Thermal Barrier Trapping Current in a Tandem Mirror

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ABSTRACT

An analytic expression for the trapping current in the thermal barrier cell of a tandem mirror due to pitch-angle scattering has been obtained. The overall system is assumed to be maintained in steady state by particle injection in the central cell and charge exchange pumping is assumed in the barrier cell. Trapping of ions in the thermal barrier due to Coulomb interactions sets an irreducible minimum against which any pumping mechanism must compete. A series of boundary value problems in the various regions of phase space is obtained based on the smallness of the bounce time with respect to the collision time. A high barrier mirror ratio, long central cell and deep barrier electrostatic potential are implicit in the analysis. For conditions of interest a Lorentz collision operator describes reasonably well the kinetic problem which we have solved using a square-well approximation. Analytic results agree with numerical results within expected limits on the order of the inverse of the barrier mirror ratio (~ 10 to 20%). A boundary layer mechanism has been proposed which limits the trapping current value as the pumping rate is increased.

1. INTRODUCTION

The thermal barrier of a tandem mirror [1] is a region of depressed magnetic field and electrostatic potential (Fig. 1) that provides thermal isolation between the electrons in the central cell and the electrons in the plugs, thereby making selective heating of the plug electrons possible. An extra coil placed between the central cell and plug expands the magnetic field lines in the barrier region, which decreases the density of particles and creates a potential well. The passing particles streaming from the central cell are trapped in this magnetic and electrostatic well by Coulomb scattering

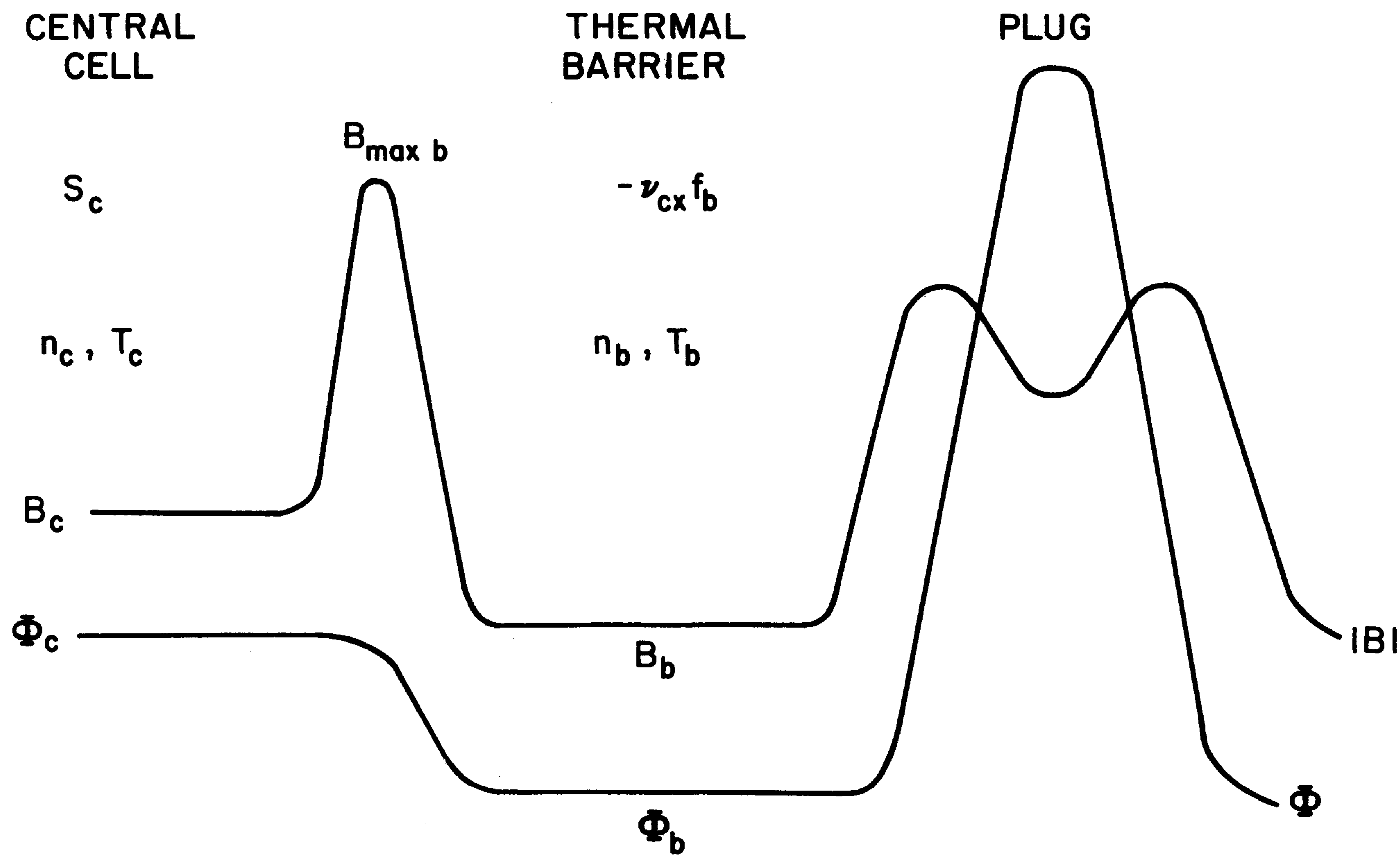

Figure 1 Sketch of the axial magnetic field and electrostatic potential profiles for a tandem mirror with thermal barrier cells between the plugs and the central solenoid. Indicated are the densities, temperatures and sources/sinks of the central cell and thermal barrier.

and this process occurs at the classical rate, at least at low density [2]. If such a scattering and filling of the thermal barrier proceeds unimpeded, the density between the mirrors fills to the ambient density and the potential variation disappears. Maintenance of the potential profile therefore requires removal of those ions that become trapped. An effective thermal barrier implies a large barrier mirror ratio ($R_b = B_{maxb}/B_b$) and a rapid pumpout of the trapped ions [1]. A basic question, then, is to know the trapping rate against which any pumping scheme must compete.

Numerical studies using Fokker-Planck and Monte Carlo codes have been performed and empirical expressions have been obtained [3-5] for the trapping rate. In this paper we derive an analytic expression for the trapping rate in the thermal barrier of a tandem mirror using a Lorentz collision operator in a square well approximation. The result is found to agree with the numerical results [4,5] within ~ 10 to 20%, which is the anticipated accuracy of the analytical calculations. In the analysis it is assumed that the thermal barrier configuration [3] is stable and also that the particles remain on the same flux tube throughout the history of their containment [3]. Therefore, our calculation sets an irreducible minimum for the trapping current, namely, that due to Coulomb interactions.

The basic problem that we need to solve is one in which a long central cell with density n_c, temperature T_c, magnetic field B_c and electrostatic potential ϕ_c is maintained in steady state by a source of particles S_c and adjacent thermal barrier cells with density n_b, temperature T_b, magnetic field depression B_b and electrostatic potential dip ϕ_b are maintained in steady state by neutral beam charge exchange of frequency ν_{cx} (Fig. 1). Magnetic field peaks ($B = B_{maxb}$) separate the barrier regions from the central cell.

We wish to obtain the trapping current into the thermal barriers. This is defined as the number of ions that are trapped per second.

It is convenient to define two parameters which will determine the solution to the problem: η is the ratio of the number of particles in the barrier region to the number of particles in the central cell (typically $\eta \sim 10^{-2} \ll 1$, i.e., "long" central cell) and $\lambda_{cx} = \nu_s^b/\nu_{cx}$ is the ratio between the collisional and charge exchange frequencies in the barrier region (typically, $\lambda_{cx} < 1$ and it decreases as the pumping rate is increased).

In Section 2 we will introduce the kinetic equation that we solve in Section 3 in the various phase space regions of the tandem mirror. In Section 4 we obtain an expression for the trapping current and present some results. In Section 5 we summarize the conclusions of this work.

2. KINETIC EQUATION

The distribution function of particles in the plasma is given by the solution of the kinetic equation $df/dt = C(f) + S$, where d/dt is the derivative along particle trajectories, $C(f)$ is the Coulomb collision operator and S is a source of particles. Since the gyrofrequency (Ω) of the particles is very large compared to any other frequency characterizing the problem, we can average the kinetic equation over the gyrophase angle (expansion in $O(\omega_b/\Omega)$, ω_b = bounce frequency), to obtain the drift kinetic equation which for low β (β = particle pressure/magnetic pressure) in an axisymmetric plasma can be written as

$$\frac{\partial f}{\partial t} + \vec{v}_\parallel \cdot \vec{\nabla} f - q\vec{\nabla}\phi \cdot (\vec{v}_\parallel + \vec{v}_D) \frac{\partial f}{\partial E} = C(f) + S \quad , \tag{1}$$

where, consistent with the guiding center picture of particle motion, $f = f(E,\mu,s,t)$ (we suppress a parametric field line dependence (r,θ) and consider azimuthal symmetry in velocity space after the gyrophase average); $v_{\parallel} = [2/m(E - \mu B(s) - q\phi(s))]^{1/2}$, $B(s)$ and $\phi(s)$ are the magnetic field and electrostatic potential at the position s along a field line, E and μ are the energy and magnetic moment adiabatic invariants; and $\overline{v}_D$ is the drift velocity. Further, we can expand the drift kinetic equation (1) in $O(\nu_s/\omega_b)$ [6], except in a thin boundary layer region around the loss boundary in phase space (in which the effective collision frequency is of the order of the bounce frequency), whose treatment will not be considered here. The lowest order distribution function satisfies the Vlasov equation whose characteristics are the trajectories of the collisionless particle motion. That is, the distribution function when expressed in terms of constants of motion does not change along a field line. In a steady state situation the bounce average of the first order equation yields the consistency condition $\oint ds/v_{\parallel}\ [C(f) + S] = 0$, where the integral extends to the quasi-periodic orbit of a particle and the integrand is evaluated at the velocity space mid-plane variables v and $\zeta = (\vec{v} \cdot \vec{B})/vB = v_{\parallel}/v$ at $s = 0$, which is the point of minimum $B(s)$ through which all particles pass; alternatively it could be evaluated using the quantities $E = 1/2\ mv^2$ and $\mu = mv_{\perp}^2/2B$.

Assuming that the magnetic field and electrostatic potential change occurs abruptly at the ends of the cell ("square-well" approximation) the last condition reduces to

$$C(f) = -S \quad , \tag{2}$$

where S represents any source or sink introduced to maintain a steady state situation. The two mentioned conditions for the lowest order distribution function determine the boundary value problem to be solved in Section 3.

