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Abstract

A direct calculation of the parallel viscosity $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle$ in the plateau regime for a large aspect ratio, axisymmetric tokamak is presented.

I. Introduction

The recent development of the moment (or fluid) equation approach[1,2] to neoclassical transport theory has provided a physical and comprehensive procedure for calculating the transport properties of an axisymmetric tokamak in long mean free path regimes, namely, the banana and plateau regimes. The crucial step in the moment equation approach is to obtain the expressions for the average parallel viscosity tensors $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle$ and $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\Theta}_a \rangle$ in terms of the plasma flows. Direct calculations of these parallel viscosities in terms of the plasma flows in the banana regime have been reported in Ref. 2. However, direct calculations of parallel viscosities in the plateau regime are not available. In this paper, we present direct calculations of the parallel viscosities in the plateau regime in the large aspect ratio limit.

II. Calculations of Parallel Viscosities

To first order in the gyroradius expansion,[4] the viscous stress tensor $\overleftrightarrow{\pi}_a$ is of the diagonal form $\overleftrightarrow{\pi}_a = (P_{\parallel a} - P_{\perp a})(\hat{n}\hat{n} - \overleftrightarrow{I}/3)$, where $P_{\parallel a} = \int d^3v \; m_a v_\parallel^2 f_a$, $P_{\perp a} = \int d^3v \; (m_a v_\perp^2/2) f_a$, m_a and f_a are the mass and particle distribution function for species a, $v_\parallel$ ($v_\perp$) is the parallel (perpendicular) speed, $\hat{n} = \vec{B}/B$ is the unit vector and $\overleftrightarrow{I}$ is the unit tensor. The parallel viscosity is then

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle = \langle (P_{\perp a} - P_{\parallel a}) \, \hat{n} \cdot \vec{\nabla} B \rangle \ . \qquad (1)$$

To zeroth order in the gyroradius expansion, $f_a = f_{ao}$ is a Maxwellian.[3] Thus, to obtain a nonzero $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle$, we need to solve the drift kinetic equation to obtain the first order gyroradius correction f_{a1} to f_{ao}.

In an axisymmetric tokamak with $\vec{B} = I(\psi)\,\vec{\nabla}\phi + \vec{\nabla}\phi \times \vec{\nabla}\psi$,[5] the drift kinetic equation for f_{a1} is

$$v_\parallel\, \hat{n} \cdot \vec{\nabla} f_{a1} + \vec{v}_d \cdot \vec{\nabla}\psi f'_{ao} - \frac{e_a v_\parallel E_\parallel}{T_a} f_{ao} = C_a(f_{a1}) \ . \tag{2}$$

Here, $\vec{v}_d \cdot \vec{\nabla}\psi = -\, I(\hat{n} \cdot \vec{\nabla}B)(v_\parallel^2 + v^2)/2\Omega_a B$, $\Omega_a = e_a B/m_a c$, $E_\parallel$ is the parallel electric field, $C_a(f_{a1})$ is the linearized collision operator, and the prime denotes $\partial/\partial\psi$. We consider a model collision operator[5]

$$C_a(f_{a1}) = \nu_d^a L f_{a1} + \Delta\nu^a \frac{v_\parallel u_{a1}}{v^2} f_{ao} + \frac{2v_\parallel}{v_{ta}^2} \left(\sum_b r_{ba} \nu_s^{ab}\right) f_{ao} \ . \tag{3}$$

The notation used in Eq. (3) and the rest of the paper is defined in Ref. 5. For this collision operator the conventional solution[5] to Eq. (2) can be written as

$$f_{a1} = -\frac{I v_\parallel}{\Omega_a} f'_{ao} + \frac{v_\parallel}{v} \left(\frac{Iv}{\Omega_a} \frac{f'_{ao}}{f_{ao}} + \frac{\bar{u}_{a1}}{v} \frac{\Delta\nu^a}{\nu_d^a} + \sum_b \frac{2\bar{v} r_{ba}}{v_{ta}^2} \frac{\nu_s^{ab}}{\nu_d^a} + \frac{e_a \bar{E}_\parallel v}{T_a \nu_d^a} \right)$$

$$\times \frac{B}{\langle B \rangle^{1/2}} f_{ao} + h_{a1} \ , \tag{4}$$

where h_{a1} is the localized (in velocity space) solution.

