ell} d\ell' v_\parallel^{-1}C_a(f_{ala\nu}^{(-1)})$. Physically this is related to the conservation of particles by the Coulomb collision operator.

Since the elements of $f_{a1c}^{(o)}$ and f_{a1t} are either odd in $v_\parallel$ or do not depend on the field line variable ℓ, $f_{a1c}^{(o)}$ and f_{a1t} will not contribute to the parallel viscosity. In order to find the nonzero contributions to the parallel viscosity we have to solve the higher order drift kinetic equation

$$v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1}^{(1)} + e_a E_\parallel v_\parallel \frac{\partial f_{ao}}{\partial E} = C_a(f_{a1}^{(o)}) \ , \tag{57}$$

where $f_{a1}^{(o)} = f_{a1c}^{(o)}$ for circulating particles and $f_{a1}^{(o)} = f_{a1t}$ for trapped particles. Recognizing that by taking the $\langle B\int d\vec{v} m_a v_\parallel L_j^{(3/2)} \ldots\rangle$ moments of Eq. (57), the left-hand side of Eq. (57) $v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1}^{(1)}$ will yield the parallel viscosities,[1] e.g.

$$\langle B\int d\vec{v} m_a v_\parallel v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1}^{(1)}\rangle = \langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle \ ,$$

$$\langle B\int d\vec{v} m_a v_\parallel L_1^{(3/2)} v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1}^{(1)}\rangle = -\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle \ ,$$

we do not have to solve for $f_{a1}^{(1)}$ explicitly, but only need to evaluate the moments $\langle B\int d\vec{v} m_a v_\parallel L_j^{(3/2)} C_a(f_{a1}^{(o)})\rangle$ in Eq. (57). The final form of the parallel viscosity will be greatly simplified if we subtract $[H(1 - \lambda) f_c^{-1} V_\parallel \langle B^2\rangle^{1/2}/B^2 \cdot \text{Eq. (47)}]$ from Eq. (57) to eliminate the $E_\parallel$ term and those momentum balance terms in the collision operator $C_a(f_{a1}^{(o)})$.[1] We then obtain

$$v_\parallel \hat{n}\cdot\vec{\nabla} f_{a1}^{(1)} = C_a(f_{a1}^{(o)}) - H(1-\lambda) f_c^{-1} v_\parallel \frac{\langle B^2\rangle^{1/2}}{B^2} \cdot \Big(\langle \frac{B}{v_\parallel} C_a(F_{a1c})\rangle + \langle \frac{B}{v_\parallel} C_a(g)\rangle\Big) - \Big(e_a E_\parallel v_\parallel \frac{\partial f_{ao}}{\partial E} - e_a \langle E_\parallel B\rangle \frac{\partial f_{ao}}{\partial E}\Big) . \tag{58}$$

Taking the $\langle B\int d\vec{v} m_a v_\parallel L_j^{(3/2)} \ldots\rangle$ moments of Eq. (58), we obtain

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle = \nu_a n_a m_a \langle B^2\rangle \frac{f_t}{f_c}\Big(\mu_{a1}\overline{U}_a + \frac{2}{5}\mu_{a2}\frac{\overline{q}_a}{P_a}\Big) , \tag{59a}$$

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle = \nu_a n_a m_a \langle B^2\rangle \frac{f_t}{f_c}\Big(\mu_{a2}\overline{U}_a + \frac{2}{5}\mu_{a3}\frac{\overline{q}_a}{P_a}\Big) , \tag{59b}$$

where $f_t \equiv 1 - f_c$, $\nu_a \equiv \tau_{aa}^{-1}$, and

$$\overline{U}_a = \frac{u_{\parallel a}}{B} + \int_{\ell_{max}}^{\ell} \frac{d\ell'}{B}\, \vec{\nabla}\cdot(\vec{u}_{\perp a})_1 + \frac{cT_a}{e_a\langle B^2\rangle}\Big(\frac{P_a'}{P_a} + \frac{e_a\Phi'}{T_a}\Big)G , \tag{60a}$$

