A Theoretical Model for Hot Plasma Flowing to a Wall With Recycling

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Fusion Technology Institute
University of Wisconsin
1500 Engineering Drive
Madison, WI 53706

http://fti.neep.wisc.edu

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A.W. Bailey and G.A. Emmert

Department of Nuclear Engineering
University of Wisconsin
Madison, Wisconsin 53706 U.S.A.

Abstract

Plasma flow along field lines to material walls is a common property of divertors and limiters where the plasma is neutralized at a target plate. A theoretical model to describe the hot plasma flowing to the wall, where it is neutralized, and the cold plasma produced by ionization of thermal neutrals has been developed. A necessary ingredient in this model is the electrostatic potential, which turns out to be positive near the wall (relative to further in the plasma) before dropping the sheath. The treatment is kinetic rather than fluid-like and largely analytic. The results agree well with a numerical kinetic calculation by Morse[1] for a uniform field case. The model is being extended to allow non-uniform magnetic fields, as is appropriate for bundle divertors.

I. INTRODUCTION

In divertors and limiters, plasma flows to material walls along field lines where it is neutralized and a cold plasma is produced by ionization of the neutral gas. This cold plasma adds to the ion density near the wall. For quasi-neutrality to be maintained the electron density must also be increased, which from the Boltzmann relation implies that if enough cold ions are present then the potential must become positive near the wall (relative to further into the plasma) before dropping in the sheath (see Fig. 1). This potential rise accelerates the cold ions back along field lines into the main body of the plasma.

Knowledge of the magnitude of this potential rise is important for a proper theoretical understanding of these flows. In order to produce proper models of particle and heat flow for transport code calculations and divertor and limiter design calculations this understanding is required. Since the characteristic distances in flows of this nature are generally less than the collision mean free paths for ions, a fluid model of the plasma is inadequate and a kinetic treatment is needed. Morse et al.[1] have attacked the problem numerically: they treat a 1-D problem and model only the parallel velocity of the ions. They include the effects of molecular interactions and charge exchange in their calculations. A simpler, analytic method of solution is presented here, which neglects the details of atomic and molecular interactions, but which can be straightforwardly extended to include the effects of non-uniform magnetic fields, as in bundle divertors. Previous analyses of bundle divertors have been made by Nicolai[2] (fluid treatment) and by Emmert and Bailey[3] (kinetic treatment) for the case in which there is insufficient neutral gas in the divertor chamber for a rising electric potential in the divertor chamber, but an analysis has not previously been done for the case in which a large cold ion density produces a positive potential.

II. SOLUTION WITH A UNIFORM MAGNETIC FIELD

Consider first a plasma flowing to a wall along field lines in a uniform magnetic field. The hot ions will possess a Maxwellian velocity distribution which is truncated due to the fact that particles with parallel velocities (with $v_x < 0$) great enough to overcome the potential barrier near the wall will not be reflected (Fig. 1). The hot ion density will be

$$n_{i_H}(x) = \int d^3vf = A \int d^3ve^{-E/T_i} \tag{1}$$

$$= 2\pi A \int_{-\infty}^{\sqrt{(2e/M)(\phi_s-\phi(x))}} dv_{\parallel} \int_0^{\infty} dv_{\perp}\, v_{\perp} e^{-Mv_{\parallel}^2/2T_i}\, e^{-Mv_{\perp}^2/2T_i}\, e^{-e\phi(x)/T_i} .$$

Here $v_{\parallel}$ and $v_{\perp}$ are the parallel and perpendicular ion velocities, M is the ion mass, T_i is the ion temperature in units of energy, $\phi(x)$ is the electric potential at the point x along a field line, and ϕ_s is the electric potential at the plasma side of the sheath at the wall. The upper velocity limit is set by noting that, for particles with just high enough velocity to reach the wall,

$$\frac{1}{2} Mv_{\parallel}^2 + e\phi(x) = e\phi_s . \tag{2}$$

Performing the integration indicated in Eq. (1) gives

$$n_{i_H}(x) = A'e^{-\psi(x)}\left(1 + \text{erf}\sqrt{\psi_s - \psi(x)}\right) \tag{3}$$

where

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{4}$$

is the error function and

$$A' = \frac{A}{2}\left(\frac{2\pi T_i}{M}\right)^{3/2} \quad , \quad \psi(x) = \frac{e\phi(x)}{T_i} \quad , \quad \psi_s = \frac{e\phi_s}{T_i} \ . \tag{5}$$

Ions produced by ionization of neutral gas evolved from the wall will be born with energies which are negligible compared to the electric potential, and so their thermal motion can be ignored. The cold ions can be considered to simply have velocities at a point such that their kinetic energies match the difference in potential energy between that point and the point at which they were created. The cold ion density will therefore be

$$n_{i_C}(x) = \Gamma_w(1 - f_{pump}) \sqrt{\frac{M}{2T_i}} \int_0^x dx' \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}} \tag{6}$$

where Γ_w is the flux of hot ions to the wall, f_{pump} is the fraction of ions striking the wall which are pumped away and do not reappear as cold ions, and $h(x)$ is a function representing the spatial profile of the ionization of the neutral gas, normalized such that

$$\int_0^\infty dx'\ h(x') = 1 \ . \tag{7}$$

The flux of hot ions to the wall will be

$$\Gamma_w = -2\pi A\ e^{-e\phi_s/T_i} \int_{-\infty}^{0} dv_\parallel\ v_\parallel e^{-Mv_\parallel^2/2T_i} \int_0^\infty dv_\perp\ v_\perp e^{-Mv_\perp^2/2T_i} = \frac{2\pi T_i A}{M}\ e^{-\psi_s} \ . \tag{8}$$

The cold ion density will be

$$n_{i_C}(x) = \frac{A'}{\sqrt{\pi}} (1 - f_{pump}) e^{-\psi_s} \int_0^x dx' \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}} . \tag{9}$$

The effects of cold electrons produced by ionization which have not rethermalized are ignored in this analysis. The electrons are assumed to possess a Boltzmann distribution at a single temperature such that

$$n_e(x) = n_o e^{\tau\psi(x)} \tag{10}$$

where n_o is the electron density deep in the plasma and

$$\tau = \frac{T_i}{T_e} . \tag{11}$$

It is convenient to define a parameter λ such that

$$\lambda = \frac{n_o}{A'} . \tag{12}$$

λ is the ratio of the total ion density to the density of ions approaching the wall, evaluated far from the wall such that the potential goes to zero, e.g., in the divertor throat. A large value of λ thus implies a high density of cold ions and a small value of λ implies a low density of cold ions.