If near and many-body collisions can be neglected (accuracy of $O(1/\ln \Lambda)$, $\ln \Lambda$ = Coulomb logarithm), the Fokker-Planck collision operator for Coulomb interactions [7] is appropriate. This operator predicts that significant changes in the distribution function of the charged particles will occur in times of the order of $v_{av}^3/n\Gamma$ where v_{av} is the average speed of the particles of interest, n is the local density, $\Gamma \equiv 4\pi q^4 \ln\Lambda/m^2$, q = electric charge and m = particle mass [6]. A Fokker-Planck operator valid for distribution functions azimuthally symmetric in velocity space is given in Ref. [7]; it can be expressed in terms of the Rosenbluth potential $g(\vec{v}) = \int d\vec{v}' f(\vec{v}')|\vec{v} - \vec{v}'|$ (Ref. [7]), which is the mean value of the relative velocity between the test and field particles of velocities $\vec{v}$ and $\vec{v}'$, respectively. This operator is nonlinear and integro-differential. For an analytic treatment of it, a "known" background distribution function is usually assumed so that it becomes a differential operator. Assuming that the distribution function of field particles is separable [6,8] in the form $f(v,\zeta) = f(v)M(\zeta)$ and piecewise differentiable in the interval $\zeta\epsilon[-1,+1]$, we can express it as a series expansion of Legendre polynomials in the form [7]

$$F(x,\zeta) = \sum_{n=0}^{\infty} a_n(x)\ P_n(\zeta) \quad ,$$

with

$$M(\zeta) = \sum_{n=0}^{\infty} M_n P_n(\zeta) \quad ,$$

$$a_n(x) = M_n F(x) \quad ,$$

$$M_n = \frac{\int_{-1}^{+1} M(\zeta) P_n(\zeta)\, d\zeta}{\int_{-1}^{+1} P_n^2(\zeta)\, d\zeta} \quad ,$$

and where $x = v/v_t$ and $F = fv_t^3/n$ are dimensionless quantities. After substituting this expansion into the definition of $g(\vec{v})$ and integration of the series term by term [7], the dimensionless Rosenbluth potential $y = g/nv_t$ takes the form

$$y(x,\zeta) = - \sum_{n=0}^{\infty} C_n y_n(x) P_n(\zeta) \quad , \tag{3}$$

where
$$C_n = \frac{M_n}{4n^2 - 1} \quad ,$$

$$y_n(x) = \int d\vec{x}'\, G_n(x,x')F(x') \quad ,$$

$$G_n(x,x') = \frac{x_<^n}{x_>^{n-1}} \left(1 - d_n \frac{x_<^2}{x_>^2}\right) \quad ,$$

$$d_n = \frac{n - 1/2}{n + 3/2} \quad ,$$

and $x_<$ ($x_>$) is the smaller (larger) of x and x'. If f(v) is a Maxwellian distribution then we obtain

$$y_n(x) = \langle G_n(x,x') \rangle = \frac{2}{\sqrt{\pi}} \{x^{1-n}[\gamma(\frac{n+3}{2},x^2) - \frac{n}{x^2}\gamma(\frac{n+5}{2},x^2)]$$

$$+ x^n[\Gamma(2 - \frac{n}{2}, x^2) - d_n x^2 \Gamma(1 - \frac{n}{2}, x^2)]\} \; ,$$

where $$\langle A(x)\rangle = \int d\vec{x}\, F_{max}(x)A(x) \quad , \tag{4}$$

with $F_{max}(x) = e^{-x^2}/\pi^{3/2}$ is an energy average of the function $A(x)$ and $\gamma(\alpha,x)$ and $\Gamma(\alpha,x)$ are the incomplete gamma functions [9]. For $n = 0$, y_0 is

$$y_0(x) = \langle G_0(x,x')\rangle = \psi\phi + \frac{\phi'}{2} \quad , \tag{5}$$

where $\phi(x) = \mathrm{erf}x$ and $\psi(x) = x + 1/2x$.

The particle confinement time of the ions in the central cell of a tandem mirror is much greater than the collision time (for $\eta << 1$ and due to the good electrostatic plugging) and therefore the distribution function for the ions in the central cell is nearly Maxwellian and we can take $y \simeq y_0$ in $O(\eta)$. For a central cell plasma consisting of singly-charged ions of one atomic species with a separable distribution function of the form $F(x,\zeta) = F_{max}(x)M(\zeta)$ and $y \simeq y_0$, the Fokker-Planck collision operator reduces to a pitch-angle operator whose dimensionless form is

$$C(F) = \frac{y'}{2x^3} LF \quad , \tag{6}$$

where $C(F) = (v_t^3/n\nu_s)\, C(f)$, $\nu_s = n\Gamma/v_t^3$, v_t = particle thermal velocity and $y' = \partial y/\partial x$ is

$$y' = y_0' = \psi'\phi + \frac{\phi'}{2x} \quad , \tag{6'}$$

or alternatively, $y'(x) = \phi(x) - G(x)$ where $G(x) = (\phi - x\phi')/2x^2$ is the Chandrasekhar function tabulated by Spitzer [10] and $L = \partial/\partial\zeta\, [(1 - \zeta^2)\, \partial/\partial\zeta]$

is the Legendre differential operator. The collision operator describing the barrier passing particles, which are those particles entering into the barrier region from the central cell (Fig. 2), is within $O(n)$ the same used for the central cell passing particles, with $y' \simeq 1$ for a typical $x_o^2 = \phi_b/T_p > 1$ (ϕ_b = barrier electrostatic dip defined as a positive quantity in energy units; T_p = passing particle temperature), i.e., the barrier passing distribution function is imposed by the central cell passing particles.

For a more general situation, such as that in the barrier trapped region, we proceed as follows. Since $y(x,\zeta)$ is an integral function of $F(x,\zeta)$, it is much less anisotropic than $F(x,\zeta)$ ($C_n \sim M_n/n^2$ in Eq. (3)). Otherwise the barrier mirror ratio $R_b = B_{maxb}/B_b$ is typically much greater than unity. Therefore, $y(\vec{x})$ is within $O(R_b^{-1})$ nearly isotropic in the populated barrier region and we can write the Fokker-Planck operator in the form [8]

$$C(F) = 4\pi F^2 + \frac{y''}{2} F'' + \frac{y'}{x^2} F' + \frac{y'}{2x^3} LF \quad .$$

The first term can be neglected within $O(e^{-x_o^2})$. Besides, due to the effect of the charge exchange pumping in a barrier with $R_b >> 1$, the angular derivatives of the distribution function are much larger than the speed derivatives for most of the particles which are those with $x > x_o$. Thus $C(F) \simeq (y'/2x^3)\, LF$ for $x > x_o > 1$.

Next, we consider some typical functional forms for $y(x,\zeta)$. The barrier distribution function is centered about the vertices of the two-sheets hyperboloid surface representing the boundary contour in velocity space (Fig. 2). For a background distribution function of the form

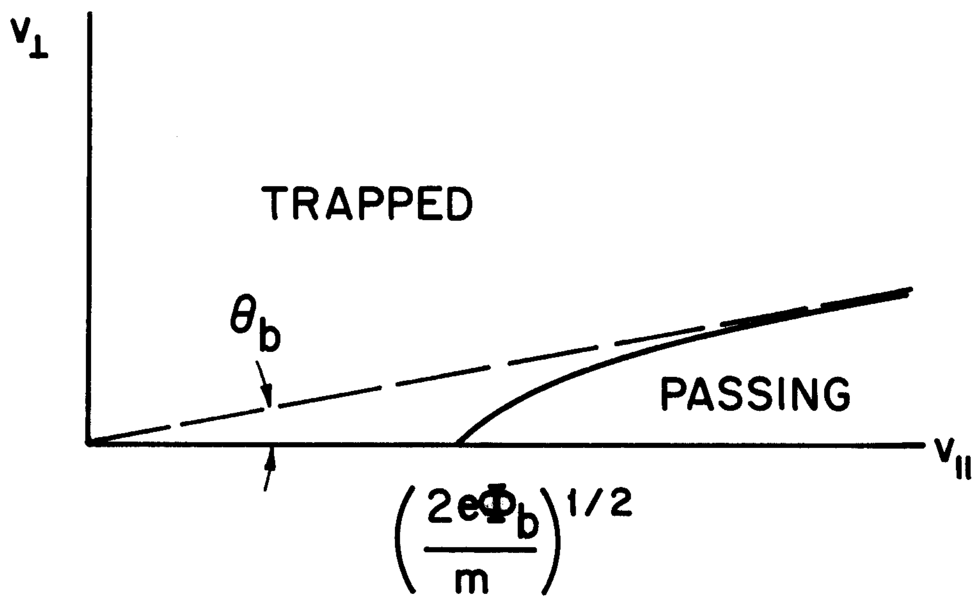

Figure 2 Velocity space in the thermal barrier cell (the plug potential has been assumed much greater than the barrier potential).

$$F(\vec{x}) = \frac{1}{2} \sum_{+,-} \delta(\vec{x} \pm \vec{x}_0) \quad ,$$

where $\sum\limits_{+,-}$ means the sum of the (+) term and the (-) term and $\vec{x} = \pm\vec{x}_0$ are the hyperboloid vertices in velocity space ($\vec{x} = \vec{v}/v_t$) , we obtain

$$y(\vec{x}) = \frac{1}{2} \sum_{+,-} (x^2 + x_0^2 \pm 2\zeta x x_0)^{1/2} \quad .$$

On the $v_\parallel$ axis its first derivative is $y'(x, \pm 1) = 1$ for $x > x_0$ and this is also approximately true within a region of width $\sim R_b^{-1}$ around the $v_\parallel$ axis in velocity space. For a background of the form

$$F(\vec{x}) = \frac{1}{2\pi^{3/2}} \sum_{+,-} e^{-(\vec{x} \pm \vec{x}_0)^2} \quad ,$$

the Rosenbluth potential $y(\vec{x})$ and its first derivative can be obtained by substituting x for $s_\pm = |\vec{x} \pm \vec{x}_0|$ in the expressions given in Eqs. (5) and (6') for a Maxwellian centered at $\overline{x} = 0$ [11], yielding

$$y'(\vec{x}) = \frac{1}{2} \sum_{+,-} [\phi(s_\pm) - G(s_\pm)] \frac{x \pm \zeta x_0}{s_\pm} \quad .$$

On the $v_\parallel$ axis $y'(x, \pm 1) \simeq 1$ for $x > x_0 > 1$ (for a narrow Maxwellian background of width $\sim R_b^{-1}$). Thus, we will assume that for the case of the barrier region (where typically $R_b \gg 1$ and $x > x_0 > 1$) $y' \simeq 1$.