$$\overline{q}_a = \frac{q_{\parallel a}}{B} + \int_{\ell_{max}}^{\ell} \frac{d\ell'}{B}\, \vec{\nabla}\cdot(\vec{q}_{\perp a})_1 + \frac{5}{2}\frac{cT_a'}{e_a\langle B^2\rangle} P_a G , \tag{60b}$$

$$G = \frac{3}{4}\frac{\langle B^2\rangle}{B_{max}^2}\Big\langle \int_0^1 d\lambda\, \frac{\lambda\Big\{B^2\int_{\ell_{max}}^{\ell} \frac{d\ell'}{B}\,\vec{\nabla}\cdot\Big(\frac{\vec{B}\times\vec{\nabla}V}{B^2}\Big) - \Big\langle\Big(1-\lambda\frac{B}{B_{max}}\Big)^{1/2}\int_{\ell_{max}}^{\ell}\frac{d\ell'}{B}\,\vec{\nabla}\cdot\Big[\frac{\vec{B}\times\vec{\nabla}V}{\big(1-\lambda\frac{B}{B_{max}}\big)^{1/2}}\Big]\Big\rangle\Big\}}{f_t\langle(1-\lambda B/B_{max})^{1/2}\rangle}\Big\rangle . \tag{60c}$$

In the toroidal symmetry case $\partial B/\partial\zeta = 0$, and Eq. (59) will reproduce the axisymmetric tokamak results. The constants μ_{aj} are defined in Ref. (1). For an electron-ion plasma $\mu_{i1} = 0.53$, $\mu_{i2} = -0.62$, $\mu_{i3} = 1.36$ for ions, and $\mu_{e1} = 1.53$, $\mu_{e2} = -2.12$, $\mu_{e3} = 4.62$ for electrons.

VII. Plasma Currents in a Nonaxisymmetric Torus

From quasi-neutrality we have

$$\vec{\nabla}\cdot\vec{J} = 0 \ , \tag{61}$$

and from lowest order force balance equation

$$\vec{J}\times\vec{B} = c\vec{\nabla}P \ , \tag{62}$$

where $P = P_e' + P_i'$. It is well-known that the plasma current lines $\vec{J}$ are straight lines in the Hamada coordinates[10,11] and can be written as

$$\vec{J} = j_t\vec{\nabla}V\times\vec{\nabla}\theta + j_p\vec{\nabla}\zeta\times\vec{\nabla}V \tag{63a}$$

where $j_t = j_t(V)$ and $j_p = j_p(V)$ are functions of the magnetic flux coordinate V only. The current $\vec{J}$ can also be expressed as

$$\vec{J} = j_{\parallel}\hat{n} + \vec{j}_{\perp} \tag{63b}$$

where $\vec{j}_{\perp} = c\vec{B}\times\vec{\nabla}P/B^2$. Taking the poloidal ($\vec{\nabla}\theta\cdot$Eq. (63)) and toroidal ($\vec{\nabla}\zeta\cdot$Eq. (63)) components of Eq. (63) we obtain

$$j_p = j_{\parallel}\hat{n}\cdot\vec{\nabla}\theta + \vec{j}_{\perp}\cdot\vec{\nabla}\theta = \vec{J}\cdot\vec{\nabla}\theta \ , \tag{64a}$$

$$j_t = j_{\parallel}\hat{n}\cdot\vec{\nabla}\zeta + \vec{j}_{\perp}\cdot\vec{\nabla}\zeta = \vec{J}\cdot\vec{\nabla}\zeta \ . \tag{64b}$$

From Eq. (64) we can express parallel current $j_{\parallel}$ as

$$j_{\parallel} = \frac{j_p}{\chi'} B - \frac{cP'}{\chi'} \frac{\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta}{B} , \tag{65a}$$

$$j_{\parallel} = \frac{j_t}{\psi'} B - \frac{cP'}{\psi'} \frac{\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta}{B} . \tag{65b}$$

