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OPTIMIZATION CONSIDERATIONS IN THE DESIGN OF A TOKAMAK REACTOR

by

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## I. Introduction

The University of Wisconsin fusion study group has been investigating CTR feasibility for a Tokamak reactor<sup>(1)</sup>. This paper is concerned with the optimization studies that have been carried out as a part of this program. The present paper differs from previous work<sup>(2,3)</sup> in several ways: (1) it starts from an existing conceptual design, (2) it limits  $\beta_p$  to  $\sqrt{A}$ , (3) it optimizes both the power and, more importantly, the cost per unit power. Only relative costs will be discussed since absolute costs depend on the plasma assumptions, fueling and heating schemes, and technology advances.

The first section will discuss power maximization and minimum cost with respect to the plasma radius for a fixed major radius. The second section will add the constraint of fixed power and the third section will consider the additional constraint imposed by the air or iron core transformer.

It will be shown that the core constraint makes the cost relatively independent of the maximum magnetic field strength for reactor powers greater than 5000 megawatts thermal and that the cost actually increases with magnetic field strength for reactor powers less than 5000 Mw. As a result, the maximum magnetic field strength should be determined by the desired neutron wall loading.

## II. Optimization with Respect to the Plasma Radius for a Fixed Major Radius

The power is expressed by the following:

$$(1a) \quad P = f \frac{n^2}{4} \langle \sigma v \rangle Q 2\pi^2 a^2 R$$

where  $n$  is the ion density,  $\langle \sigma v \rangle$  the fusion reaction rate,  $Q$  the amount of energy per fusion event,  $a$  and  $R$  the plasma and major radius, respectively, and  $f$ , a factor to account for the ion spatial profile in temperature and density. For a  $\beta$ -limited reactor, this expression becomes

$$(1b) \quad P = \frac{C_o \beta_p^2 B_t^4 a^2 R}{q^4 A^4}$$

Using the relationship  $B_t R = \text{constant}$  and the assumption that

$\beta_p^{\max} = \sqrt{A}$  at  $r = a$  yields

$$(1c) \quad P = C_o \frac{B_m^4}{q^4} \left(1 - \frac{\tau_b}{R} - \frac{a}{R}\right)^4 \frac{a^5}{R^2}$$

with

$$C_o = \frac{\pi^2 f \langle \sigma v \rangle Q}{32 \mu_0^2 k^2 T^2}$$

and where  $B_t$  is the toroidal magnetic field at the plasma axis and  $B_m$  is the maximum field value at the superconductor. Here  $A$  is the aspect ratio  $= \frac{R}{a}$ ,  $\tau_b$  is the vacuum gap plus blanket and shield thickness, and  $\beta_p$  is the ratio of particle pressure to poloidal magnetic field pressure. The stability factor,  $q$ , is defined as the reciprocal of the rotational transform angle  $i$ ;  $q = \frac{2\pi}{i} = \frac{1}{A} \frac{B_t}{B_p}$ . The Tokamak geometry is shown in Figure 1.

The present study uses the conservative limit  $\beta_p^{\max} = \sqrt{A}$  which, if the bootstrap current materializes, will enable Tokamaks to operate steady state. The analysis will also assume a fixed value for  $T$  which

implies the temperature is independent of variations in the other parameters or that the other parameters can be varied to maintain the temperature at the desired value. Conn<sup>(4)</sup>, in studying steady-state solutions for Tokamak systems using self consistent energy balance and diffusion equations, has found the temperature to be very insensitive to variations in the plasma radius for operating temperatures of 10-15 keV. While the fixed temperature assumption is no longer valid for operating temperatures above 30 keV, the conclusions of this paper are still found to be qualitatively correct.