In summary, we will describe the diffusion in velocity space by a Fokker-Planck collision operator which in the central cell takes the form (in $O(\eta)$)

$$C(F) \simeq \frac{y'}{2x^3} LF \quad , \tag{7}$$

with y' given in Eq. (6') and in the thermal barrier region takes the form

$$C(F) \approx \frac{1}{2x^3} LF \quad , \tag{7'}$$

which is approximately correct within $O(\eta, R_b^{-1})$ for $x > x_0 > 1$. For a high barrier mirror ratio the barrier passing particles have a small perpendicular velocity so that a relatively small change in pitch-angle ($\theta = \cos^{-1} (v_{\parallel}/v)$) of the velocity vector can cause them to become trapped. Thus, the primary effect of collisions is pitch-angle scattering and pitch-angle trapping will dominate energy trapping. This is what leads us to the simplification of the collision operator to the Lorentz form.

We consider a low β plasma with two components (electrons and ions). The electron speed is assumed to be very large compared to the ion speed so that the electrons contribute negligibly to the ion collision operator. However, electrons cancel the ion space charge making possible the vanishing of the electric field in the central regions of the central cell and barrier cells and creating an electric field in the mirror regions to preserve quasi-neutrality in both cells.

3. BULK ANALYSIS

The particle distribution function will be determined in each of the various phase space regions (Fig. 3) by solving the kinetic equation (2) with pitch-angle collision operators given in Eqs. (7) and (7'). Those particles which bounce between the plugs are called passing particles and those constrained to move in either the central cell or the barrier region are called trapped particles. Particle injection maintains a steady-state collisional distribution function in the central cell and in the barrier passing region.

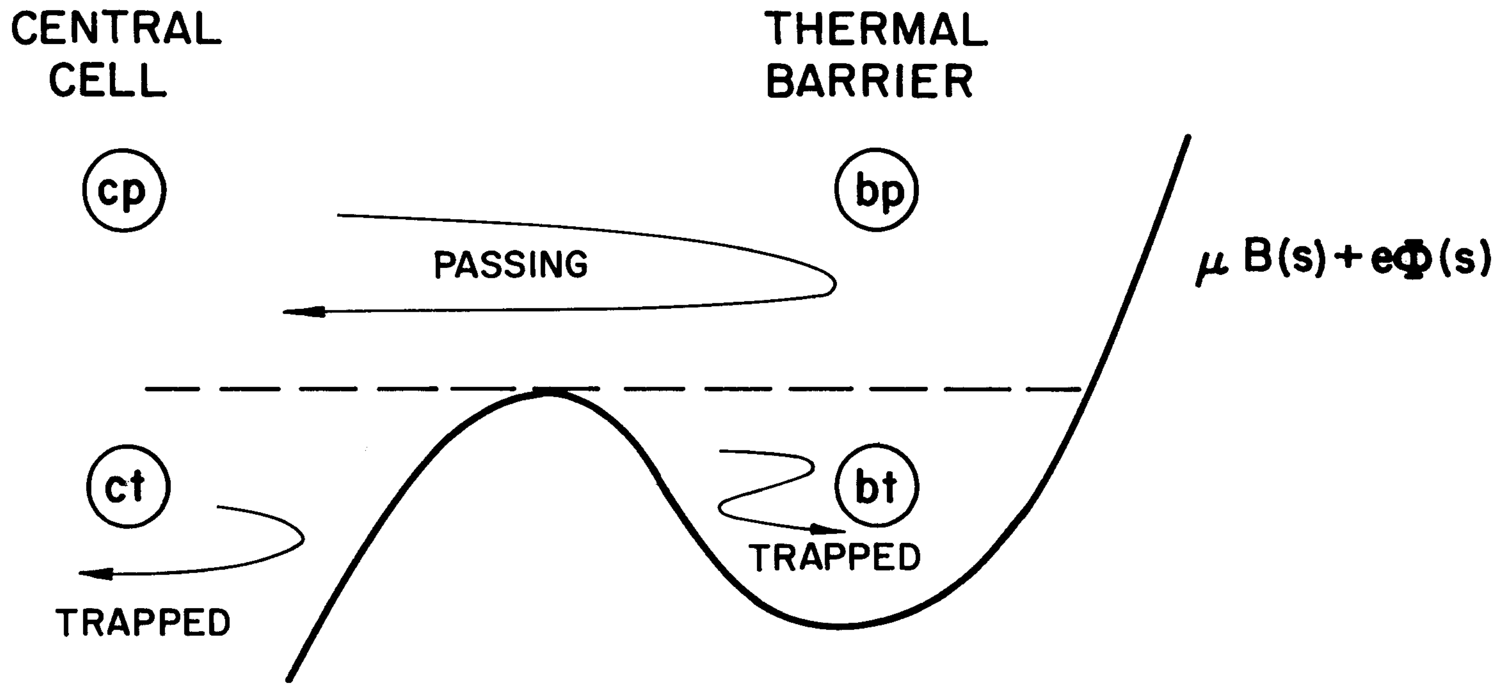

Figure 3 Effective parallel potential $\mu B(s) + e\phi(s)$ in the thermal barrier. Represented are the various phase space regions for passing and trapped particles in the central cell and barrier cell.

The distribution function of particles trapped in the barrier region is determined by the competition between collisional filling-in and the pumping process due to charge exchange off the neutral beams.

The distribution function will be the solution of the following set of boundary value problems:

Central cell trapped particles:

$$\nu_s^c \frac{y'}{2x^3} LF_{ct}(x,\zeta) = -S_{ct}(x,\zeta) \quad , \tag{8}$$

$$\dot{F}_{ct}(\zeta=0) = 0 \quad , \quad F_{ct}(\zeta=\zeta_c) = F_o \quad .$$

Central cell passing particles:

$$\nu_s^c \frac{y'}{2x^3} LF_{cp}(x,\zeta) = -S_{cp}(x,\zeta) \quad , \tag{9}$$

$$\dot{F}_{cp}(\zeta=1) = 0 \quad , \quad F_{cp}(\zeta=\zeta_c) = F_o \quad .$$

Barrier passing particles:

$$\nu_s^b \frac{y'}{2x^3} LF_{bp}(x,\zeta) = -S_{bp}(x,\zeta) \quad , \tag{10}$$

$$\dot{F}_{bp}(\zeta=1) = 0 \quad , \quad F_{bp}(\zeta=\zeta_b) = F_o \quad .$$

Barrier trapped particles:

$$\nu_s^b \frac{y'}{2x^3} LF_{bt}(x,\zeta) = \nu_{cx} F_{bt}(x,\zeta) \quad , \tag{11}$$

$$\dot{F}_{bt}(\zeta=0) = 0 \quad , \quad F_{bt}(\zeta=\zeta_b) = F_o \quad .$$

Here, ν_s^c and ν_s^b are the collision frequencies in the central cell and barrier, respectively, and ν_{cx} is the charge exchange pumping rate in the barrier region. The dimensionless distribution functions F_{ct}, F_{cp}, F_{bt} and F_{bp} correspond to the trapped and passing regions of the central cell and barrier, respectively. The condition $\dot{F} = \partial F/\partial\theta = 0$ at $\theta = \pi/2$ is due to the symmetry of the angular distribution function of trapped particles. Otherwise, since the distribution function is azimuthally symmetric in velocity space (after the gyrophase average), the condition of symmetry at $\theta = 0$ and $\theta = \pi$ must also be imposed on the passing particle distribution function ($\partial F/\partial\theta \rightarrow -\partial F/\partial v_{\parallel}$ vsinθ as θ approaches the $v_{\parallel}$ axis). The remaining free constants of the problem are obtained by matching the value of the distribution function at the loss boundary with the isotropic level existent throughout the machine [12] and that will be determined by the continuity conditions given below. This matching is consistent with our $O(\nu_s/\omega_b)$ analysis, which neglects the effects of the collisional boundary layer around the contours between the various regions in velocity space. The values ζ_c and ζ_b are defined by the equations of conservation of the energy (E) and magnetic moment (μ). In the central cell, $1 - \zeta_c^2 = R_c^{-1}$ where $R_c = B_{maxb}/B_c$ and the central cell electrostatic potential has been chosen as the reference potential ($\phi_c = 0$). Equating the (E,μ) values of the central cell particles entering the barrier region with mean energy T_p at the barrier peak ($B = B_{maxb}$) to those of these particles at the bottom of the barrier region ($B = B_b$) we obtain $1 - \zeta_b^2 = R_b^{-1}(1 + x_o^2)^{-1}$, where $R_b = B_{maxb}/B_b$ and $x_o^2 = \phi_b/T_p$. Thus, we can define an effective mirror ratio for the barrier particles as $R_b(1 + x_o^2)$ which permits consideration of the barrier ambipolar effects in velocity space while assuming a separable boundary contour [13,14]. Otherwise, the ambipolar effects can be taken into account by the more rigorous method of solving the collision operator using hyperbolic coordinates like

$$x = (v^2 + \alpha^2)^{1/2} \quad ,$$

$$y = \frac{v\zeta}{(v^2 + \alpha^2)^{1/2}} \quad ,$$

with $\alpha^2 = v_0^2/(R_b - 1)$ and $v_0^2 = 2q\phi_b/m$ so that the loss boundary contour $v_\parallel^2 - (R - 1)v_\perp^2 = v_0^2$ maps into the coordinate surface $y = \zeta_c$.

The boundary conditions imposed in Eq. (11) are those describing the barrier trapped particles with $v > v_0$. We will not consider a small fraction of barrier trapped particles with $v < v_0$ which are close to the loss boundary vertices and whose main effect is to provide for a speed gradient of the distribution function that produces some trapping due to energy scattering which is neglected in our analysis ($O(R_b^{-1})$ correction). The dimensionless particle sources ($S(x,\zeta) = S(v,\zeta)\ v_t^3/n$) S_{ct}, S_{cp}, and S_{bp} maintain in steady state the trapped and passing regions of the central cell and the barrier passing region, respectively.