Defining the Pfirsch-Schlüter current j_{ps} which has zero average surface ($\langle j_{ps}B\rangle = 0$) as

$$j_{ps} = - \frac{cP'}{\chi'} \left(\frac{\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta}{B} - \frac{\langle\vec{B}\cdot\vec{\nabla}V\cdot\vec{\nabla}\theta\rangle}{\langle B^2\rangle} B\right) , \tag{66a}$$

$$j_{ps} = - \frac{cP'}{\psi'} \left(\frac{\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta}{B} - \frac{\langle\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta\rangle}{\langle B^2\rangle} B\right) , \tag{66b}$$

we obtain

$$j_{\parallel} = j_{ps} + \frac{j_p}{\chi'} B - \frac{cP'}{\chi'} \frac{\langle\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta\rangle}{\langle B^2\rangle} B = j_{ps} + \frac{\langle\vec{J}\cdot\vec{B}\rangle}{\langle B^2\rangle} B , \tag{67a}$$

$$j_{\parallel} = j_{ps} + \frac{j_t}{\psi'} B - \frac{cP'}{\psi'} \frac{\langle\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta\rangle}{\langle B^2\rangle} B = j_{ps} + \frac{\langle\vec{J}\cdot\vec{B}\rangle}{\langle B^2\rangle} B . \tag{67b}$$

Note that the definitions of j_{ps} given in Eqs. (66a) and (66b) can be shown to be identical. The rest of the parallel current $\langle\vec{J}\cdot\vec{B}\rangle B/\langle B^2\rangle$ given in Eqs. (67),

$$\frac{\langle\vec{J}\cdot\vec{B}\rangle}{\langle B^2\rangle} B = \frac{j_p'}{\chi'} B - \frac{cP'}{\chi'} \frac{\langle\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta\rangle}{\langle B^2\rangle} B , \tag{68a}$$

$$\frac{\langle\vec{J}\cdot\vec{B}\rangle}{\langle B^2\rangle} B = \frac{j_t'}{\psi'} B - \frac{cP'}{\psi'} \frac{\langle\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\zeta\rangle}{\langle B^2\rangle} B , \tag{68b}$$

can also be shown to be equivalent and will be determined from the flux surface averaged parallel momentum and heat balance equations. In general, the parallel current $\langle \vec{J}\cdot\vec{B} \rangle$ consists of the classical Spitzer current $\langle j_s B \rangle$ driven by the parallel electric field $\langle E_\parallel B \rangle$ and the neoclassical current $\langle j_{nc} B \rangle$ driven by the parallel viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$, and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$ (Ref. 18). The bootstrap current $\langle j_{bs} B \rangle$ can be obtained from the neoclassical current $\langle j_{nc} B \rangle$ by setting $E_\parallel = 0$ and will be discussed in the next section.

VIII. Parallel Plasma Flows and Bootstrap Currents

The parallel plasma flows and bootstrap currents can be calculated from the flux-surface-averaged parallel momentum and heat flux balance equations[1,24]

$$\langle B(F_{\parallel a1} + n_a e_a E_\parallel) \rangle = \langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle \ , \tag{69a}$$

$$\langle BF_{\parallel a2} \rangle = \langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle \ . \tag{69b}$$

The expressions for the parallel viscosities $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a \rangle$ in both the Pfirsch-Schlüter and banana regimes were derived in Sections V and VI. The parallel friction forces for a simple electron-ion plasma are derived in Ref. (20) and are

$$F_{\parallel e1} = -F_{\parallel i1} = \ell_{11}^{e}(u_{\parallel i} - u_{\parallel e}) + \frac{2}{5}\,\ell_{12}^{e}\,\frac{q_{\parallel e}}{p_e} \ , \tag{70a}$$

$$F_{\parallel e2} = -\ell_{12}^{e}(u_{\parallel i} - u_{\parallel e}) + \frac{2}{5}\,\ell_{22}^{e}\,\frac{q_{\parallel e}}{p_e} \ , \tag{70b}$$

$$F_{\parallel i2} = -\frac{2}{5}\,\ell_{22}^{i}\,\frac{q_{\parallel i}}{p_i} \ , \tag{70c}$$

where $\ell_{11}^e = n_e m_e \nu_{ei}$, $\ell_{12}^e = 1.5\ \ell_{11}^e$, $\ell_{22}^e = 4.66\ \ell_{11}^e$, $\ell_{22}^i = \sqrt{2}\ n_i m_i \nu_{ii}$.