By maximizing (1c) with respect to the plasma radius,  $a$ , the maximum power is found to occur at  $a = \frac{5}{9} R(1 - \frac{\tau_b}{R})$ . The maximum power is given by

$$(2) \quad P^{\max} = C_o \frac{B_m^4}{q^4} \left(1 - \frac{\tau_b}{R}\right)^9 \left(\frac{4}{9}\right)^4 \left(\frac{5}{9}\right)^5 R^3$$

The relative power as a function of plasma radius is shown in Figure 2a which is a plot for a reactor having  $R = 12.5\text{m}$  and  $\tau_b = 2.5\text{m}$ , parameters from Reference 1. As will be shown in a later section, core space limitations as well as neutron wall loadings may limit the plasma radius to values less than the unconstrained optimum.

To minimize the relative cost/power, the following assumptions appear justified

- (1) the cost of the reactor is primarily the magnet cost
- (2) the magnet cost is proportional to the energy stored in the magnetic field,  $E_s$ .

Lubell<sup>(5)</sup> has shown that the magnet cost is proportional to  $E_s^{0.8}$  and

Boom<sup>(6)</sup> has shown, for more limited cases, that the cost is proportional

to  $E_s^{1.2}$ . The stored magnetic energy can be expressed as

$$(3a) \quad E_s \sim B_t^2 R^3 \left[ 1 - \sqrt{1 - \left(\frac{r_m}{R}\right)^2} \right]$$

and approximated by

$$(3b) \quad E_s \sim B_t^2 R r_m^2 = B_m^2 [R - r_m]^2 \frac{r_m^2}{R}$$

where  $r_m$  is the magnet inner radius and is equal to  $(a + \tau_b)$ . The approximate expression is accurate to 10% for  $r_m/R$  ratios less than 0.7.

Combining (3b) with (1c) the cost per unit power can be expressed as

$$(4) \quad \text{Cost/Power} = K = \frac{C_1 r_m^2 R^5}{B_m^2 q^4 (R - r_m)^2 a^5}$$

This expression can be minimized with respect to the plasma radius,  $a$ , and a minimum found where

$$(5) \quad a^{\min} = \frac{3R}{10} \left[ 1 + \sqrt{1 + \frac{40\tau_b}{9R}} \right] - \tau_b$$

Figure (2b) shows the variation of  $K$  with " $a$ ". These studies have indicated that the optimum aspect ratios are found to have values between 1.7-2.5.

The variation of power and cost per power as a function of the major radius,  $R$ , is shown in Figure 3. It can be seen that while the power increases by a factor of eight in going from  $R = 10m.$  to  $R = 15m.$  in accordance with equation (2), the cost per power decreased by only a factor of two. Therefore, there is little economic gain in building

Tokamak reactors with  $R > 15m$ .

The same analysis has been done for the limit  $\bar{\beta}_p = A$  with similar conclusions.

### III. Optimization for a Fixed Power

We now look for a set of reactor parameters  $(R,a)$  that minimizes the cost (the stored magnetic energy) subject to the constraint of fixed power. We choose a major radius,  $R = aA$ , solve the power equation (1c) for the plasma radius,  $a$ , and then calculate the stored magnetic energy,  $E_s$ , given by equation (3b). We repeat the procedure, generating a locus of constant power points in the  $(R,a)$  plane and look for a minimum in  $E_s$ . Figures 4, 5, and 6 show the variation of Cost/Power vs. Aspect Ratio at a fixed power of 5000 Mw for different values of  $\tau_b$ , the blanket and shield thickness;  $B_m$ , the maximum magnetic field at the superconductors; and,  $q$ , the stability factor. A minimum can be found and as shown in the figures, occurs at aspect ratios 1.6-1.7. This minimum in the aspect ratio has been found to be independent of the power (1-10 GW), maximum magnetic field strength (6-20 T), and thickness of blanket and shield (1.5m.-2.5m.). The use of the more accurate expression for  $E_s$  makes the minimum more pronounced.

A plot of neutron wall loading,  $P_{nw}$ , is shown in Figure 7 for two values of the stability factor,  $q$ . The peak in the neutron wall loading occurs for aspect ratios 2.5-3.5 over a wide range of power, magnetic field, and blanket-shield thickness. It will be shown in the next section that the core constraint forces the parameters for the optimum cost reactor to be close to the maximum neutron wall loading.