We will search for solutions of the form $F(x,\zeta) = F_{max}(x)\ M(\zeta)$ in the central cell and of the form $F(x,\zeta) = F_{max}(x)\ \exp\ (x_0^2)\ M(x,\zeta)$, $x > x_0$ in the barrier region. We will also consider separable source functions of the form $S(x,\zeta) = S(x)\ \sigma(\zeta)$. In considering a general function $\sigma(\zeta)$ we introduce a trivial factor in the denominator of the normalization coefficients A's (defined below) of the form

$$\bar{\sigma}_t = \frac{\int_0^{\zeta_c} d\zeta \int_{\zeta_c}^{\zeta} \frac{-2d\zeta}{1-\zeta^2} \int_0^{\zeta} d\zeta\ \sigma(\zeta)}{\int_0^{\zeta_c} d\zeta\ \ell n\ (\frac{1-\zeta^2}{1-\zeta_c^2})} ,$$

in the case of trapped particles [15] and

$$\bar{\sigma}_p = \frac{\int_{\zeta_c}^{1} d\zeta \int_{\zeta_c}^{\zeta} \frac{d\zeta}{1-\zeta^2} \int_{\zeta}^{1} d\zeta\ \sigma(\zeta)}{\int_{\zeta_c}^{1} d\zeta\ \ell n\ (\frac{1+\zeta}{1+\zeta_c})} ,$$

in the case of passing particles, as can be seen by going through the analysis below for a general $\sigma(\zeta)$. Therefore, we will consider isotropic sources ($\sigma(\zeta) = 1$, $\bar{\sigma} = 1$), distributed as $\frac{S_{ct}}{\zeta_c} = \frac{S_{cp}}{1-\zeta_c} = S_c$ = central cell source (any other source distribution introduces again an unimportant factor in the A's defined below). We will also assume that the charge exchange pumping frequency ν_{cx} is constant over the phase space of the trapped barrier region. The charge exchange process does not alter significantly the passing distribution function since on the one hand those particles charge exchanged in the trapped region of the barrier are a source for the passing region and this is an $O(n)$ effect on the particles over the entire machine and, on the other hand, with regard to those particles charge exchanged in the passing region we note that in an analysis of $O(\nu_s/\omega_b)$ the collisional response and the charge exchange pumping are much slower processes than the bouncing of particles in the barrier region. The sources and sinks which maintain the system in steady-state are localized in real space. This fact does not influence the solution to the problem due to the rapid bouncing of the barrier particles.

In the subsequent velocity space integrations we will set the central cell electrostatic potential equal to zero and we will assume that the plug potential is much greater than the barrier potential.

The distribution function of particles trapped in the central cell, which is the solution to Eq. (8), takes the form

$$f_{ct}(v,\zeta) = [n^{a}_{ct}\, A_{ct}\, \ln\left(\frac{1-\zeta^2}{1-\zeta_c^2}\right) + \frac{n^{i}_{ct}}{\zeta_c}]\, f_{max}\,(v)\, \theta\,(\zeta_c - |\zeta|)\ , \qquad (12)$$

where $f_{max} = e^{-v^2/v_t^2}/\pi^{3/2}\, v_t^3$; $\theta(x > 0) = 1$ and $\theta(x < 0) = 0$ (Heaviside step function); the density superscripts "a" and "i" refer to the anisotropic and isotropic density components, respectively; $A^{-1}_{ct} = \ln\,[(1 + \zeta_c)/(1 - \zeta_c)] - 2\zeta_c$; and $\int d\vec{v}\ f_{ct}\,(v,\zeta) = n^{a}_{ct} + n^{i}_{ct} = n_{ct}$ = density of trapped particles in the central cell. In the passing region of the central cell the solution to Eq. (9) is

$$f_{cp}\,(v,\zeta) = [n^{a}_{cp}\, A_{cp}\, \ln\left(\frac{1 + |\zeta|}{1 + \zeta_c}\right) + \frac{n^{i}_{cp}}{1-\zeta_c}]\, f_{max}\,(v)\, \theta\,(|\zeta| - \zeta_c)\ , \qquad (13)$$

with $A^{-1}_{cp} = 2\,\ln\,(2/1 + \zeta_c) - 1 + \zeta_c$ and $\int d\vec{v}\ f_{cp}\,(v,\zeta) = n^{a}_{cp} + n^{i}_{cp} = n_{cp}$ = density of passing particles in the central cell. In the barrier passing region the solution to Eq. (10) is

$$f_{bp}\,(v,\zeta) = [n^{a}_{bp}\, A_{bp}\, \ln\left(\frac{1 + |\zeta|}{1 + \zeta_b}\right) + \frac{n^{i}_{bp}}{1-\zeta_b}]$$

$$H\left(\frac{\phi_b}{T_p}\right) f_{max}\,(v)\, e^{\phi_b/T_p}\, \theta(|\zeta| - \zeta_b)\, \theta(v-v_o)\ , \qquad (14)$$

with $A^{-1}_{bp}\,(\zeta_b) = 2\,\ln\,(2/1 + \zeta_b) - 1 + \zeta_b$ and $H^{-1} = \langle 1\rangle^*$ is given in Eq. (A.2) where

$$\langle A(x)\rangle^* = \int_{x>x_o} d\vec{x}\, F_{max}(x)\, e^{x_o^2} A(x) \ , \tag{15}$$

$x_o^2 = \phi_b/T$; $\frac{1}{2} m\, v_o^2 = \phi_b$; for $x_o = 0$, $\langle A(x)\rangle^* = \langle A(x)\rangle$ (cf. Eq.(4)); and $\int d\vec{v}\, f_{bp}(v,\zeta) = n_{bp}^a + n_{bp}^i = n_{bp}$ = density of passing particles at the bottom of the barrier electrostatic well:

$$n_{bp} = [n_{bp}^a(o)\, \frac{A(0)}{A(\zeta_b)} + n_{bp}^i(o)\,(1 - \zeta_b)]\, \frac{H(0)}{H(x_o^2)} \ ,$$

where $n_{bp}^a(o)$ and $n_{bp}^i(o)$ are densities at the barrier peak ($B = B_{maxb}$). Since $n_{bp}^a(o) \sim O(\eta)$, (cf., Eq. (24)), we can write n_{bp} in the form

$$n_{bp} = \frac{n_c^i}{\sqrt{\pi}\, R_b\, x_o}\, [1 + O\,(\eta,\, x_o^{-2},\, R_b^{-1}\, x_o^{-2})] \ , \tag{15'}$$

where $n_c^i = n_{cp}^i + n_{ct}^i$ = isotropic central cell density. The confinement time of the passing ions is typically much longer than the collision time in the central cell and then their distribution function is approximately Maxwellian [in $O(\eta)$] with temperature determined by the central cell.

The distribution function of particles trapped in the barrier region is the solution of a Legendre boundary value problem with complex eigenvalues where the differential equation has the form (cf. Eq. (11))

$$\frac{\partial}{\partial\zeta}(1 - \zeta^2)\frac{\partial}{\partial\zeta} M(x,\zeta) = [\lambda_{cx}\, \frac{y'}{2x^3}]^{-1}\, M(x,\zeta) \ .$$

A real and bounded general solution satisfying the boundary conditions is

$$f_{bt}(v,\zeta) = n_{bt} \frac{K_\tau(\zeta) + L_\tau(\zeta)}{K_\tau(\zeta_b)+L_\tau(\zeta_b)} G(\frac{\phi_b}{T_t}) f_{max}(v) e^{\phi_b/T_t} \theta(\zeta_b-|\zeta|) \theta(v-v_o) , \quad (16)$$

where $K_\tau(\zeta)$ and $L_\tau(\zeta)$ are Conical functions [16] of index τ such that

$$\frac{1}{4} + \tau^2 = [\lambda_{cx} \frac{y'}{2x^3}]^{-1} . \quad (17)$$

The Conical functions $K_\tau(\zeta)$ and $L_\tau(\zeta)$ coincide with the Spherical Harmonics $P_{-(1/2)+i\tau}(\zeta)$ and $P_{-(1/2)+i\tau}(-\zeta)$ for real τ [17,18], respectively. The functions $K_\tau(\zeta)$ and $L_\tau(\zeta)$ are linearly independent solutions of the Legendre differential equation [19] and effectively correspond to first and second kind Legendre functions $P_{-(1/2)+i\tau}(z)$ and $Q_{-(1/2)+i\tau}(z)$. They are regular functions in the complex plane cut along $|\arg(1 \pm z)| < \pi$ and they also are entire and even functions of the index τ. These functions are real functions (representable by real hypergeometric series) for real τ [16]. At very low energies $\tau^2 + \frac{1}{4} \simeq 0$ and the Conical functions become very close to a constant, i.e., low energy particles are so collisional that they "fill" the barrier trapped region. An equivalent situation is attained if the pumping rate is decreased ($\nu_{cx} \to 0$). The normalization factor G is defined as

$$G^{-1} = < \int_0^{\zeta_b} \frac{K_\tau(\zeta) + L_\tau(\zeta)}{K_\tau(\zeta_b)+L_\tau(\zeta_b)} d\zeta >^* , \quad (18)$$

and it is evaluated in Appendix A. Also $\int d\vec{v} f_{bt}(v,\zeta) = n_{bt}$ = density of trapped particles at the bottom of the barrier electrostatic well. The distribution function of the particles trapped in the barrier cell becomes localized to a narrow region near the loss boundary in velocity space when the charge exchange pumping rate is fairly strong (cf. Eq. (A.5), Fig. 4), decaying exponentially as

$$M(\zeta) \sim e^{-a(\zeta_b-\zeta)/\lambda_{cx}^{1/2}} , \tag{19}$$

(cf. Eq. (A.6)), where $a = (2x^3/(1 - \zeta_b^2)y')^{1/2}$ is typically much greater than unity so that the angular characteristic width of the trapped barrier distribution function is $\Delta\zeta << 1$. A convenient measure of the smallness of this width is $\lambda_{cx}^{1/2}(\Delta\zeta \sim \lambda_{cx}^{1/2}, \lambda_{cx} \sim \nu_{cx}^{-1})$. The trapped distribution function gradient near the loss boundary becomes very steep as indicated explicitly in Eq. (A.7). Since our analysis is collisionless in a bounce time $[O(\nu_s/\omega_b)]$, the derivative of the distribution function solution in the barrier cell has a discontinuity at the loss boundary which otherwise will be smoothed out by boundary layer effects [20]. In a bulk analysis the normal derivative of the barrier distribution function is not continuous at the loss boundary because the flow of particles through it is not conservative (particles that become trapped in the barrier region are pumped out by charge exchange). When the pumping rate is large this discontinuity introduces a significant error in the trapping current evaluation.