In order to calculate the parallel viscosity in terms of the plasma flow, we put f_{a1} into a different form, namely,

$$f_{a1} = -\frac{I v_\parallel}{\Omega_a} f'_{ao} + \frac{v_\parallel}{v} S_a(\psi,v) \frac{B}{\langle B^2 \rangle^{1/2}} f_{ao} + h_{a1} \ , \tag{5}$$

where the function $S_a(\psi,v)$ is used to replace the terms inside the parentheses of Eq. (4). The choice of this particular form for f_{a1} is motivated by the fact that all terms inside the parentheses of Eq. (5) are functions only of the flux coordinate ψ and speed v. The function $S_a(\psi,v)$ will be expanded in terms of Laguerre polynomials[2] $L_j^{(3/2)}$ of order (3/2), i.e.,

$$S_a(\psi,v) = \frac{2v}{v_{ta}^2} (A_o L_o^{(3/2)} + A_1 L_1^{(3/2)} + \ldots) , \tag{6}$$

where $L_o^{(3/2)} = 1$, $L_1^{(3/2)} = 5/2 - x_a^2$, and $x_a = v^2/v_{ta}^2$. If the energy dependence of $S_a(\psi,v)$ is sufficiently smooth, we can neglect $j \geq 2$ terms. Since the localized solution h_{a1} is smaller than the rest of the terms on the right side of Eq. (5) by a factor of order[5] $\varepsilon(\nu_d^a/\omega_{ta})^{-1/3}$, we can determine $S_a(\psi,v)$ correct to order $\varepsilon(\nu_d^a/\omega_{ta})^{-1/3}$ without knowing h_{a1}. To determine $S_a(\psi,v)$ take the $v_\parallel L_o^{(3/2)}$ and $v_\parallel L_1^{(3/2)}$ moments of Eq. (5), and find $A_o = u_a \langle B^2\rangle^{1/2}$, $A_1 = -(2/5)(q_a/P_a)\langle B^2\rangle^{1/2}$, where $u_a = (u_{\parallel a} - V_{1a})/B$, $q_a = (q_{\parallel a} - 5\, P_a V_{2a}/2)/B$, $u_{\parallel a} = \int d^3v\, v_\parallel f_{a1}$, $q_{\parallel a} = \int d^3v\, v_\parallel (m_a v^2/2) f_{a1}$, $V_{1a} = -IcT_a(P_a'/P_a + e_a\Phi'/T_a)/e_aB$, $V_{2a} = -IcT_a'/e_aB$, $P_a = n_aT_a$, and Φ is the equilibrium potential. It can be shown[2] that u_a and q_a are only functions of the flux coordinate ψ.

Substituting Eq. (5) into Eq. (2), we obtain an equation for h_{a1}

$$v_\parallel\, \hat{n} \cdot \vec{\nabla} h_{a1} - \frac{1}{2}(v^2 - v_\parallel^2)(\hat{n} \cdot \vec{\nabla} B)\frac{S_a}{v}\langle B^2\rangle^{-1/2} f_{ao} - \nu_d^a L h_{a1}$$

$$\frac{1}{2}(v^2 - v_\parallel^2)\, \hat{n} \cdot \vec{\nabla}(\ln B)\frac{\partial h_{a1}}{\partial v_\parallel} = \nu_d^a \frac{v_\parallel}{v} f_{ao}\Big(\frac{e_a E_\parallel v}{T_a \nu_d^a} + \sum_b \frac{\Delta\nu^{ab}}{\nu_d^a}\frac{u_{a1}}{v} + \sum_b 2\frac{v r_{ba}}{v_{ta}^2}\frac{\nu_s^{ab}}{\nu_d^a}$$

$$+ I\frac{v}{\Omega_a}\frac{f_{ao}'}{f_{ao}} - S_a\frac{B}{\langle B^2\rangle^{1/2}}\Big) . \tag{7}$$