The scale length for potential variation is the mean free path for ionization of the cold gas. For typical parameters, this is long compared with the Debye length; consequently, the plasma can be assumed to be quasi-neutral outside the sheath, so that

$$n_{i_C} = n_e - n_{i_H} \tag{13}$$

or

$$\int_0^x dx' \frac{(1 - f_{pump})}{\sqrt{\pi}} e^{-\psi_s} \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}}$$
$$= \lambda e^{\tau\psi(x)} - e^{-\psi(x)}\left(1 + \text{erf}\left(\sqrt{\psi_s - \psi(x)}\right)\right) . \tag{14}$$

Let

$$\eta(x) = \psi_s - \psi(x) , \tag{15}$$

then assuming η varies monotonically with x

$$\int_0^\eta d\eta' \frac{(1 - f_{pump})}{\sqrt{\pi}} e^{-\psi_s} \frac{h(\eta')}{\sqrt{\eta' - \eta}} \frac{dx}{d\eta'} = \lambda e^{\tau\psi_s} e^{-\tau\eta} - e^{-\psi_s} e^{\eta} \left(1 + \text{erf}(\sqrt{\eta})\right) . \tag{16}$$

An Abel inversion gives[4]

$$\frac{(1 - f_{pump})}{\sqrt{\pi}} e^{-\psi_s} h(\eta) \frac{dx}{d\eta}$$
$$= \frac{1}{\pi} \frac{d}{d\eta} \int_0^\eta d\eta' \frac{\lambda e^{\tau\psi_s} e^{-\tau\eta'} - e^{-\psi_s} e^{\eta'} \left(1 + \text{erf}(\sqrt{\eta'})\right)}{\sqrt{\eta - \eta'}} . \tag{17}$$

Performing the integration gives

$$\frac{(1 - f_{pump})}{\sqrt{\pi}} e^{-\psi_s} h(\eta) \frac{dx}{d\eta} = \frac{1}{\pi} \frac{d}{d\eta} \Big[\frac{2\lambda}{\sqrt{\tau}} e^{\tau\psi_s} e^{-\tau\eta} D(\sqrt{\tau\eta}) - \sqrt{\pi}\, e^{-\psi_s} e^{\eta} \mathrm{erf}(\sqrt{\eta}) - \sqrt{\pi}\, e^{-\psi_s}(e^{\eta} - 1)\Big] \tag{18}$$

where

$$D(x) = \int_0^x e^{t^2}\, dt \tag{19}$$

is Dawson's integral. Integrating the cold ion distribution from the edge of the sheath into the plasma yields unity (i.e., all cold ions appear somewhere); so integrating Eq. (18) with respect to η from 0 to ψ_s gives

$$(1 - f_{pump}) e^{-\psi_s} = \frac{2\lambda}{\sqrt{\pi\tau}} D(\sqrt{\tau\psi_s}) - \mathrm{erf}(\sqrt{\psi_s}) - 1 + e^{\psi_s} . \tag{20}$$

This is a transcendental equation for the potential just outside the sheath edge. Different values of the cold ion density, and hence of λ, will give different values for this potential. There will, however, be a minimum value λ below which the potential would not be monotonic. A non-monotonic potential profile would result in ions being trapped in the potential well by collisions. These trapped particles would then increase the potential until the well would disappear. Hence a non-monotonic potential profile is unphysical and cannot be allowed. The minimum value for λ is obtained by performing the differentiation indicated in Eq. (18), giving

$$\frac{(1 - f_{pump})}{\sqrt{\pi}} e^{-\psi_s} h(\eta) \frac{dx}{d\eta} = \frac{1}{\pi} [\lambda e^{\tau\psi_s} (\frac{1}{\sqrt{\eta}} - 2\sqrt{\tau}\, e^{-\tau\eta} D(\sqrt{\tau\eta}))$$

$$-\sqrt{\pi}\, e^{-\psi_s} e^{\eta} \mathrm{erf}(\sqrt{\eta}) - \frac{e^{-\psi_s}}{\sqrt{\eta}} - \sqrt{\pi}\, e^{-\psi_s} e^{\eta}] \ . \tag{21}$$

Insisting that $dx/d\eta > 0$ at $\eta = \psi_s$ gives the requirement,

$$\lambda > \{\mathrm{erf}(\sqrt{\psi_s}) + \frac{e^{-\psi_s}}{\sqrt{\pi\psi_s}} + 1\} \ / \ \{\frac{e^{\tau\psi_s}}{\sqrt{\pi}} (\frac{1}{\sqrt{\psi_s}} - 2\sqrt{\tau}\, e^{-\tau\psi_s} D(\sqrt{\tau\psi_s}))\} \ . \tag{22}$$

Figure 2 depicts the profiles for several values of λ for the case of $T_i = T_e$ and no pumping. The ion distribution functions away from the wall are shown in Fig. 3. Only for the minimum value of λ is the distribution continuous. The discontinuity is caused by ions produced near $\eta = \psi_s$ which travel very slowly. For $\lambda = \lambda_{min}$ continuity results from $dx/d\eta$ being 0 at $\eta = \psi_s$, implying infinitesimally few ions with very low velocity. The regions of valid solution of Eq. (16) for $T_i = T_e$ subject to the constraint of Eq. (22) are shown in Fig. 4. The result is in good agreement with the limits of solution found by Morse et al.[1] The value of the potential rise is also in agreement.

III. SOLUTION FOR A BUNDLE DIVERTOR

For a pumped limiter or poloidal divertor the assumption of a uniform field is reasonable. In a bundle divertor, however, there will be a strong focusing of the magnetic field at the throat of the divertor where the diversion coils are located. This effect can be modeled by considering a stepped magnetic field as shown in Fig. 5. In this case allowance must be made for a jump in the potential due to the discontinuous magnetic field. Solution is obtained as before except that now quasi-neutrality must be required separately in the divertor chamber and in the divertor throat. Solution is now required for the normalized potential, ψ_d, immediately downstream of the drop in the magnetic field as well as the normalized potential at the sheath edge, ψ_s. Simultaneous solution of two nonlinear transcendental equations is thus required. Their derivation is given in the Appendix.

There will be three conditions limiting how small λ, and thus the cold ion density, can be: the requirement of monotonicity of the potential in the divertor, the requirement of a positive value for ψ_d, and the minimum value of λ for which a root of both equations exists. The potential for the case of $T_i = T_e$, no pumping, $\lambda = 3$, and various values of the downstream mirror ratio, $R_d = B_{throat}/B_{divertor\ chamber}$, are shown in Figs. 6 and 7.

IV. CONCLUSIONS

As has been shown, the potential rise experienced by a plasma flowing to a wall can be derived analytically giving values in excellent agreement with numerical treatments. Unlike the numerical treatments, however, solution of these analytical expressions requires minimal computer time. They are thus far more useful in time-dependent tokamak transport codes such as BALDUR[5] or WHIST[6] where a large number of evaluations may be required.

ACKNOWLEDGMENT

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REFERENCES

1. P. Gierszewski, P. McKenty, J. McCullen, and R. Morse, "Structure of Wall Plasmas Near Divertor Neutralizer Plates or Limiters," submitted for publication.