A plot of the minimum cost/power at three power levels, as a function of maximum magnetic field strength at the superconductors is shown in Figure 8. The cost decreases monotonically with magnetic field. This behavior is altered dramatically when a core constraint is imposed as described in the next section.

#### IV. Optimization at a Fixed Power with Core Constraint

Since all Tokamaks require an air or iron core transformer to create the toroidal plasma current, space must be provided for this transformer, its windings and the toroidal magnet windings. The following constraint must be satisfied

$$(6) \quad r_o + H + \tau_b + a \leq R$$

where  $r_o$  is the radius of the transformer windings,  $H$  is the combined thickness of the primary and toroidal magnet windings,  $\tau_b$  is the vacuum gap plus blanket and shield thickness, and "a" the plasma radius (see Figure 1). For an air-core superconductor winding,  $r_o$  can be computed as follows<sup>(7)</sup>.

$$(7) \quad \pi r_o^2 B_{sp} = \Phi_c = 1/2 L_p I_p$$

where  $B_{sp}$  is the superconducting primary field and  $L_p$ ,  $I_p$  are the plasma ring inductance and current, respectively. Since

$$(8) \quad B_p = \frac{\mu_0 I_p}{2\pi a} = \frac{B_t}{qA}$$

and

$$(9) \quad L_p \approx \mu_0 R (\ln 8A - 1.75)$$

then

$$(10) \quad r_o = a \left[ \frac{B_m (R - r_m)}{q B_{sp}} (\ln 8A - 1.75) \right]^{1/2}$$

For the family of  $(R,a)$  generated in the previous section, the  $(R,a)_{\min}$  is chosen that satisfies the constraint expressed by Equation 6.

The effect of the core constraint is to increase the minimum cost, move the minimum cost reactor to higher aspect ratios and to introduce an aspect ratio dependence on magnetic field strength; the larger the magnetic field, the larger the aspect ratio for minimum cost.

Figure 9 shows these trends. For power levels less than 5000 Mw(th), the cost/power actually increases with magnetic field strength. By comparing Figure 9 with Figure 8, it can be seen that the cost of a 1000 Mw(th) reactor is a factor of three higher and the 10,000 Mw(th) reactor 50% higher at 12T because of the core constraint. Physically, the larger magnetic field strength requires smaller major and plasma radii to satisfy the power constraint. In order to satisfy the core constraint, a set of  $(R,a)$  values must be chosen that is far from optimum. A similar plot for a larger stability value,  $q = 1.75$ , is shown in Figure 10. The same qualitative behavior exists except it is not as severe as for the  $q = 1.5$  case. However, when  $q = 2.5$ , the core constraint has only a small effect. In this case, since the power is inversely proportional to  $q^4$ , a larger major and plasma radius must be chosen to satisfy the power constraint. This larger  $(R,a)$  more easily satisfies the core constraint but the cost/power is more than twice the  $q = 1.5$  case.

The core constraint has also moved the minimum cost reactor to higher neutron wall loadings. Materials studies have indicated that the neutron wall loading may be the limiting constraint in the design of a power reactor. In both Figure 9 and Figure 10, the neutron wall loadings are shown in parenthesis.

Figure 11 and Figure 12 show the sensitivity of the results to variations in the blanket-shield thickness and primary and toroidal windings, respectively. The absolute cost/power is very sensitive to these variations because the smaller thickness allows the same magnetic field on axis for a lower field at the magnet and consequently, a lower cost. Also, the same behavior with magnetic field exists as in Figure 9; for the 2000 Mw reactor an increase in cost for an increase in magnetic field while the cost of the 5000 Mw reactor is almost independent of the field. Since it is unreasonable to expect blanket-shield plus vacuum gap plus insulation thickness to be less than 1.5m. and since the toroidal plus primary windings thickness will be at least 1 m., the present results are realistic.