Since $|F_{\parallel i2}| \gg |\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_i\rangle|$ and $|F_{\parallel e2}| \gg |\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_e\rangle|$ for a large aspect ratio torus, from Eqs. (69b) and (70) we have, to the lowest order in $(|\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle|/|F_{\parallel a2}|)$,

$$u_{\parallel i} = u_{\parallel e} = u_{\parallel} \ , \tag{71a}$$

$$q_{\parallel e} = q_{\parallel i} = 0 \ . \tag{71b}$$

Due to conservation of momentum and quasi-neutrality, we have from Eqs. (69a) and (70a)

$$\sum_{i,e} \langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle = 0 \ . \tag{72}$$

Since $|\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_i\rangle|/|\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_e\rangle| \sim \sqrt{m_i/m_e} \gg 1$, the lowest order version of Eq. (72) is

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_i\rangle = 0 \ . \tag{73}$$

From the expressions of ion parallel viscosities in the Pfirsch-Schlüter and banana regimes and Eq. (73), we obtain the parallel plasma flow speeds in the Pfirsch-Schlüter $u_{\parallel ps}$ and banana $u_{\parallel b}$ regimes

$$u_{\parallel ps} = -\tilde{G}_{ps} \frac{cT_i}{eB} \left(\frac{P_i'}{P_i} + \frac{e\Phi'}{T_i} + 1.69 \frac{T_i'}{T_i}\right) \ , \tag{74a}$$

$$u_{\parallel b} = -\tilde{G}_b \frac{cT_i}{eB} \left(\frac{P_i'}{P_i} + \frac{e\Phi'}{T_i} - 1.17 \frac{T_i'}{T_i}\right) \ , \tag{74b}$$

where

$$\tilde{G}_{ps} = \frac{\langle \hat{n}\cdot\vec{\nabla}B \frac{1}{B}\frac{\partial B}{\partial\theta}\rangle}{\langle(\hat{n}\cdot\vec{\nabla}B)^2\rangle} \vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}\theta + (\theta \rightarrow \zeta) \ , \tag{75a}$$

$$\tilde{G}_b = G - \int_{\ell_{max}}^{\ell} \frac{d\ell'}{B} \frac{\vec{B}\times\vec{\nabla}V\cdot\vec{\nabla}B^2}{B^4} \ . \tag{75b}$$

The notation $(\theta \rightarrow \zeta)$ used in Eq. (75a) means this term is the same as the first term except that θ changes to ζ. In the axisymmetric tokamak limit $(\partial B/\partial\zeta = 0)$, it can be shown that $\tilde{G}_{ps} = \tilde{G}_b = B_t/B_p$ where B_p (B_t) is the poloidal (toroidal) magnetic field strength and thus Eq. (74) will reproduce the axisymmetric tokamak results.

Utilizing the expressions of the parallel friction forces given in Eqs. (70a) and (70b), we obtain an expression for the parallel current

$$\langle j_\parallel B\rangle = \langle ne(u_{\parallel i} - u_{\parallel e})B\rangle = \frac{\sigma_s}{ne} \langle F_{\parallel e1}B + \frac{\ell_{12}^e}{\ell_{22}^e} F_{\parallel e2}B\rangle \ , \tag{76}$$

where $\sigma_s = (ne)^2\ell_{22}^e/[\ell_{22}^e\ell_{11}^e - (\ell_{12}^e)^2]$ is the classical Spitzer conductivity. Employing the parallel momentum and heat flux balance equations Eq. (69), we can write Eq. (76) explicitly in terms of the parallel electric field and viscosities

$$\langle j_\parallel B\rangle = \frac{\sigma_s}{ne} \left(ne\langle E_\parallel B\rangle + \langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_e\rangle + \frac{\ell_{12}^e}{\ell_{22}^e} \langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_e\rangle\right) \ . \tag{77}$$

The current driven by the parallel electric field in Eq. (77) is the classical Spitzer current

$$\langle j_{\parallel s} B\rangle = \sigma_s \langle E_\parallel B\rangle \ . \tag{78a}$$

The current driven by the parallel viscosities in Eq. (77) is the neoclassical current

$$\langle j_{nc} B\rangle = \frac{\sigma_s}{ne} \left(\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_e\rangle + \frac{\ell_{12}^e}{\ell_{22}^e} \langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_e\rangle\right) \ . \tag{78b}$$