2. A. Nicolai, J. Nuclear Materials 93&94 (1980) 231.

3. G.A. Emmert and A.W. Bailey, J. Nuclear Materials 93&94 (1980) 238.

4. C.E. Pearson, Handbook of Applied Mathematics (Van Nostrand Reinhold, New York, 1974) p. 615.

5. D.E. Post et al., in Proc. 7th Int. Conf. on Plasma Physics and Controlled Nuclear Fusion, Innsbruck, 1978, Vol. 1 (IAEA, Vienna, 1979) p. 471.

6. W.A. Houlberg, A.T. Mense, and S.A. Attenberger, Nucl. Fusion 20 (1980) 811.

APPENDIX - DERIVATION OF THE BUNDLE DIVERTOR SOLUTION

The hot ion distribution in the throat of a bundle divertor will be a truncated Maxwellian in velocity space. Part of velocity space will be depopulated due to the fact that particles capable of reaching the target plate of the divertor do not return. On the boundary between the populated and depopulated regions of velocity space

$$\frac{1}{2} Mv^2_{\parallel throat} + \frac{1}{2} Mv^2_{\perp throat} = \frac{1}{2} Mv^2_{\perp sheath\ edge} + e\phi_s \tag{A1}$$

where ϕ_s is the potential just outside the sheath relative to the main body of the plasma, or equivalently

$$\frac{1}{2} Mv^2_{\perp throat} \left(1 - \frac{1}{R_d}\right) = e\phi_s - \frac{1}{2} Mv^2_{\parallel throat} \tag{A2}$$

where R_d is the downstream mirror ratio, $B_{throat}/B_{divertor\ chamber}$. So on the boundary

$$v_{\perp throat} = \sqrt{\gamma\left(\frac{2}{M} e\phi_s - v^2_{\parallel throat}\right)} \tag{A3}$$

where $\gamma = R_d/(R_d - 1)$. The hot ion density in the divertor throat will be

$$n_{i_{H_{throat}}} = 2\pi A \left\{\int_{-\infty}^{0} dv_\parallel \int_{0}^{\infty} dv_\perp\, v_\perp e^{-Mv_\parallel^2/2T_i}\, e^{-Mv_\perp^2/2T_i} + \int_{0}^{\sqrt{(2/M)e\phi_s}} dv_\parallel \int_{0}^{\sqrt{\gamma(2/M)e\phi_s - v_\parallel^2}} dv_\perp\, v_\perp e^{-Mv_\parallel^2/2T_i}\, e^{-Mv_\perp^2/2T_i}\right\} \tag{A4}$$

Doing the integration gives

$$n_{i_{H_{throat}}} = A'\{1 + \mathrm{erf}(\sqrt{\psi_s}) - \sqrt{R_d - 1}\ e^{-\gamma\psi_s}\ \mathrm{erf}(\sqrt{(\gamma - 1)\psi_s})\} \ . \tag{A5}$$

The hot ion density in the divertor chamber can be computed by integrating over the velocity distribution of those ions with $v_\parallel > 0$ at the throat, or equivalently

$$v_\perp^2(R_d - 1) < v_\parallel^2 + \frac{2}{M} e\phi$$

in the divertor chamber. So in the divertor chamber

$$n_{i_H}(x) = 2\pi A e^{-e\phi(x)/T_i} \int_{-\infty}^{\sqrt{(2e/M)(\phi_s-\phi(x))}} dv_\parallel \int_0^{\sqrt{(v_\parallel^2/(R_d-1))+(2e\phi(x)/M(R_d-1))}} dv_\perp\ v_\perp e^{-Mv_\perp^2/2T_i}\ e^{-Mv_\parallel^2/2T_i} \tag{A6}$$

or

$$n_{i_H}(x) = A'[e^{-\psi}(1 + \mathrm{erf}(\sqrt{\psi_s - \psi})) - \frac{1}{\sqrt{\gamma}} e^{-\gamma\psi}(1 + \mathrm{erf}(\sqrt{\gamma(\psi_s - \psi)}))] \ . \tag{A7}$$

As in the case of constant magnetic field the cold ion density in the divertor chamber will be

$$n_{i_C}(x) = \Gamma_w(1 - f_{pump}) \sqrt{\frac{M}{2T_i}} \int_0^x dx' \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}} \tag{A8}$$

but now the hot ion flux to the wall will be

$$\Gamma_w = -2\pi A e^{-e\phi_s/T_i} \int_{-\infty}^{0} dv_\parallel v_\parallel e^{-Mv_\parallel^2/2T_i} \int_0^{\sqrt{(v_\parallel^2/(R_d-1))+(2e\phi_s/M(R_d-1))}} dv_\perp v_\perp e^{-Mv_\perp^2/2T_i} \quad \text{(A9)}$$

or

$$\Gamma_w = 2\pi\left(\frac{T_i}{M}\right) A\left[e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}\right] . \quad \text{(A10)}$$

So the cold ion density will be

$$n_{i_C}(x) = (1 - f_{pump}) \frac{A'}{\sqrt{\pi}} \left(e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}\right) \int_0^x dx' \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}} . \quad \text{(A11)}$$

The equation for quasi-neutrality in the divertor chamber is thus (assuming a Boltzmann electron distribution)

$$\int_0^x dx' \frac{(1 - f_{pump})}{\sqrt{\pi}} \left(e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}\right) \frac{h(x')}{\sqrt{\psi(x') - \psi(x)}} = \lambda e^{\tau\psi(x)}$$

$$- e^{-\psi(x)}\left(1 + \text{erf}\left(\sqrt{\psi_s - \psi(x)}\right)\right) + \frac{1}{\gamma} e^{-\gamma\psi(x)}\left(1 + \text{erf}\left(\sqrt{\gamma(\psi_s - \psi(x))}\right)\right) . \quad \text{(A12)}$$

Solving as before by Abel inversion gives

$$(1 - f_{pump})\left(e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}\right) h(\eta) \frac{dx}{d\eta} = \frac{d}{d\eta} \left[\frac{2\lambda}{\sqrt{\pi\tau}} e^{\tau\psi_s} e^{-\tau\eta} D(\sqrt{\tau\eta}) - e^{-\psi_s} e^{\eta} \text{erf}(\sqrt{\eta})\right.$$

$$\left. - e^{-\psi_s}(e^{\eta} - 1) + \gamma^{-3/2} e^{-\gamma\psi_s} e^{\gamma\eta} \text{erf}(\sqrt{\gamma\eta}) + \gamma^{-3/2} e^{-\gamma\psi_s}(e^{\gamma\eta} - 1)\right] . \quad \text{(A13)}$$

If all of the ionization of neutral gas is assumed to take place between the sheath edge and the magnetic field jump, the cold ion distribution can be integrated out to give

$$(1 - f_{pump})(e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}) = \frac{2\lambda}{\sqrt{\pi\tau}} e^{\tau\psi_d} D(\sqrt{\tau\eta_d}) - e^{-\psi_d} \operatorname{erf}(\sqrt{\eta_d}) - e^{-\psi_d}$$

$$+ e^{-\psi_s} + \gamma^{-3/2} e^{-\gamma\psi_d} \operatorname{erf}(\sqrt{\gamma\eta_d}) + \gamma^{-3/2} e^{-\gamma\psi_d} - \gamma^{-3/2} e^{-\gamma\psi_s} \tag{A14}$$

where $\eta_d = \psi_s - \psi_d$ and $\psi_d = e\phi_d/T_i$, ϕ_d being the potential just downstream of the drop in the magnetic field. This is one transcendental equation for ψ_d and ψ_s.