The unknown density levels of the distribution function expressions given in Eqs. (12)-(14), (16) are determined by the following relations. Since we are assuming that the effects of collisions in a single period of the particle motion are small enough to be neglected and that a quasi-steady-state exists, the distribution function is a constant along field lines. The distribution function depends only on the constants of motion which in the adiabatic theory of a cylindrically symmetric system are: the magnetic moment, $\mu = mv_\perp^2/2B$, the

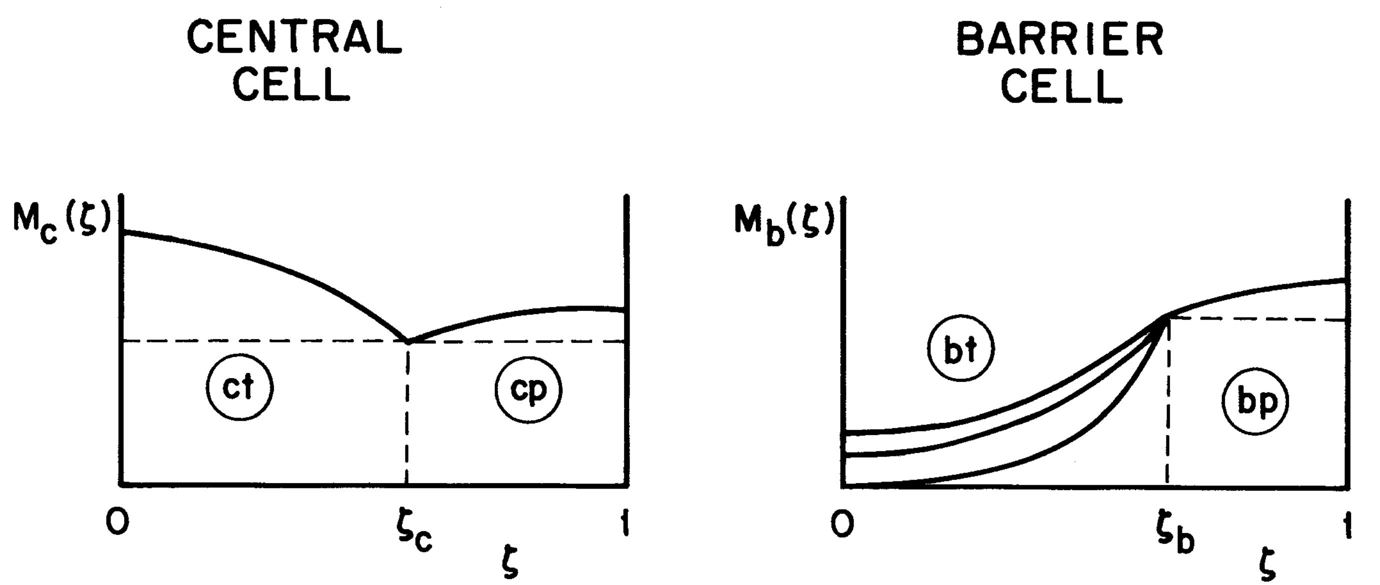

Figure 4 Angular distribution function in the central cell and in the thermal barrier. Several solutions in the trapped region of the barrier cell (for different pumping rates) have been sketched.

energy, $E = mv^2/2 + \phi$, and the magnetic flux, $\psi = \pi r^2 B$. There is an implicit radial dependence in the distribution function which has been omitted in our analysis. We can write the statements of energy and magnetic moment conservation as

$$v_{CC}^2 = v_{BC}^2 - v_o^2 ,$$

$$\frac{1-\zeta_{CC}^2}{1-\zeta_c^2} = \frac{1-\zeta_{BC}^2}{1-\zeta_b^2} ,$$

where the subscripts CC and BC stand for central cell and barrier cell, respectively, and v_o, ζ_b and ζ_c define the boundary contours as stated before. Then we impose the conditions

$$f_{cp}(0,\zeta_c) = f_{bp}(v_o,\zeta_b) , \tag{20}$$

$$f_{cp}(0,1) = f_{bp}(v_o,1) ,$$

which ensures the continuity of the distribution function at the barrier peak ($B = B_{maxb}$). Also, from the continuity of the distribution function at the boundary contours of the central cell and barrier cell phase spaces we have

$$f_{ct}(0,\zeta_c) = f_{cp}(0,\zeta_c) , \tag{20'}$$

$$f_{bp}(v_o,\zeta_b) = f_{bt}(v_o,\zeta_b) .$$

The distribution function in the barrier passing region, which is determined by the distribution function in the central cell passing region, provides a boundary condition for the barrier trapped distribution function at the boundary contour.

In steady-state the particle currents satisfy

$$J_{ct} + J_{cp} + J_{bp} + J_{bt} = 0 \ , \tag{21}$$

where $J \equiv \frac{\partial N}{\partial t} = \int d\psi \oint \frac{d\ell}{B} j$ with $\int d\psi \oint \frac{d\ell}{B} = V$ = cell volume (V_c = central cell volume, V_b = barrier volume); here N is the number of particles in the cell; $d\psi$ is the differential magnetic flux; $d\ell$ is the field line differential length; $j = - \int d\vec{v}\ C(f) = \int d\vec{S}_v \cdot \vec{\Gamma}$ is the loss of particles per second and per cm^3 where $C(f) = - \vec{\nabla}_v \cdot \vec{\Gamma}$, the volume integral extends over the region of velocity space occupied by the particles of interest and the surface integral is defined on the surface surrounding that volume. Although the arguments presented here are rigorously valid only for individual force tubes we will assume that the conditions are sufficiently uniform over the cross-section of the cells so that we can refer the calculations to the cells as a whole. Then we obtain

$$j_{ct} = A_{ct}\ I\ \zeta_c\ n^a_{ct}\ \nu^c_s \ , \tag{22}$$

$$j_{cp} = \frac{A_{cp}}{2} I\ (1 - \zeta_c)\ n^a_{cp}\ \nu^c_s \ ,$$

$$j_{bp} = \frac{A_{bp}}{2} I^*\ (1 - \zeta_b)\ n^a_{bp}\ \nu^b_s \ ,$$

$$j_{bt} = - n_{bt}\ \nu_{cx} \ ,$$

where $I = \langle \frac{y'}{x^3} \rangle / \langle 1 \rangle = 1.388$ [15] and $I^* = \langle \frac{y'}{x^3} \rangle^* / \langle 1 \rangle^* = \frac{\Gamma(0,x_0^2)}{\Gamma(\frac{3}{2},x_0^2)}$ ($y' \simeq 1$ in the barrier region) with $\Gamma(\alpha,x)$ being the incomplete gamma function [9]. Since we are considering a collisional and stable plasma in which an equilibrium situation has been reached by particle injection and charge exchange pumping, the condition given in Eq. (21) can be equivalently formulated as

$$S_{ct} + S_{cp} + S_{bp} = S_{bt} , \tag{23}$$

where $S = \int d\psi \oint \frac{d\ell}{B} \int d\vec{v}\ S(v,\zeta)$ = particle source (the velocity space integral extends over the region of velocity space occupied by the particles of interest) and $S_{bt} = \int d\psi \oint \frac{d\ell}{B} \int d\vec{v}\ \nu_{cx}\ f_{bt}$ = charge exchange sink. With uniform conditions in the radial and axial directions the charge exchange sink takes the form $\nu_{cx}\ n_{bt}\ V_b$. We analyze those particles which are well confined by the plug electrostatic potential and assume that other losses different from the ones considered here (e.g., Pastukhov losses, radial losses, etc.) are balanced otherwise.

Substituting the distribution functions of Eqs. (12)-(14), (16) and the currents of Eq. (22) into the continuity conditions given in Eqs. (20), (20') and (21) and solving, we obtain

$$n_{ct}^{a} = \frac{n_{bt}}{A_{ct}\ I\ \lambda_{cx}}\ \eta + O(\eta^2) , \tag{24}$$

$$n_{ct}^{i} = \zeta_c\ G\ n_{bt} ,$$

$$n_{cp}^{a} = \frac{2n_{bt}}{A_{cp}\ I\ \lambda_{cx}}\ \eta + O(\eta^2) ,$$

$$n^i_{cp} = (1 - \zeta_c)\, G n_{bt} \; ,$$

$$n^a_{bp} = \frac{2D\; n_{bt}}{A_{bp}\; I\; H\lambda_{cx}}\, \eta + O(\eta^2) \; ,$$

$$n^i_{bp} = (1 - \zeta_b)\, \frac{G}{H}\, n_{bt} \; ,$$

$$n_{bt} = \frac{n_c}{G}\, [1 - \frac{1}{A_c\; IG\; \lambda_{cx}}\, \eta] + O(\eta^2) \; ,$$

where $D = \ell n(2/(1 + \zeta_c))/\ell n(2/(1 + \zeta_b))$; $n_c = n_{ct} + n_{cp}$ is the central cell density; $A_c^{-1} = A_{ct}^{-1} + 2A_{cp}^{-1}$; and terms of $O(\eta^2)$ have been neglected. For simplicity we have assumed that the plasma is isothermal with a temperature determined otherwise by energy balance throughout the machine. The parameter $\eta = (V_b/V_c)(\nu_s^b/\nu_s^c)$ is, then, the ratio of the total number of particles in the barrier region to the number of particles in the central cell.