To calculate the bootstrap current, we use the expressions for the parallel flow speeds given in Eq. (74) to evaluate the parallel viscosities in Eq. (78b) and obtain

$$\langle j_{bs} B\rangle = -2.95 \frac{f_t}{f_c} c\tilde{G}_b \Big\{P'\Big(1 + \frac{\ell_{12}^e}{\ell_{22}^e} \frac{\mu_{e2}}{\mu_{e1}}\Big) + nT_i\Big[\Big(\frac{\mu_{i2}}{\mu_{i1}} + \frac{\ell_{12}^e}{\ell_{22}^e} \frac{\mu_{e2}}{\mu_{e1}} \frac{\mu_{i2}}{\mu_{i1}}\Big) + \Big(\frac{\mu_{e2}}{\mu_{e1}} + \frac{\ell_{12}^e}{\ell_{22}^e} \frac{\mu_{e3}}{\mu_{e1}}\Big) \frac{T_e'}{T_i'}\Big]\Big\} \ . \tag{79}$$

The bootstrap current in the Pfirsch-Schlüter regime is proportional to $(\omega_{te}/\nu_e)^2$, where ω_{te} is the electron transit frequency, and thus is negligible. Again, in the axisymmetric tokamak limit, Eq. (79) will reproduce the usual tokamak results[1] since $\tilde{G}_b = B_t/B_p$ in this limit.

In the banana regime the geometrical factor $\tilde{G}_b$ probably has to be evaluated numerically for an arbitrary nonaxisymmetric system. However, in the Pfirsch-Schlüter regime the geometrical factor $\tilde{G}_{ps}$ can be easily evaluated for an arbitrary nonaxisymmetric torus as long as a model for the magnetic field B is given. We will consider two interesting examples in this regime. For simplicity, we will neglect the heat flux and radial electric field in the present discussion.

For a rippled tokamak with model field $B = B_0(1 - \varepsilon\cos\theta - \delta\cos N\zeta)$ in which δ is the toroidal ripple strength, ε is the inverse aspect ratio, and N is the number of toroidal bumps, we have

$$u_{\parallel ps} = -\frac{cT_i}{eB_p}\left(\frac{1 - (N\delta)^2}{1 + \alpha^{-2}}\right)\left(\frac{P_i'}{P_i} + \frac{e\Phi'}{T_i} + 1.69\,\frac{T_i'}{T_i}\right) \qquad (80)$$

where $\alpha = \varepsilon/Nq\delta$, and $q = \varepsilon B_t/B_p$. If $\delta = 0$ (this corresponds to perfect toroidal symmetry), $u_{\parallel ps} = -(cT_i/eB_p)(P_i'/P_i + e\Phi'/T_i + 1.69\,T_i'/T_i)$ which is the usual tokamak result.[1,14] The parallel flow speed is reduced for $\alpha < 1$, which occurs for the relatively low ripple level of $\delta > \varepsilon/nq \sim 10^{-2}$ in a typical tokamak. In the limit $N\delta \gg 1$ (this corresponds to perfect poloidal symmetry) $u_{\parallel ps} = (cT_i/eB_p)(B_p/B_t)^2(P_i'/P_i + e\Phi'/T_i + 1.69\,T_i'/T_i)$, which is a factor of $(B_p/B_t)^2$ smaller than the toroidal symmetry value and the parallel flow is oppositely directed (i.e., $u_\parallel$ changes sign). The geometrical factor $\tilde{G}_b$ also has a similar qualitative behavior: namely, in the toroidally symmetric system, $\tilde{G}_b = B_t/B_p$, and as we increase the toroidal bumpiness of the system to the extent that the system is almost poloidally symmetric $\tilde{G}_b = -B_p/B_t$, which is again smaller than the toroidal symmetry value by a factor of $(B_p/B_t)^2$ and has an opposite sign. Thus, both the plasma parallel flow speed $u_{\parallel b}$ and bootstrap current $j_{\parallel b}$ can be reduced if we increase the toroidal bumpiness of the system.