A second equation relating ψ_d and ψ_s is obtained by requiring quasi-neutrality in the divertor throat. Using the value for the hot ion density in the throat given by Eq. (A5), the equation of quasi-neutrality is

$$R_d \int_0^{\psi_s} d\eta' \frac{1 - f_{pump}}{\sqrt{\pi}} (e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s}) \frac{h(\eta)}{\sqrt{\psi_s - \eta'}} \frac{dx}{d\eta'} = \lambda - 1 - \operatorname{erf}(\sqrt{\psi_s})$$

$$+ \sqrt{R_d - 1}\, e^{-\gamma\psi_s} \operatorname{erf}(\sqrt{(\gamma - 1)\psi_s}) . \tag{A15}$$

Now $h(\eta) \frac{dx}{d\eta}$ is zero for $\eta > \eta_d$ and is given by carrying out the differentiation in Eq. (A13) for $\eta < \eta_d$:

$$h(\eta) \frac{dx}{d\eta} = \{\lambda \frac{e^{\tau\psi_s}}{\sqrt{\pi}} [\frac{1}{\sqrt{\eta}} - 2\sqrt{\tau}\, e^{-\tau\eta} D(\sqrt{\tau\eta})] - e^{-\psi_s} e^{\eta} \operatorname{erf}(\sqrt{\eta}) - e^{-\psi_s} \frac{1}{\sqrt{\pi\eta}}$$

$$- e^{-\psi_s} e^{\eta} + \gamma^{-1/2} e^{-\gamma\psi_s} e^{\gamma\eta} \operatorname{erf}(\sqrt{\gamma\eta}) + \gamma^{-1} e^{-\gamma\psi_s} \frac{1}{\sqrt{\pi\eta}} \tag{A16}$$

$$+ \gamma^{-1/2} e^{-\gamma\psi_s} e^{\gamma\eta}\} \;/\; \{(1 - f_{pump})(e^{-\psi_s} - \frac{1}{\gamma} e^{-\gamma\psi_s})\} .$$

Using this expression, the integration in Eq. (A15) can be performed yielding

$$R_d\{\lambda[\frac{e^{\tau\psi_s}}{\pi}(\sin^{-1}(\frac{2\eta_d - \psi_s}{\psi_s}) + \frac{\pi}{2}) - e^{\tau\psi_s} + \frac{2e^{\tau\psi_s}}{\pi}\sin^{-1}(\sqrt{\frac{\psi_d}{\psi_s}})$$

$$+ \frac{2e^{\tau\psi_d}}{\pi}\alpha(\sqrt{\frac{\eta_d}{\psi_d}}, \tau\psi_d)] - [-e^{-\psi_s} + \frac{2e^{-\psi_s}}{\pi}\sin^{-1}(\sqrt{\frac{\psi_d}{\psi_s}})$$

$$+ \frac{2e^{-\psi_d}}{\pi}\alpha(\sqrt{\frac{\eta_d}{\psi_d}}, -\psi_d) + \frac{e^{-\psi_s}}{\pi}(\sin^{-1}(\frac{2\eta_d - \psi_s}{\psi_s}) + \frac{\pi}{2}) + \text{erf}(\sqrt{\psi_s})$$

$$- \text{erf}(\sqrt{\psi_d})] + \frac{1}{\gamma}[-e^{-\gamma\psi_s} + \frac{2e^{-\gamma\psi_s}}{\pi}\sin^{-1}(\sqrt{\frac{\psi_d}{\psi_s}}) + 2\frac{e^{-\gamma\psi_d}}{\pi}\alpha(\sqrt{\frac{\eta_d}{\psi_d}}, -\gamma\psi_d) \tag{A17}$$

$$+ \frac{e^{-\gamma\psi_s}}{\pi}(\sin^{-1}(\frac{2\eta_d - \psi_s}{\psi_s}) + \frac{\pi}{2}) + \text{erf}(\sqrt{\gamma\psi_s}) - \text{erf}(\sqrt{\gamma\psi_d})]\}$$

$$= \lambda - 1 - \text{erf}(\sqrt{\psi_s}) + \sqrt{R_d - 1}\, e^{-\gamma\psi_s}\, \text{erf}(\sqrt{(\gamma - 1)\psi_s})$$

where

$$\alpha(x,y) = \int_0^x \frac{e^{yt^2}}{1 + t^2} dt . \tag{A18}$$

This is the second transcendental equation for ψ_d and ψ_s which must be solved simultaneously with Eq. (A14).

Figure Captions

Figure 1: Conceptual potential profile and hot ion distribution function for hot plasma flowing to a wall with cold plasma recycling.

Figure 2: Profile of $\eta = \psi_s - \psi$ as a function of the integrated normalized cold ion source for $T_e = T_i$ and no pumping at various values of the parameter λ.

Figure 3: Ion distribution function for $T_e = T_i$ and no pumping. The function is continuous only for $\lambda = \lambda_{min} = 2.63$.

Figure 4: Minimum value of λ for a monotonic solution as a function of the pumping fraction for $T_e = T_i$. Shown is the limit set by Eq. (22) and the limit determined numerically by Morse et al.[1]

Figure 5: Conceptual model of the magnetic field and potential profiles in a bundle divertor. The jump in the magnetic field causes a jump in the electric potential.

Figure 6: Profiles of $\eta = \psi_s - \psi$ as a function of the integrated normalized cold ion source for $T_e = T_i$, no pumping, and $\lambda = 3$ at various values of the downstream mirror ratio $R_d = B_{throat}/B_{divertor\ chamber}$.

Figure 7: Normalized potential at the sheath edge, ψ_s, and normalized potential just downstream of the magnetic field drop, ψ_d, as function of the downstream mirror ratio $R_d = B_{throat}/B_{divertor\ chamber}$. Values are for $\lambda = 3$, $T_e = T_i$, and no pumping.