To obtain Eq. (7), we have used the model collision operator Eq. (3). We will solve Eq. (7) in the large aspect ratio limit, i.e., $\varepsilon \ll 1$ with the usual tokamak coordinates (r,θ,ϕ). In the plateau regime, $(v_\parallel/v) \sim (\nu_d^a/\omega_{ta})^{1/3} \ll 1$, and the driving term on the right side of Eq. (7) is smaller than that on the left side by a factor $(\nu_d^a/\omega_{ta})^{4/3}$ and hence can be neglected. Neglecting the $(v_\parallel/v)^2 \sim (\nu_d^a/\omega_{ta})^{2/3}$ term and the mirror force term (the term that is proportional to $\partial h_{a1}/\partial v_\parallel$) which is important in calculating the friction force[5] but is not important in calculating parallel viscosities, we obtain the usual plateau regime equation

$$v_\parallel \; \hat{n} \cdot \vec{\nabla} h_{a1} - \frac{v}{2} (\hat{n} \cdot \vec{\nabla} B) \frac{S_a}{\langle B^2\rangle^{1/2}} f_{ao} = \nu_d^a L h_{a1} \; , \tag{8a}$$

whose solution is

$$h_{a1} = \frac{\varepsilon}{2} S_a \left(\frac{\nu_d^a}{\omega_{ta}}\right)^{-1/3} \frac{B}{\langle B^2\rangle^{1/2}} f_{ao} \cdot \int_0^\infty \sin(\theta - p\tau)\, e^{-\tau^3/6} \, d\tau \; . \tag{8b}$$

Since $(v_\parallel/v) \sim (\nu_d^a/\omega_{ta})^{1/3} \ll 1$ in the plateau regime, the expression for $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle$ can be simplified to

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle \simeq \langle \int d^3 v \, \frac{m_a v^2}{2} f_{a1} \, (\hat{n} \cdot \vec{\nabla} B) \rangle \; . \tag{9}$$

With Eqs. (8b) and (9), we obtain the parallel viscosity in terms of u_a and q_a

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_a \rangle = 3 \, \langle (\hat{n} \cdot \vec{\nabla} B)^2 \rangle \left(\mu_{a1} \, u_a + \frac{2}{5} \mu_{a2} \frac{q_a}{P_a} \right) , \tag{10}$$

where $\mu_{a1} = \frac{\sqrt{\pi}}{3} \frac{P_a}{\omega_{ta}} \Gamma(3)$, $\mu_{a2} = \frac{\sqrt{\pi}}{3} \frac{P_a}{\omega_{ta}} [\Gamma(4) - \frac{5}{2} \Gamma(3)]$, and Γ is the Gamma function. To obtain Eq. (10), we have converted $\sin\theta$ into $(\hat{n} \cdot \vec{\nabla} B)(Rq/\varepsilon B)$, as is appropriate for the large aspect ratio limit employed here. Similarly, the parallel heat viscosity $\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle$ in the plateau regime is

$$\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\Theta}_a\rangle = 3\,\langle(\hat{n}\cdot\vec{\nabla}B)^2\rangle\,(\mu_{a2}\,u_a + \frac{2}{5}\,\mu_{a3}\,\frac{q_a}{P_a})\ , \tag{11}$$

where $\mu_{a3} = \frac{\sqrt{\pi}}{3}\frac{P_a}{\omega_{ta}}\,[\Gamma(5) - 5\Gamma(4) + \frac{25}{4}\,\Gamma(3)]$.

III. Discussion and Conclusion

The parallel viscosities obtained in Eqs. (10) and (11) are the same as those given in Ref. 2. In Ref. 2, $\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_a\rangle$ and $\langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\theta}_a\rangle$ were obtained indirectly from the calculation of the particle and heat fluxes with the model collision operator given in Eq. (3). Note that the result obtained here in Eqs. (10) and (11) depends only on the pitch angle scattering part of the collision operator and hence is not very sensitive to the detailed form of the collision operator utilized. The direct method developed in this paper provides a logically correct way of calculating the parallel viscosity for the plateau regime in the moment equation approach[2] to neoclassical transport theory.

Acknowledgement

This work was supported by the U.S. Department of Energy Contract No. DE-AC02-80-ER-53104.

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