Before we discuss a physical picture to explain these qualitative features, we consider another example, namely a stellarator with model field $B = B_0[1 - \varepsilon_t\cos\theta - \varepsilon_h\cos(\ell\theta - m\zeta)]$ where ε_t is the inverse aspect ratio, ε_h

is the helical perturbation, and ℓ and m are the poloidal and toroidal field periods of the windings. The parallel flow speed in the Pfirsch-Schlüter regime is then

$$u_{\parallel ps} = -\frac{cT_i}{eB_p} \frac{(1 + \ell^2\varepsilon_h^2/\varepsilon_t^2) - \ell mq\varepsilon_h^2/\varepsilon_t^2 - m^2\varepsilon_h^2}{1 + (\ell\varepsilon_h\varepsilon_t - mq\varepsilon_h/\varepsilon_t)^2} \left(\frac{P_i'}{P_i} + \frac{e\Phi'}{T_i} + 1.69\frac{T_i'}{T_i}\right) , \quad (81)$$

where $q = \varepsilon_t B_t/B_p$. For both toroidal ($\varepsilon_h \to 0$) and helical ($\varepsilon_t \to 0$) symmetry cases $u_{\parallel ps} = -(cT_i/eB_p)(P_i'/P_i + e\Phi'/T_i + 1.69\ T_i'/T_i)$. However, if $\varepsilon_t \sim \varepsilon_h$, $m\varepsilon_h \ll 1$, and $mq \gg \ell$ as is typically the case in stellarators, $u_{\parallel ps} \simeq (cT_i/eB_p)(\ell/mq)(P_i'/P_i + e\Phi'/T_i + 1.69\ T_i'/T_i)$, which is a factor of (ℓ/mq) smaller than either its toroidal or helical symmetry value. For the PROTO-CLEO stellarator,[25] the factor ℓ/mq is in the range of 1/10 to 1/5.

We will now give a simple physical picture to explain the qualitative behavior of the plasma parallel flows $u_{\parallel ps}$ and $u_{\parallel b}$ and bootstrap current $j_{\parallel b}$. Physically, this is because in a toroidally symmetric system plasma flows freely in the toroidal direction (conservation of toroidal angular momentum). Its poloidal flow tends to slow down due to viscous damping. However, the equilibrium diamagnetic flow $\vec{u}_\perp$ has a poloidal component. In order to eliminate this poloidal diamagnetic flow, plasma flows along the field line with a speed $u_\parallel$ such that its poloidal component cancels the poloidal diamagnetic flow and the net flow velocity $\vec{u} = \vec{u}_\perp + u_\parallel\hat{n}$ is in the toroidal direction, as shown in Fig. 1. [We note here that for a toroidally symmetric system it is radial electric field Φ' that adjusts itself to cancel the poloidal component of $u_\parallel$. The $u_\parallel$ is determined by the initial toroidal angular momentum on the flux surface. Thus, after this rapid relaxation process, the radial electric field Φ' is related to the initial toroidal angular momentum.[26] However, here we are more interested in comparison of the magnitudes of $u_\parallel$ for the same Φ', n', and T' in systems with different symmetry properties and initial conditions. The adjustment of $u_\parallel$ here should be interpreted as choosing different initial conditions such that their Φ' are the same after a few viscous damping times.] As the toroidal bumpiness of the system is increased the plasma cannot flow freely in the toroidal direction without viscous dissipation. In order to minimize the viscous heating, the net flow velocity is no longer in the toroidal direction -- see Fig. 2. The net flow direction is roughly along contours of constant B in the Pfirsch-Schlüter regime. This can be understood by putting the parallel viscosity $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle$ in the Pfirsch-Schlüter regime into form

$$\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle = 3p_a\tau_a\mu_{a1}\langle \hat{n}\cdot\vec{\nabla}B\,\frac{\vec{u}\cdot\vec{\nabla}B}{B}\rangle \ . \tag{82}$$

To obtain Eq. (82), we have neglected the heat flux term. Thus, $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_i\rangle = 0$ implies roughly $\vec{u}\cdot\vec{\nabla}B \simeq 0$, which means the flow velocity $\vec{u}$ lies on contours of constant mod B to avoid viscous damping. When the toroidal bumpiness is so large that the system is virtually poloidally symmetric, plasma can flow freely in the poloidal direction without viscous dissipation. In a manner analogous to that in the toroidally symmetric system, the toroidal component of the parallel plasma flow cancels the toroidal diamagnetic flow and the net flow is in the poloidal direction, as shown in Fig. 3. From Figs. 1 and 3, we see that the parallel flow speed in a poloidally symmetric system is a factor of $(B_p/B_t)^2$ smaller than that in a toroidally symmetric system, and oppositely directed. Thus, as the bumpiness in the toroidal direction is increased, parallel plasma flows and the bootstrap current are reduced. Note that Figs.