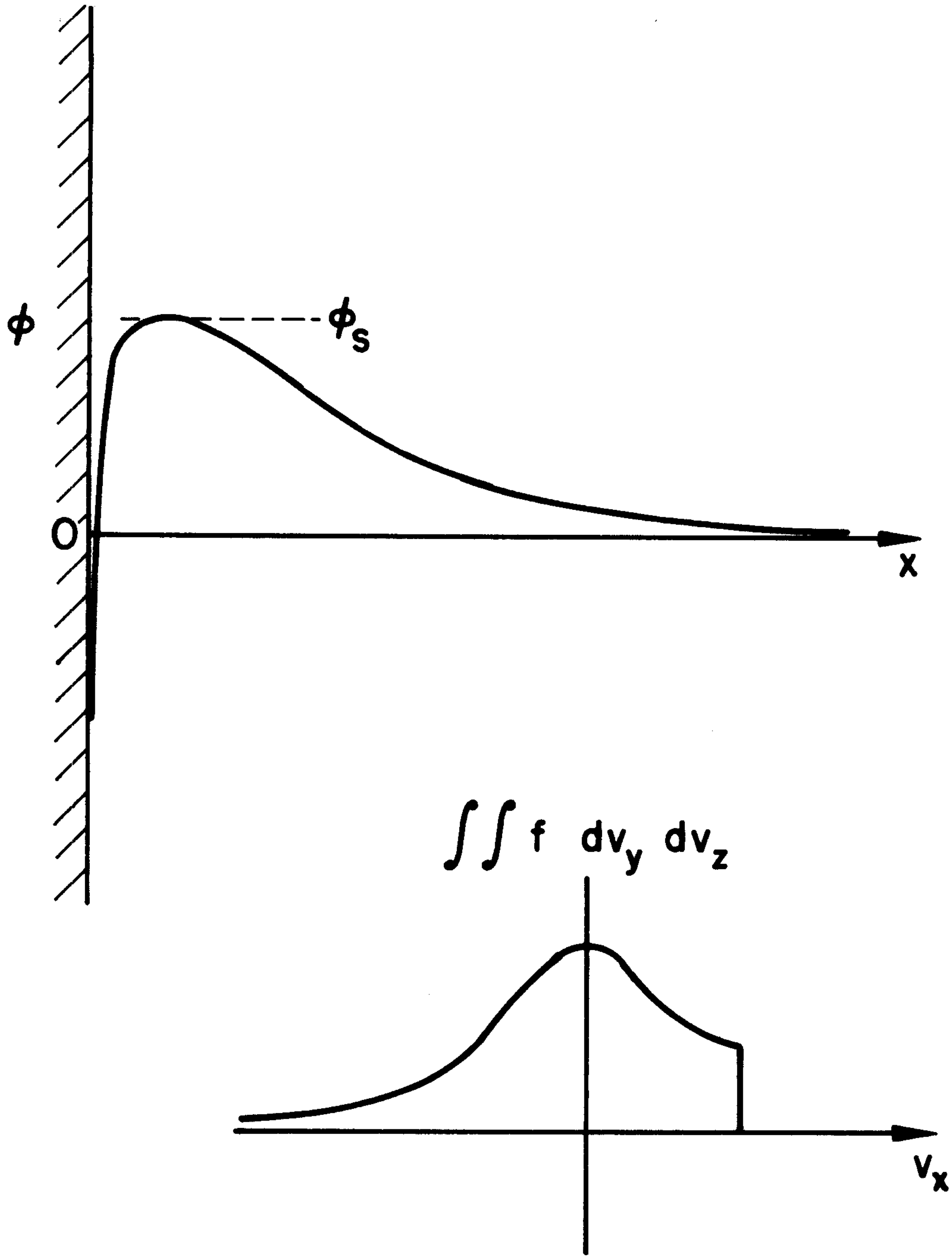
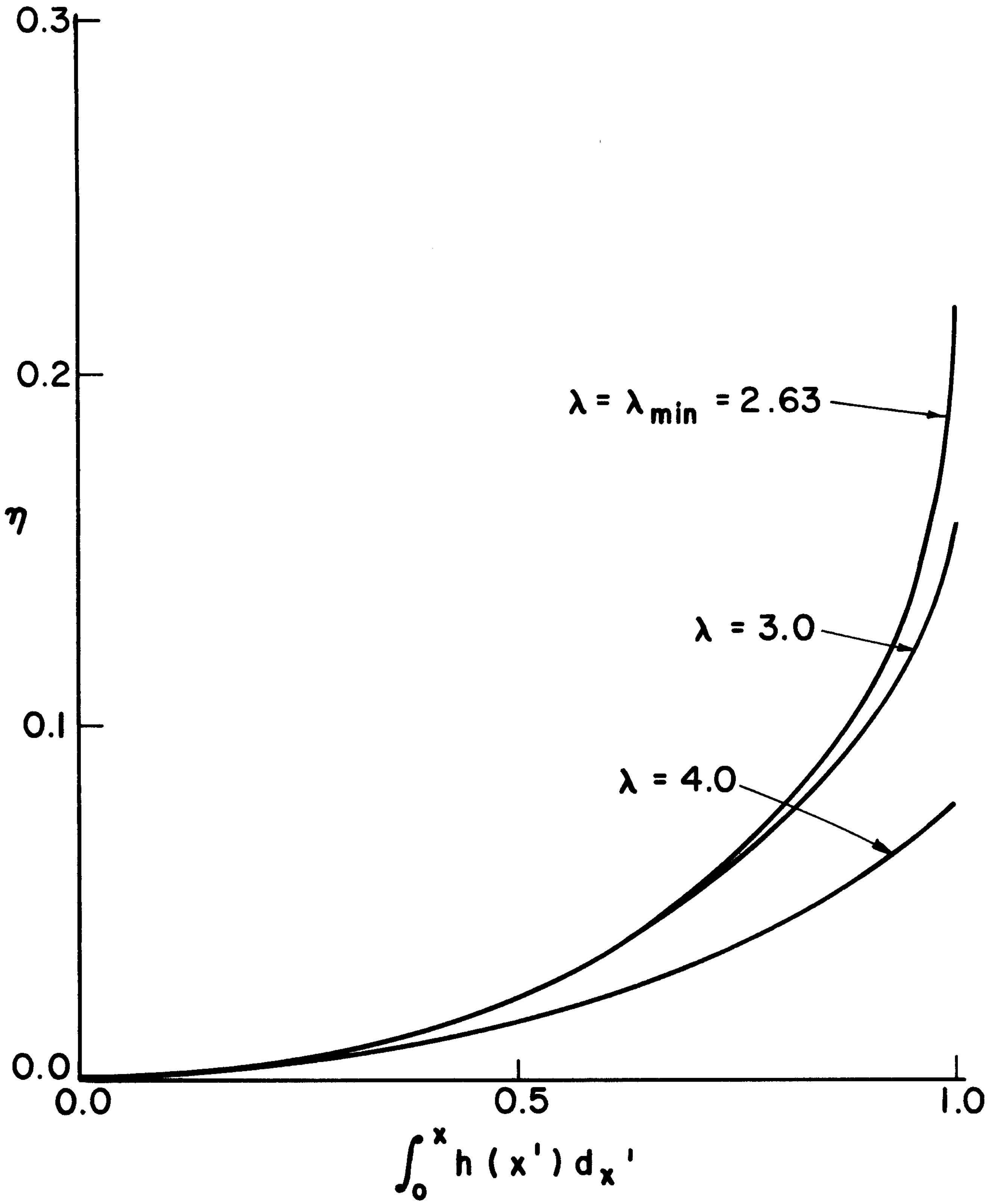