In the case of a long central cell ($\eta << 1$) the distribution functions in the central cell and barrier passing region are nearly isotropic, cf. Eq. (24). Figure 5 shows the distribution function in the barrier region when the pumping rate is increased ($\lambda_{cx} \to 0$). Then the distribution function contours wrap around the loss boundary in velocity space and the trapped density in the barrier region decreases. For some level of pumping (defined in the next section) the trapped distribution function in the barrier is localized to a boundary layer region in which our analysis of $O(\nu_s/\omega_b)$ is no longer valid. In that case a boundary layer analysis is required [20] and a half-range boundary value problem needs to be solved with inhomogeneous asymptotic conditions given by our "bulk" analysis (Eqs. (14,16)), which is valid far away from the boundary layer. As a consequence of a boundary layer analysis the

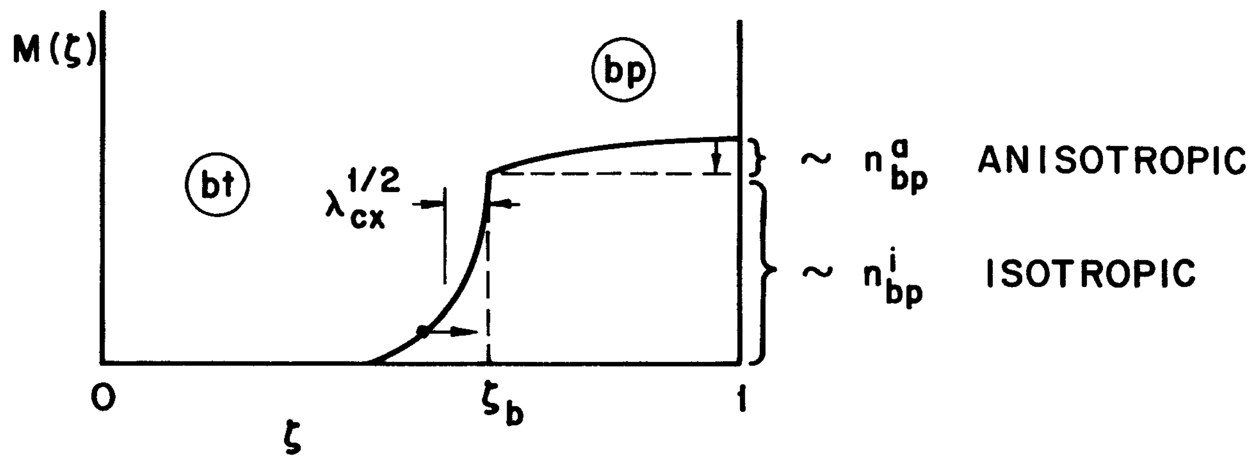

Figure 5 Angular distribution function in the thermal barrier of a tandem mirror. The arrows indicate how f_{bt} is localized to a boundary layer as $\nu_{cx} \to \infty$ and how f_{bp} becomes nearly isotropic as $\eta \to 0$.

derivative of the distribution function in the barrier region will be continuous at the loss boundary and boundary layer effects will bound the otherwise unbounded growth of the barrier trapping current (as discussed below and noted in Refs. [4,5]) by limiting the steepening of the barrier distribution function near the loss boundary.

4. TRAPPING CURRENT

As a result of the analysis carried out in the preceding section, the various density levels throughout the machine have been determined and we can write the barrier passing density as $n_{bp} = \frac{\Omega}{\lambda_{cx}} n_{bt}$ where

$$\Omega = (1 - \zeta_b)\, \lambda_{cx} \frac{G(x_o^2)}{H(x_o^2)} + \frac{2D}{A_{bp}\ I\ H}\, \eta + O(\eta^2)\ . \qquad (25)$$

Since the barrier trapping current is $j_t = n_{bt}\, \nu_{cx}$ [$-j_{bt}$ in Eq. (22)], we obtain

$$j_t = \frac{n_b\ \nu_{cx}}{1 + \frac{\Omega}{\lambda_{cx}}}\ , \qquad (26)$$

where $1 + \frac{\Omega}{\lambda_{cx}} = \frac{g}{g-1}$ and $g = n_b/n_{bp}$ is the filling ratio as defined in Ref. [1]. In the limit of a long central cell ($\eta << 1$) we can write

$$\Omega = (\lambda_{cx}\ \lambda_{cx}^*)^{1/2} + O(\eta)\ . \qquad (27)$$

Then the barrier trapping current (particles/cm^3/sec) is given by the expression

$$j_t = \frac{n_b\ \nu_{cx}}{1 + \left(\frac{\lambda_{cx}^*}{\lambda_{cx}}\right)^{1/2}}\ , \qquad (28)$$

valid for $\eta \ll 1$; here n_b is the barrier density (cm^{-3}); ν_{cx} is the charge exchange frequency (sec^{-1}); $\lambda^*_{cx} \simeq x_o/2R_b$ (for $x_o^2 > 1$), cf. Eq. (A.4); $x_o^2 = \phi_b/T_p$; ϕ_b = barrier electrostatic dip; T_p = passing particle temperature; $R_b = B_{maxb}/B_b$ = barrier mirror ratio; $\lambda_{cx} = \nu_s^b/\nu_{cx}$; $\nu_s^b = n_b\, \Gamma/v_t^3$; $\Gamma = 4\pi\, q^4\, \ell n\Lambda/m^2$ and $T_p = \frac{1}{2}\, m\, v_t^2$. Finite η corrections are discussed in Appendix B.

Fokker-Planck studies have been carried out in Refs. [4,5] for situations comparable to those described by Eq. (28). In Figs. 6 and 7 we give a comparison between the numerical results obtained for j_t in these references and the analytical results obtained using Eq. (28). Input data for the calculation are R_b, ϕ_b, n_b, ν_{cx} and T_p. We have evaluated the collisional time from the expression $1/\nu_s^b = \frac{6.27\text{x}10^{11}}{n_b\, \ell n\Lambda}\, T_p^{3/2}$ with $\ell n\Lambda = 34.9 - \ell n \sqrt{\frac{n_b}{E_i T_e}}$ as calculated in the Fokker-Planck studies [4,5] (T_e = electron temperature and E_i = average ion energy in the bottom of the barrier cell). As expected, our analytic expression provides an estimate of the trapping current that is lower than that obtained from numerical results since we have evaluated only the trapping due to pitch-angle scattering neglecting the energy scattering contribution. The average errors obtained in the cases represented in Figs. 6 and 7, which correspond to high and low barrier mirror ratios, are -6.4% and -18.6%, respectively. High barrier mirror ratios are necessary for reasonable thermal barrier cells [1] and, therefore, we expect that the trapping current due to energy scattering will be an $O(R_b^{-1}) \sim 10\%$ effect in the trapping current calculation.

We can define a trapping frequency ν_t as

$$j_t = n_{bp}\, \nu_t \,,$$

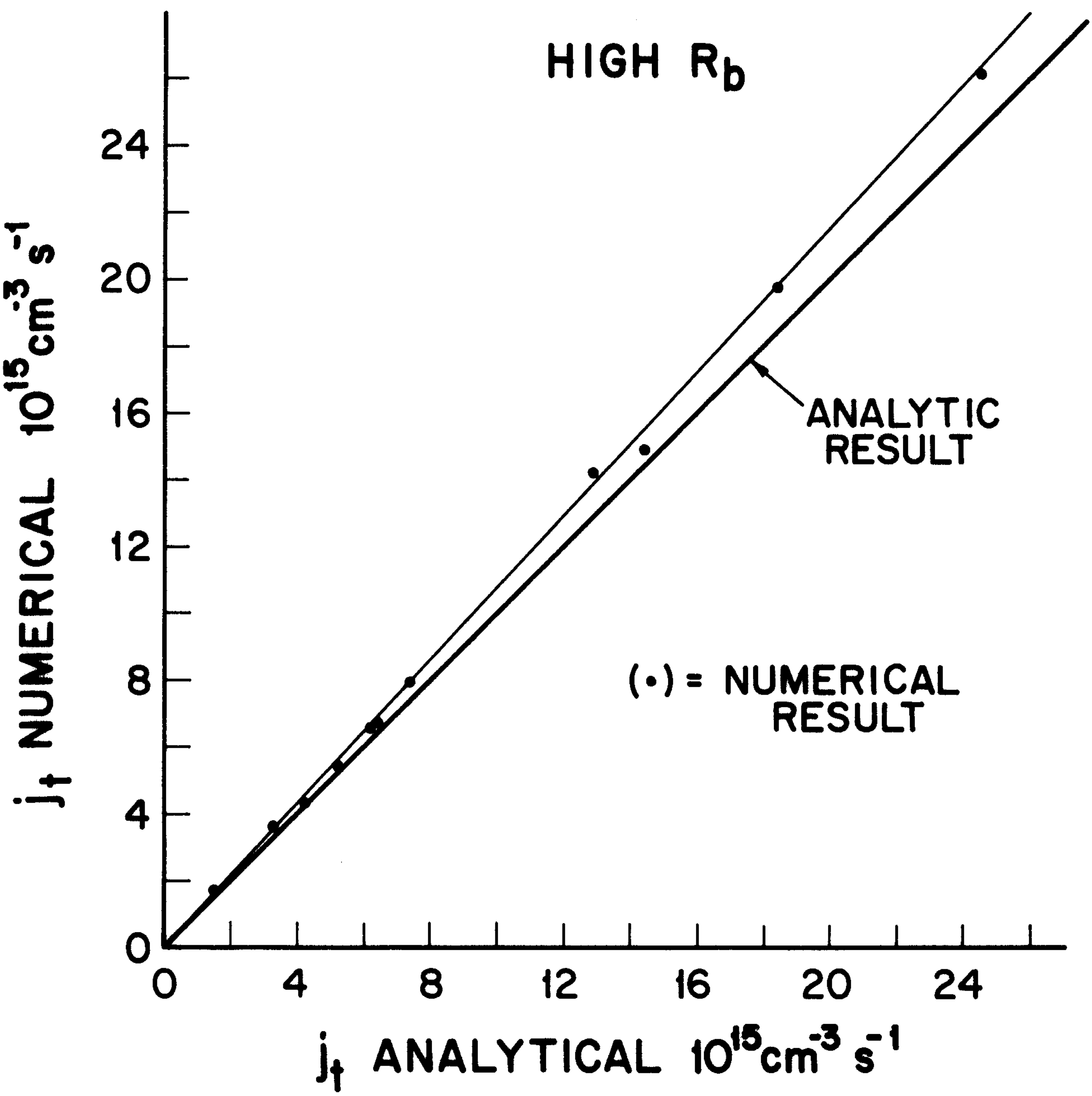

Figure 6 Comparison of numerical(*) and analytical results for the barrier trapping current. High barrier mirror ratio cases. (*) Results taken from Refs. [4] and [5].