1, 2, and 3 are valid only if there is no heat flux. If the heat flux is present, there is a residual flow in the asymmetry direction which is proportional to the temperature gradient.

IX. Relaxation of Toroidal Rotation

For a general nonaxisymmetric torus, the plasma can still rotate or flow in the toroidal direction (see Fig. 2). This rotation can be slowed down by the nonambipolar flux Γ^a_{na} (Refs. 6, 9). The force that acts on the toroidal rotation is the $\vec{J}\times\vec{B}$ force induced by the nonambipolar flux. Taking the toroidal component of Eq. (4), we obtain

$$\frac{\partial}{\partial t}\left(\sum_a n_a m_a \frac{\vec{u}_a\cdot\vec{B}_t}{\psi'}\right) = \sum_a n_a e_a \frac{\vec{B}_t\cdot\vec{E}}{\psi'} + \sum_a n_a e_a \chi' \frac{\vec{u}_a\cdot\vec{\nabla}V}{c} + \sum_a \frac{1}{\psi'} \vec{B}_t\cdot\vec{F}_{a1} - \sum_a \frac{1}{\psi'} \vec{B}_t\cdot\vec{\nabla}p_a - \sum_a \frac{1}{\psi'} \vec{B}_t\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a \;. \tag{83}$$

Due to conservation of momentum, quasi-neutrality, and ambipolarity Eq. (83) can be simplified after averaging over a flux surface to yield only[6]

$$\frac{\partial}{\partial t}\left\langle\sum_a n_a m_a \frac{\vec{u}_a\cdot\vec{B}_t}{\psi'}\right\rangle = -\sum_a \frac{e_a\chi'}{c}\,\Gamma^a_{na} \;. \tag{84}$$

Equation (84) is the governing equation of the toroidal rotation. In the steady state, we obtain the usual criterion[9]

$$\sum_a e_a \Gamma^a_{na} = 0 \;, \tag{85}$$

which is used to determine the radial electric field Φ'.

X. Concluding Remarks

We have generalized the moment equation approach to nonaxisymmetric toroidal systems under the assumption that magnetic flux surfaces exist for such a system. To simplify the calculation, we employ the usual Hamada coordinates. In that coordinate system, the first order plasma flows are straight lines. The flux-friction relationships are derived from the momentum and heat flux balance equations. We find that the major difference between axisymmetric and nonaxisymmetric systems is that there are nonaxisymmetric (and nonambipolar) particle (Γ^a_{na}) and heat (q^a_{na}) fluxes, which are driven by the toroidal viscosities. The nonambipolar particle flux Γ^a_{na} will slow down the toroidal rotation and determine the radial electric field.

With the expressions of particle and heat fluxes given in Eqs. (23) and (26), the radial electric field determined by Eq. (85), and the Ohm's law Eq. (77), a three-dimensional transport calculation for a nonaxisymmetric torus can be carried out with the usual set of flux surface averaged particle and energy conservation equations,[1,14] as long as an equilibrium is given. Of course, we still need to calculate the parallel viscosities in the plateau regime in the future.

We have calculated the parallel viscosities in both the Pfirsch-Schlüter and banana regimes and determined the parallel plasma flow speeds and bootstrap current in a general nonaxisymmetric torus. The bootstrap current is driven by the parallel viscosities. We find that both parallel plasma flows and the bootstrap current can be reduced if we increase the toroidal bumpiness of the system. These results may provide theoretical explanations for the experimental observations that the toroidal rotation is reduced as the toroidal ripple strength is increased in the ISX-B experiment[3] and the fact that no bootstrap current is observed in the PROTO-CLEO stellarator.[25]

The bootstrap current in a stellarator has also been calculated in Ref. 27. In Ref. 27, the current is estimated by the conventional "kinetic" approach,[28] and the parallel momentum balance is not taken into account. Thus it leads to a slightly different result from ours, both in the numerical coefficients and in the form of the result. Also, whereas the work in Ref. 27 is restricted to the banana regime, here we have given calculations of the flows and bootstrap currents in both the banana and Pfirsch-Schlüter regimes.