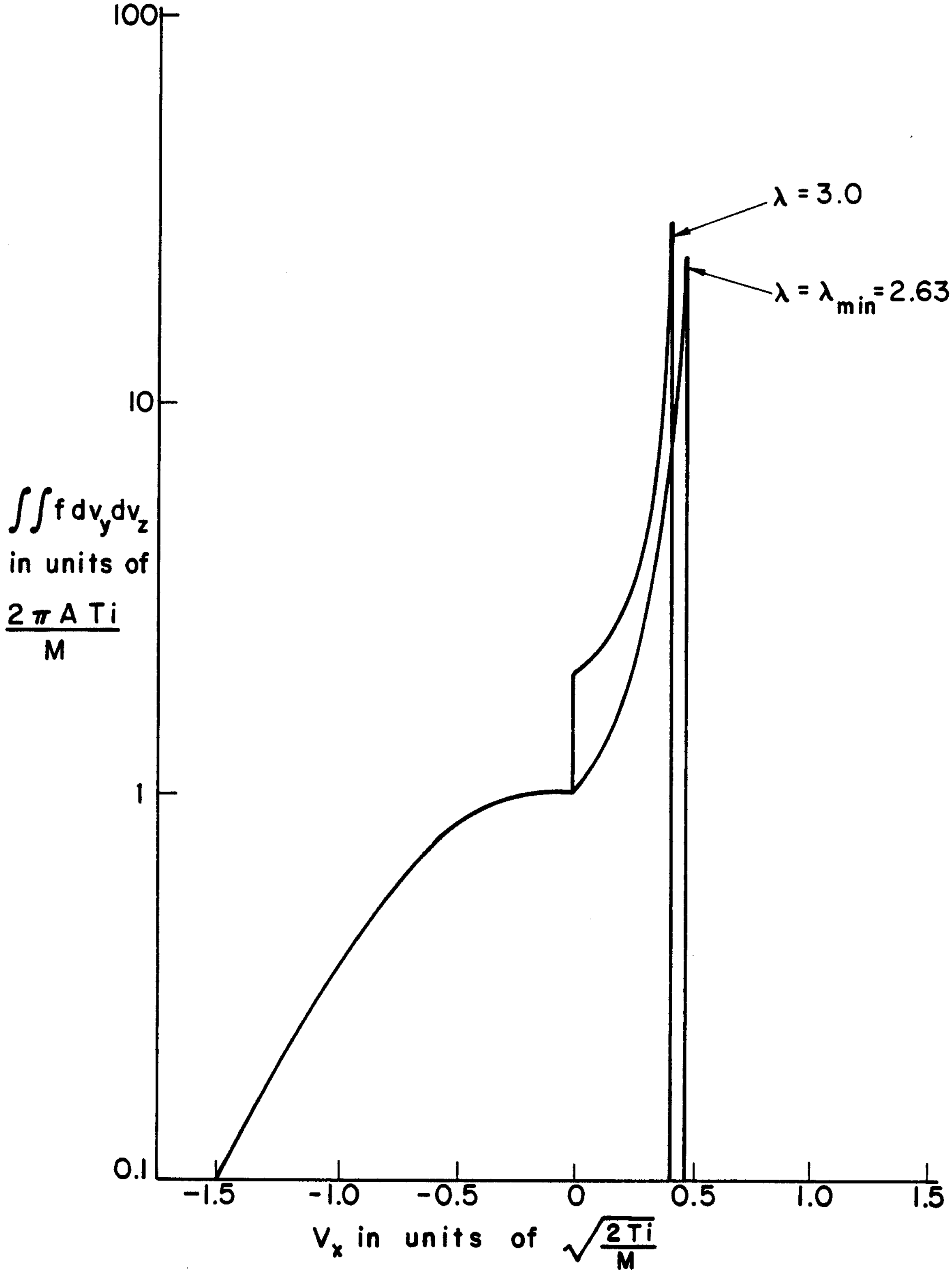