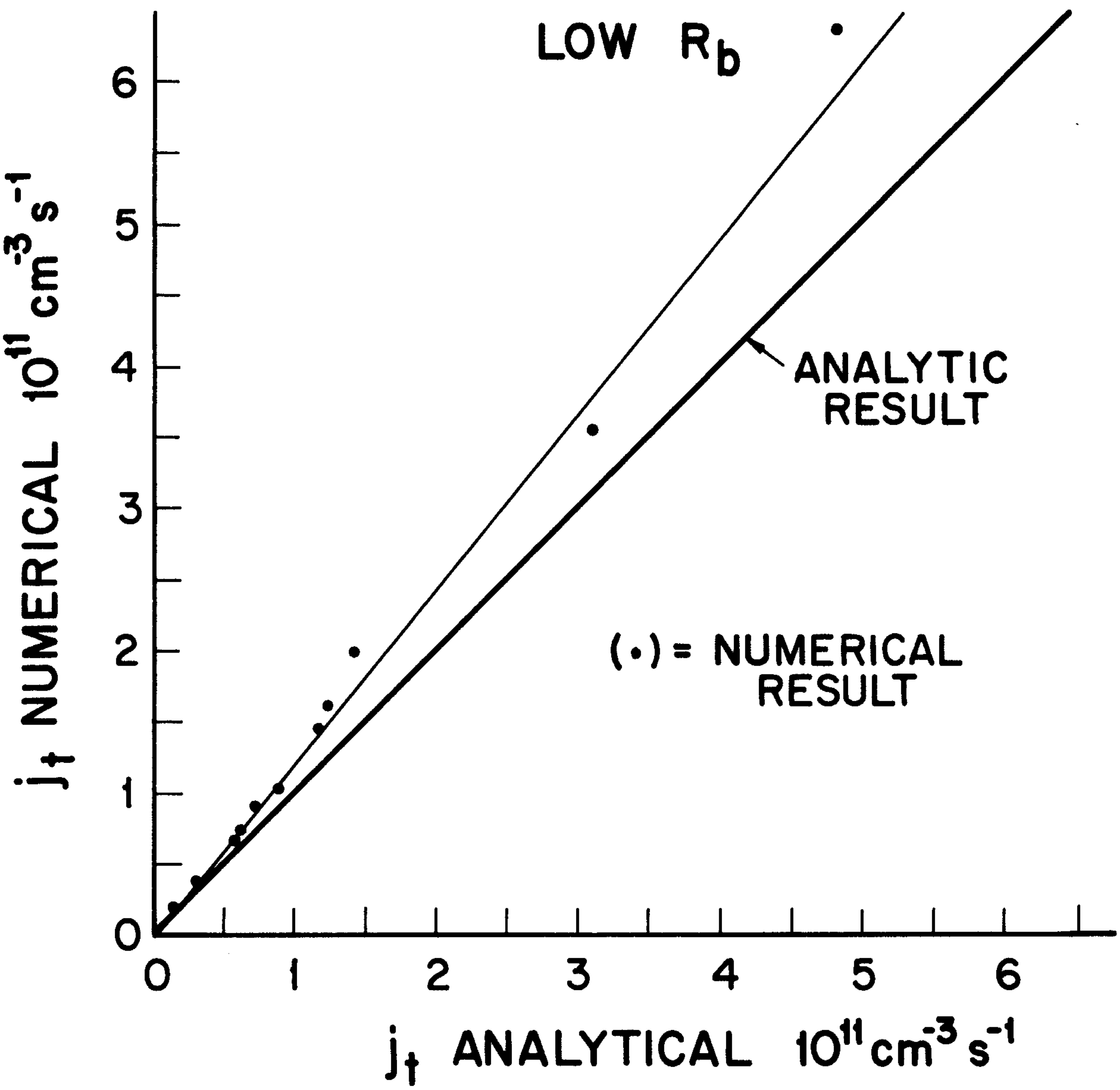

Figure 7 Comparison of numerical(*) and analytical results for the barrier trapping current. Low barrier mirror ratio cases. (*) Results taken from Refs. [4] and [5].

that is, ν_t is an effective frequency for replenishing with passing particles the trapped particles that have been pumped out. Since $j_t = n_{bt}\,\nu_{cx}$ and $n_{bt} = \frac{\lambda_{cx}}{\Omega}\,n_{bp}$, then

$$\nu_t = \frac{\nu_s^b}{\Omega} ,$$

and using Eq. (27) we obtain

$$\nu_t = \nu_{cx}\left(\frac{\lambda_{cx}}{\lambda_{cx}^*}\right)^{1/2} . \tag{29}$$

Boundary layer effects are expected to be important when $\nu_{cx} > \nu_t$ or, from Eq. (29), when $\lambda_{cx}^{1/2} < \lambda_{cx}^{*1/2}$; this condition corresponds to $g < 2$ according to the expression for g given in Eq. (26). In the barrier trapped region, collisional effects compete with charge exchange to establish a distribution function profile which has a typical angular width $\Delta\zeta \sim \lambda_{cx}^{1/2}$ (cf. Eq. (19)). This result can be also obtained as the solution to a boundary layer kinetic equation of the form

$$\nu_s^b\,\frac{y'}{2x^3}\,(1 - \zeta_b^2)\,\frac{\partial^2 f}{\partial\zeta^2} = \nu_{cx}\, f ,$$

whose $O(\nu_s/\omega_b)$ solution for $|\zeta| < \zeta_b$, with boundary conditions given in Eq. (11), is that obtained in Eq. (19). This equation describes the velocity space diffusion of the particles trapped in the barrier region when they are very localized near the loss boundary, i.e., for $\tau >> 1$ and $\theta \simeq \theta_b$ as was assumed in obtaining the expression (19). Therefore, we can visualize $\lambda_{cx}^{*1/2}$ as corresponding to a critical boundary layer width $\Delta\zeta^*$ such that if the barrier trapped distribution function is localized within it, $\Delta\zeta(\sim \lambda_{cx}^{1/2}) < \Delta\zeta^* (\sim \lambda_{cx}^{*1/2})$, boundary layer effects become significant. Alternatively, we can write

$$\frac{\lambda_{cx}}{\lambda_{cx}^*} \simeq \frac{\nu_{s\ eff}}{\nu_{cx}} ,$$

where

$$\nu_{s\ eff} = \frac{\nu_s^b}{\lambda_{cx}^*} \simeq \frac{\nu_s^b(v_o)}{\left(\frac{\theta_b}{\pi/2}\right)^2} , \tag{29'}$$

is an effective collision frequency for particles of velocity about v_o to pitch-angle scatter out of the passing barrier region, $\Delta\zeta \sim 1 - \zeta_b$, $\zeta_b = \cos\theta_b$ (by a random walk process). Then, from Eq. (29), we obtain

$$\nu_t = (\nu_{cx}\ \nu_{s\ eff})^{1/2} . \tag{30}$$

Therefore, the following statements are equivalent

$$\nu_{cx} \gtrless \nu_t ,$$

$$\nu_t \gtrless \nu_{s\ eff} ,$$

$$\lambda_{cx}^{*1/2} \gtrless \lambda_{cx}^{1/2} .$$

In general $\nu_t \neq \nu_{s\ eff}$ due to the effect that the charge exchange pumping in the trapped region has on the gradient of the distribution function at the boundary contour. Only when $\nu_{cx} = \nu_{s\ eff}$ is $\nu_t = \nu_{s\ eff}$ (cf. Eq. (30)).

The trapping current expression (28) obtained from a bulk analysis grows unboundedly as the pumping rate is increased since $j_t \sim \nu_{cx}^{1/2}$. This is because the angular gradient of the trapped distribution function in the barrier region becomes arbitrarily large at the loss boundary. An unbounded behavior has also been observed in numerical studies [4]. In order for the trapping current to remain finite as the pumping rate is increased the derivative of the barrier distribution function must be continuous across the loss boundary in velocity space. When the pumping rate is increased the characteristic width $\Delta\zeta \sim \lambda_{cx}^{1/2}$ decreases and boundary layer effects become important imposing an upper bound to the barrier trapping current and removing any sharp gradient formed near the loss boundary. We note that λ_{cx}^{*} increases as ϕ_b increases or R_b decreases. In both cases a lower $\nu_{s\ eff}$ is expected so that the condition $\nu_{cx} > \nu_t$ is more easily achieved or, in other words, boundary layer effects will be more important, which is equivalent to the initial statement of greater λ_{cx}^{*}.

The expression (28) can also be written in the following way. Using the relation $n_{bt}/n_{bp} \simeq (\lambda_{cx}/\lambda_{cx}^{*})^{1/2} + O(\eta)$ we obtain

$$j_t = \frac{n_{bp}^2}{K_i T_p^{3/2}} \frac{1}{\lambda_{cx}^{*}} [1 + (\frac{\lambda_{cx}^{*}}{\lambda_{cx}})^{1/2}] , \qquad (31)$$

where $n_{bp} = n_b/[1 + (\lambda_{cx}/\lambda_{cx}^{*})^{1/2}]$ = barrier passing density and $K_i = m^{1/2}/2^{1/2}\pi q^4 \ell n\Lambda$ ($K_i T_p^{3/2} = n\tau_{ii}, \tau_{ii}$ = ion-ion 90° scattering time). Since $n_{bp} \simeq n_c^i/\sqrt{\pi}\ R_b\ x_o$ (cf. Eq. (15')) the barrier passing density decreases as ions accelerate in the electrostatic well and the magnetic field lines expand in the magnetic well. Otherwise $\nu_{s\ eff} \sim \lambda_{cx}^{*-1} \sim R_b/x_o$ (cf. Eqs. (29', A.4)) so that $\nu_{s\ eff}$ increases as the angular width of the barrier passing region of velocity space is reduced and decreases as the barrier passing particles are accelerated. Therefore, for low pumping rate ($\lambda_{cx}^{1/2} >> \lambda_{cx}^{*1/2}$ or $g >> 1$) the trapping current behaves like $j_t \sim \phi_b^{-3/2} R_b^{-1}$ for a given central cell density. Finally, we note that as $\nu_{cx} \to 0$ the trapping current vanishes (cf. Eq. (28)) since then the barrier trapped region becomes "filled".

5. CONCLUSION

An analytic expression for the trapping current in the thermal barrier region of a tandem mirror due to pitch-angle scattering has been obtained. The expression compares well (~ - 10%) with Fokker-Planck numerical studies for barrier mirror ratios of interest. Boundary layer effects in the phase space of the thermal barrier cell might be significant for filling ratios $g < 2$.