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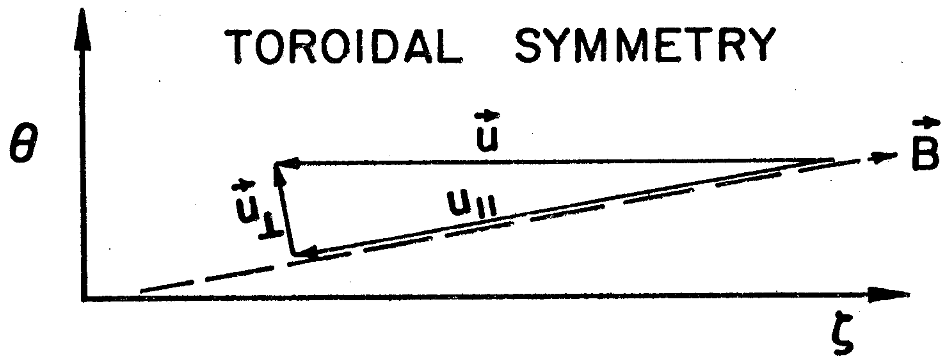

Figure 1. Flow patterns in a toroidally symmetric system.

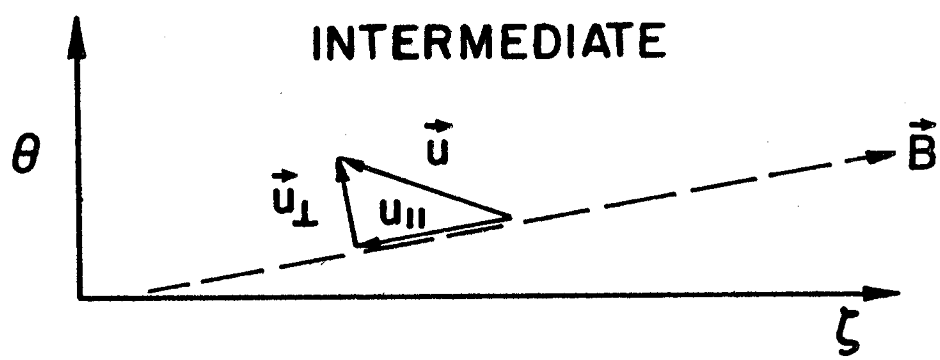

Figure 2. Flow patterns in a nonsymmetric system.

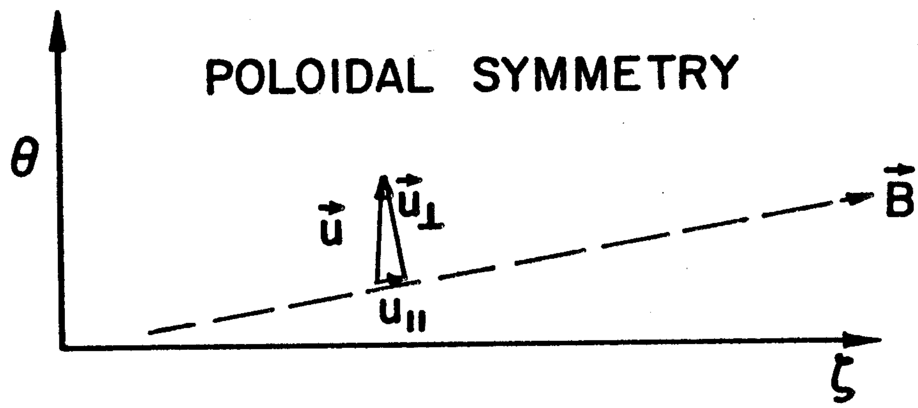

Figure 3. Flow patterns in a poloidally symmetric system.