Figure 1.



0.3
0.2
0.1
0.0
η
$\lambda = \lambda_{min} = 2.63$
$\lambda = 3.0$
$\lambda = 4.0$
0.0
0.5
1.0
$\int_0^x h(x')\,dx'$

Figure 2.



100
10
1
0.1
∫∫f dv_y dv_z
in units of
2π A Ti / M
λ = 3.0
λ = λ_min = 2.63
−1.5
−1.0
−0.5
0
0.5
1.0
1.5
V_x in units of √(2Ti/M)

Figure 3.

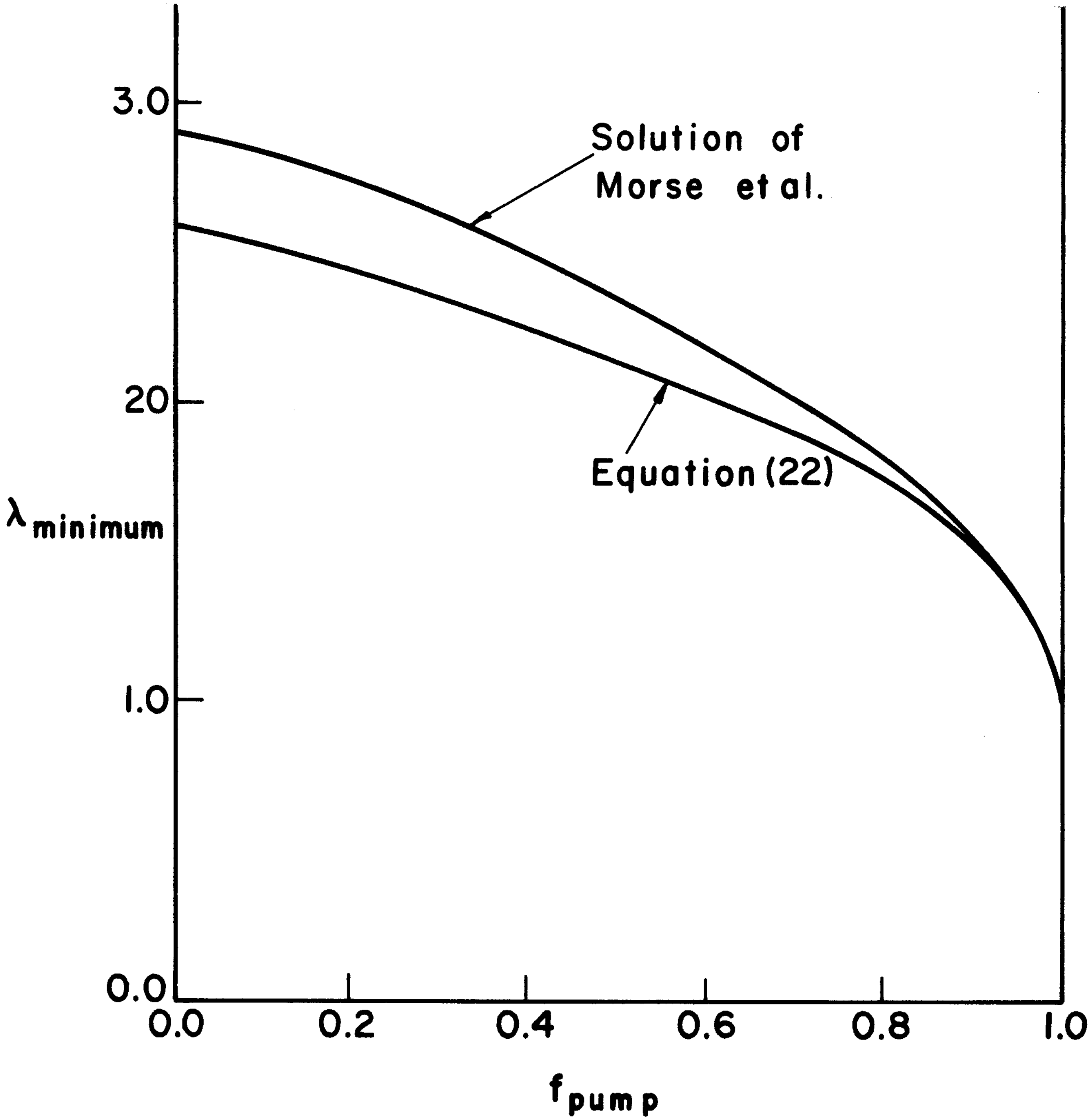

3.0
20
1.0
0.0
λ minimum
Solution of
Morse et al.
Equation (22)
0.0
0.2
0.4
0.6
0.8
1.0
f pump

Figure 4.

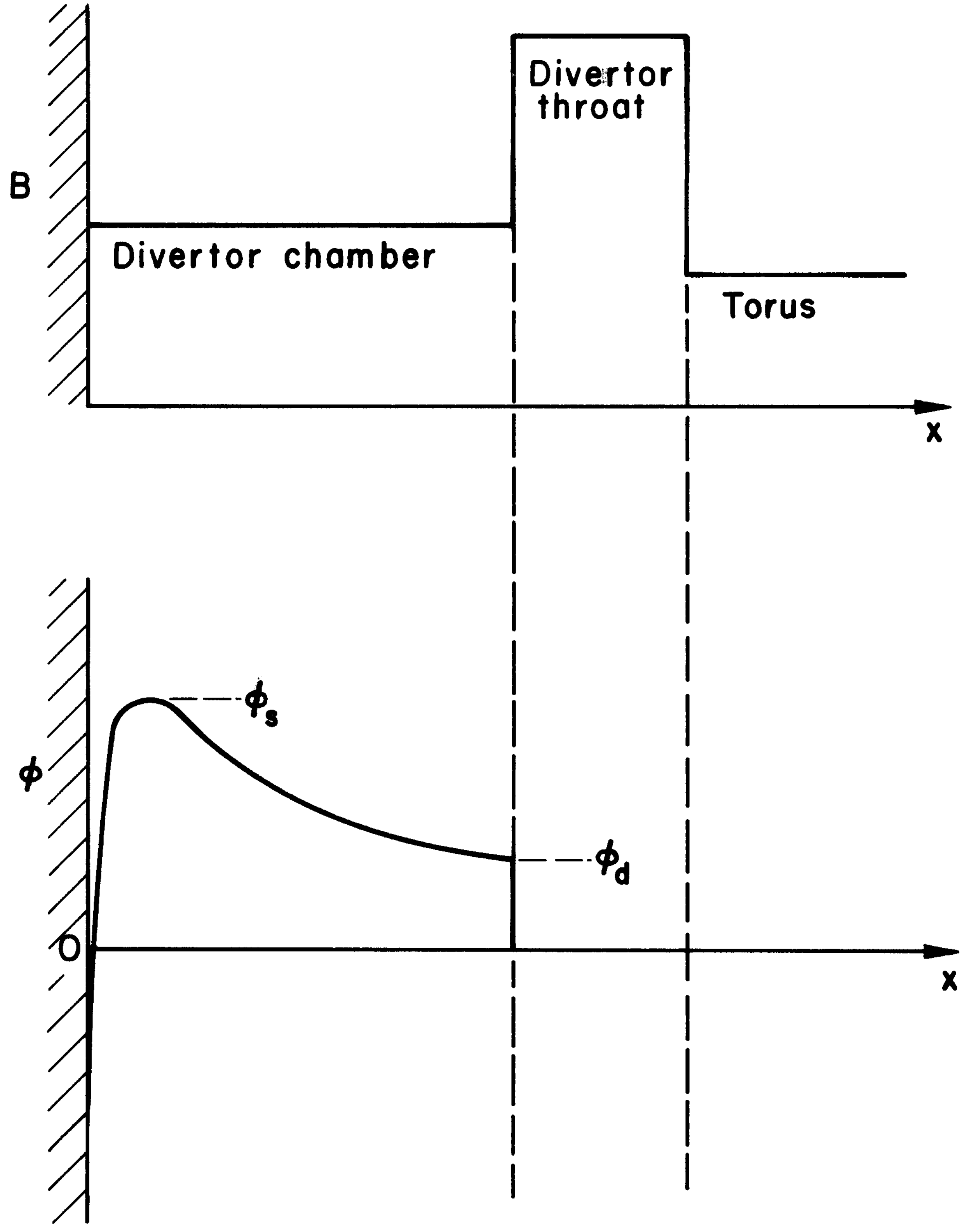


Figure 5.