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APPENDIX A

In this Appendix we derive some of the mathematical results referred to in the text. First we want to evaluate $\Omega = (1 - \zeta_b)\, \lambda_{cx}\, G(x_o^2)/H(x_o^2) + O(\eta)$ defined in Eq. (25) and where $x_o^2 = \phi_b/T_p$, $H^{-1} = \langle 1\rangle^*$ and G is given in Eq. (18). The angular integration can be performed by using standard recurrence relations for Legendre functions (identities 7.8.4 of Ref. [19] and 8.735.2 of Ref. [9]) obtaining

$$\int_0^{\zeta_b} \frac{K_\tau(\zeta) + L_\tau(\zeta)}{K_\tau(\zeta_b) + L_\tau(\zeta_b)}\, d\zeta = \frac{(1-\zeta_b^2)^{1/2}}{\frac{1}{4} + \tau^2} \frac{L_\tau^1(\zeta_b) - K_\tau^1(\zeta_b)}{L_\tau(\zeta_b) + K_\tau(\zeta_b)} ,$$

where the Conical functions $K_\tau^1(\zeta)$ and $L_\tau^1(\zeta)$ coincide with the Spherical Harmonics $P^1_{-(1/2)+i\tau}(\zeta)$ and $P^1_{-(1/2)+i\tau}(-\zeta)$, respectively [18]. Then Ω reduces to the form

$$\Omega = \lambda_{cx}^{1/2}\, \lambda_{cx}^{*1/2} ,$$

where

$$\lambda_{cx}^{*1/2} = \lambda_{cx}^{1/2}\, (1 - \zeta_b) \frac{G(x_o^2)}{H(x_o^2)} \simeq 2 \operatorname{tg} \frac{\theta_b}{2}\, f(x_o^2) , \qquad \text{(A.1)}$$

with

$$f(x_o^2) = \lambda_{cx}^{-1/2} \frac{\langle 1\rangle^*}{\left\langle \frac{y'}{x^3} \frac{L_\tau^1\,(\zeta_b) - K_\tau^1\,(\zeta_b)}{L_\tau\,(\zeta_b) + K_\tau\,(\zeta_b)} \right\rangle^*} ,$$

$$\langle 1\rangle^* = \frac{2e^{x_o^2}}{\sqrt{\pi}}\,\Gamma(\tfrac{3}{2},\, x_o^2) = \frac{2}{\sqrt{\pi}}\, x_o + e^{x_o^2}\,\mathrm{erfc}x_o\ , \qquad \text{(A.2)}$$

and y' and τ have been defined in Eqs. (6',17), respectively. The denominator of $f(x_o^2)$ can be evaluated for a typical $x_o^2 > 1$ by approximating $\frac{L_\tau^1(\zeta_b) - K_\tau^1(\zeta_b)}{L_\tau(\zeta_b) + K_\tau(\zeta_b)} \simeq \tau$ (relations 7.1 and 13.4 of Ref. [18]) and $\tau \simeq (\frac{2}{\lambda_{cx}y'})^{1/2}\ x\ ^{3/2}$ to yield

$$f(x_o^2) = \frac{\Gamma(3/2,\, x_o^2)}{2^{1/2}\ \Gamma(3/4,\, x_o^2)}\ ,$$

and asymptotically (relation 6.5.32 of Ref. [21]) we obtain

$$f(x_o^2) \simeq \frac{x_o^{3/2}}{\sqrt{2}}\,[1 + 0(x_o^{-2})]\ . \qquad \text{(A.3)}$$

Then, the parameter λ_{cx}^* takes the form

$$\lambda_{cx}^* \simeq \frac{x_o}{2R_b}\,[1 + 0\ (x_o^{-2}\ ,\ R_b^{-1})]\ . \qquad \text{(A.4)}$$

It is also of interest to evaluate the asymptotic behavior ($\tau \to \infty$) of the angular distribution function in the barrier trapped region

$$M(\zeta) = M_o\,\frac{K_\tau(\zeta) + L_\tau(\zeta)}{K_\tau(\zeta) + L_\tau(\zeta_b)}\ .$$

Since $\tau \sim \lambda_{cx}^{-1/2}$, the asymptotic behavior $\tau \to \infty$ corresponds to an increase of the pumping rate in the barrier region. At the origin

$$M(\zeta=0) = \frac{M_o}{K_\tau(\zeta_b) + L_\tau(\zeta_b)} \frac{2\sqrt{\pi}}{|\Gamma(\frac{3}{4} + i\,\frac{\tau}{2})|^2} ,$$

where $\Gamma(z)$ is the Gamma function [9] and use has been made of identities 7.6.7 of Ref. [19] and 6.1.32 of Ref. [21]. The asymptotic behavior of $M(\zeta)$ at the origin is then

$$M(0) \sim e^{-(\frac{\pi}{2} - \theta_b)\tau} , \tag{A.5}$$

where $\theta_b = \cos^{-1} \zeta_b$ and we have used the relations 6.1.45 of Ref. [21] and (10) of Ref. [17]. Also, from the relation (10) of Ref. [17] we obtain $M(\zeta) \sim e^{-(\theta - \theta_b)\tau}$ and near the loss boundary ($\theta \simeq \theta_b$)

$$M(\zeta) \sim \exp \left\{- \left(\frac{2x^3}{\lambda_{cx}(1-\zeta_b^2)y'}\right)^{1/2} (\zeta_b - \zeta)\right\} . \tag{A.6}$$

The derivative of $M(\zeta)$ at the loss boundary can be written as

$$\frac{\partial M}{\partial \zeta} (\zeta = \zeta_b) = \frac{M_o}{(1 - \zeta_b^2)^{1/2}} \frac{L_\tau^1(\zeta_b) - K_\tau^1(\zeta_b)}{L_\tau(\zeta_b) + K_\tau(\zeta_b)} ,$$

where we have used the relation 2.7 of Ref. [18], and asymptotically behaves as

$$\frac{\partial M}{\partial \zeta} (\zeta = \zeta_b) \sim \tau , \tag{A.7}$$

according to the relations 7.1 and 13.4 of Ref. [18].

APPENDIX B

In this Appendix we discuss the effects of having a finite $\eta < 1$, i.e., "short" central cell. In this case the results given in Eq. (24) become

$$n_{ct}^{a} = \frac{n_{bt}}{A_{ct}\, I\, \lambda_{cx}\, B}\, \eta \ , \tag{B.1}$$

$$n_{ct}^{i} = \zeta_c\, G\, n_{bt} \ ,$$

$$n_{cp}^{a} = \frac{2n_{bt}}{A_{cp}\, I\, \lambda_{cx}\, B}\, \eta \ ,$$

$$n_{cp}^{i} = (1 - \zeta_b)\, G\, n_{bt} \ ,$$

$$n_{bp}^{a} = \frac{2D\, n_{bt}}{A_{bp}\, I\, H\, \lambda_{cx}\, B}\, \eta \ ,$$

$$n_{bp}^{i} = (1 - \zeta_b)\, \frac{G}{H}\, n_{bt} \ ,$$

$$n_{bt} = \frac{n_c}{G + \dfrac{\eta}{A_c\, I\, \lambda_{cx}\, B}} \ ,$$

where
$$B = 1 + \frac{I^*\, D(1-\zeta_b)}{I\, H}\, \eta \ .$$

Then the trapping current is given by

$$j_t = \frac{n_b\, \nu_{cx}}{1 + \dfrac{\Omega}{\lambda_{cx}}} \ , \tag{B.2}$$

with

$$\Omega = (1 - \zeta_b)\, \lambda_{cx} \frac{G}{H} + \frac{2D}{A_{bp}\, I\, H\, B}\, \eta \, .$$

In $O(\eta)$ the source of particles that become trapped in the barrier region is the isotropic level which is an infinite source of particles ($\eta \to 0$). Corrections of $O(\eta)$ to j_t are due to the trapping of the non-Maxwellian part (loss-cone type) of the barrier passing distribution function. In the hypothetical situation where the pumping drain would dominate the flow of central cell particles into the barrier, the isotropic level throughout the machine would be lowered as $\lambda_{cx} \to 0$ ($\eta \sim 1$, "very short" central cell). In this case $G << \eta/A_c I \lambda_{cx} B$ even though $G \sim \lambda_{cx}^{-1/2}$, (cf. Eqs. (A.1, A.4)); therefore, from Eq. (B.1) we see that $n_{bt} \sim \lambda_{cx}$, the anisotropic level $n^a \sim \lambda_{cx}^0$, the isotropic level $n^i \sim \lambda_{cx}^{1/2}$ and the trapping current $j_t \sim \nu_{cx}^0$ (cf. Eq. (B.2)). Usually this will not be the case but instead, with a large volume central cell, $n_{bt} \sim G^{-1} \sim \lambda_{cx}^{1/2}$, $n^a \sim \eta$, $n^i \sim \lambda_{cx}^0$ and the trapping current $j_t \sim \nu_{cx}^{1/2}$, as obtained from Eqs. (B.1, B.2).

REFERENCES

[1] Baldwin, D.E., Logan, B.G., Phys. Rev. Lett. 43 (1979) 1318.

[2] Kesner, J., Phys. Fluids 14 (1971) 868.

[3] Baldwin, D.E., Logan, B.G., Simonen, T.C., eds. UCID-18496 (1980).

[4] LoDestro, L.L., UCAR-10060-80-2 (1980).

[5] Futch, A.H., LoDestro, L.L., UCRL-50051-81-3 (to be published).

[6] BenDaniel, D.J., Allis, W.P., Plasma Physics 4 (1962) 31.

[7] Rosenbluth, M.N., MacDonald, W.M., Judd, D., Phys. Rev. 107 (1957) 1.

[8] Bing, G.F., Roberts, J.E., Phys. Fluids 4 (1961) 1039.

[9] Gradshteyn, I.S., Ryzhik, I.M., Table of Integrals, Series and Products, Academic Press, New York (1965).

[10] Spitzer, L., Jr., Physics of Fully Ionized Gases, Interscience, New York (1962).

[11] Trubnikov, B.A., Sov. Phys. JETP 7 (1958) 926.

[12] Li, X.Z., Emmert, G.A., University of Wisconsin Report UWFDM-393 (1980).

[13] Kaufman, A.N., TID-7520 (Pt. 2) (1956) 387.

[14] Baldwin, D.E., Rev. Mod. Phys. 49 (1977) 317.

[15] Sivukhin, D.V., Reviews of Plasma Physics, edited by Leontovich, M.A., Vol. IV, Consultants Bureau, New York (1966) 93.

[16] Prasad, G., A Treatise on Spherical Harmonics and the Functions of Bessel and Lame, Part II, Benares, India (1932) Chaps. I and II.

[17] Zhurina, M.I., Karmazina, L.N., Tables of Legendre Functions $P_{-(1/2)+i\tau}(x)$, Pergamon Press, New York (1964) Part I and (1965) Part II.

[18] Zhurina, M.I., Karmazina, L.N., Tables and Formulae for the Spherical Functions $P^m_{-(1/2)+i\tau}(z)$, Pergamon Press, New York (1966).

[19] Lebedev, N.N., Special Functions and their Applications, Dover (1972) Chaps. 7 and 8.

[20] Carrera, R., Callen, J.D., Shaing, K.C., Bull. Am. Phys. Soc. 26 (1981) 902 (to be published).

[21] Abramowitz, M., Stegun, I.A., Handbook of Mathematical Functions, Dover, New York (1965).