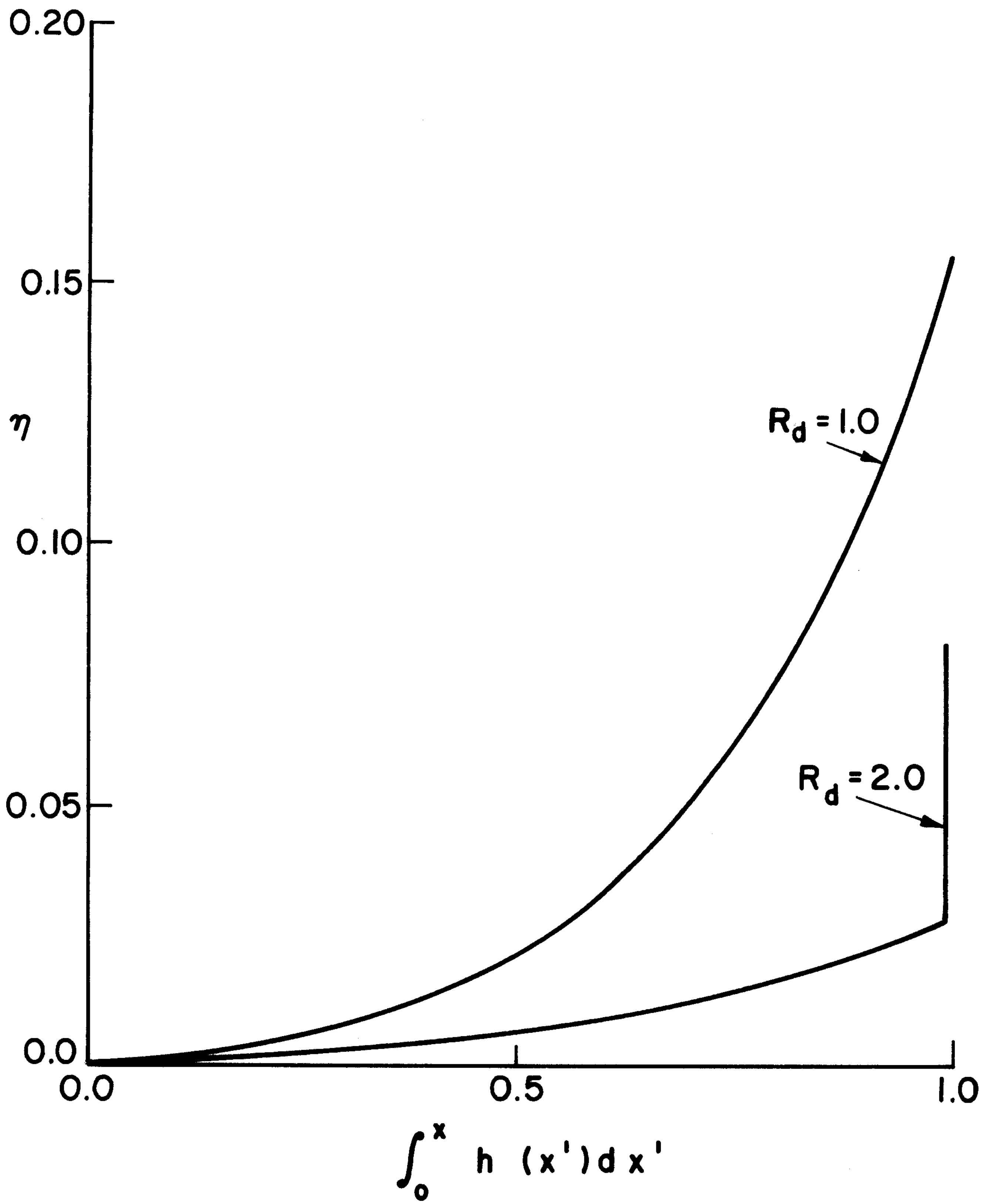


Figure 6.

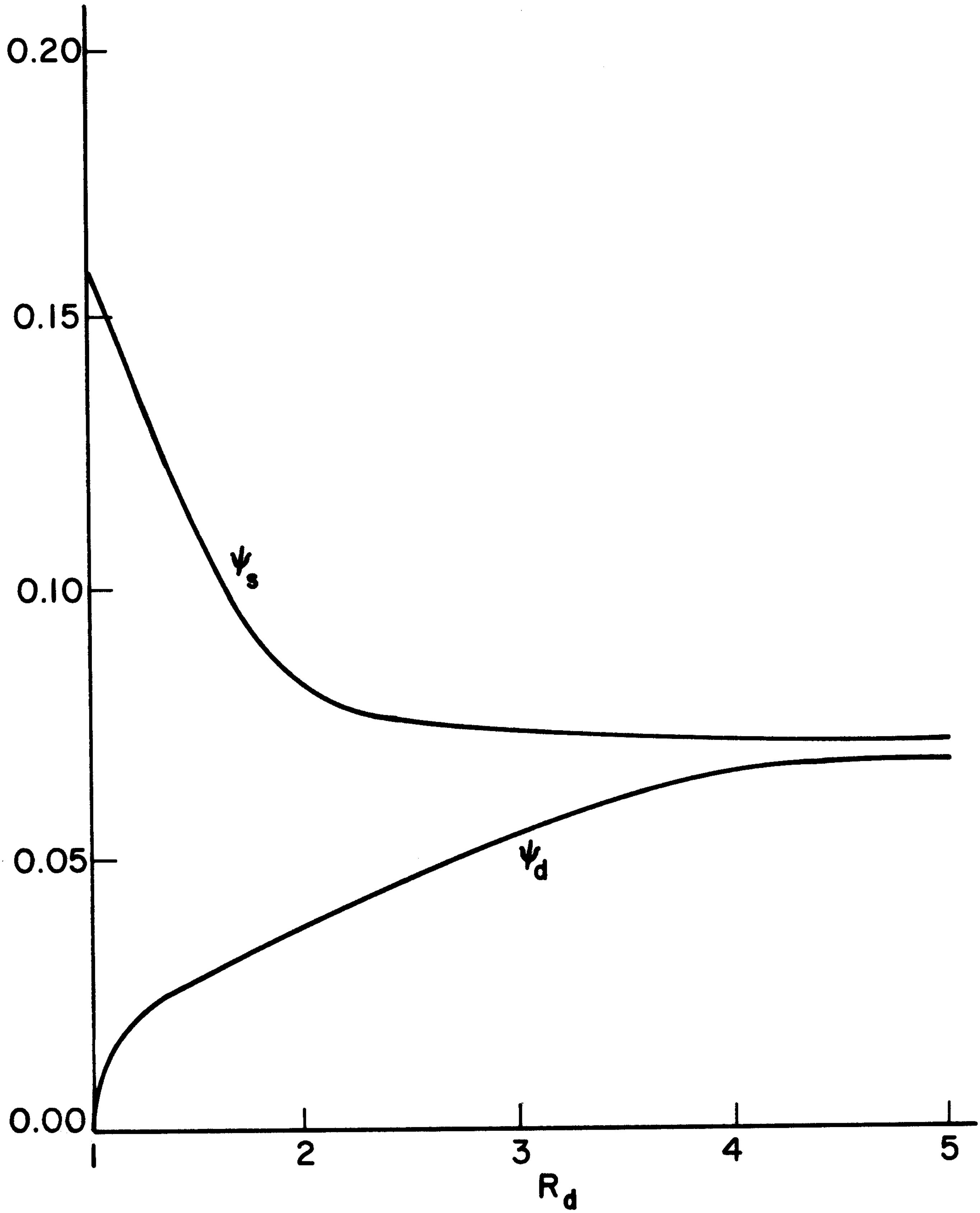


Figure 7.