Some typical parameters for a 5000 Mw (th) reactor operating in the 10-15 Kev range are shown in Table I.

#### V. Summary and Conclusion

Using the University of Wisconsin conceptual design for a Tokamak reactor, we have attempted to optimize the cost/power subject to two constraints - fixed power and core space. It has been found that the core constraint significantly increases the cost for power reactors less than 5000 Mw(th). Further, the cost/power is relatively independent of the maximum magnetic field strength for reactor powers greater than 5000 Mw(th) and the cost actually increases with increasing magnetic field for reactor powers less than 5000 Mw(th). These results coupled with the neutron wall loading constraint imposed by material damage considerations will limit the maximum magnetic field strength.

Also, the reactor should be designed with the smallest blanket-shield consistent with tritium breeding and energy deposition and with the smallest windings consistent with stress and field considerations.

#### Acknowledgements

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Table I

5000 Mw Tokamak Reactor  
(H=2.0 m.)

$B_m$ (Tesla)	8.6	8.6	8.6	9.5	7.5
$\tau_b$ (meter)	1.5	2.0	2.5	2.5	2.5
$q$	1.5	1.5	1.5	1.75	1.75
$A$	2.5	2.5	2.6	2.4	2.1
$a$ (meter)	4.0	4.4	4.6	5.2	7.0
$R$ (meter)	10.0	10.9	12.0	12.3	14.8
$P_{nw}$ (Mw/m <sup>2</sup> )	1.8	1.5	1.3	1.15	0.7
$K$ (Cost/Power)	55	75	100	125	130

## FIGURE CAPTIONS

- Figure 1 - Tokamak Geometry where  $R$  = major radius,  $a$  = plasma radius,  $\tau_b$  = vacuum gap plus blanket plus shield thickness,  $r_o$  = radius of air or iron core transformer, and  $h$  = thickness of primary and toroidal field coils.
- Figure 2 - (a) Power vs. plasma radius,  $a$  (b) Cost per power vs. plasma radius. Both curves for  $R = 12.5m$ ,  $\tau_b = 2.5m$ .
- Figure 3 - (a) Power vs. major radius  $R$  plotted at  $P^{\max}$  (b) Cost per power vs. major radius plotted for  $a^{\min}$ .
- Figure 4 - Cost/Power vs. Aspect Ratio for  $P = 5000 \text{ Mw(th)}$ ,  $q = 1.5$  and  $\tau_b = 1.5m$ . for two values of  $B_m$ , 8.6T and 12.0T.
- Figure 5 - Cost/Power vs. Aspect Ratio for  $P = 5000 \text{ Mw(th)}$ ,  $q = 1.5$  and  $\tau_b = 2.5m$ . for two values of  $B_m$ , 8.6T and 12.0T.
- Figure 6 - Cost/Power vs. Aspect Ratio for  $P = 5000 \text{ Mw(th)}$ ,  $\tau_b = 2.5m$ . and  $B_m = 8.6T$  for two values of  $q$ , 1.5 and 1.75.
- Figure 7 - Neutron Wall Loading vs. Aspect Ratio for  $P = 5000 \text{ Mw(th)}$ ,  $B_m = 8.6T$ ,  $\tau_b = 2.5$  for two values of  $q$ , 1.5 and 1.75.
- Figure 8 - Minimum Cost/Power vs. Magnetic Field with no core constraint for  $q = 1.5$ ,  $\tau_b = 2.5m$ , and  $P = 1000 \text{ Mw}$ ,  $5000 \text{ Mw}$ ,  $10,000 \text{ Mw}$ .
- Figure 9 - Cost/Power vs. Magnetic Field with core constraint for  $q = 1.5$ ,  $\tau_b = 2.5m$ ,  $H = 2.0m$  and  $P = 1000 \text{ Mw}$ ,  $2000 \text{ Mw}$ ,  $3000 \text{ Mw}$ ,  $5000 \text{ Mw}$ ,  $10,000 \text{ Mw}$ . Neutron wall loadings  $P_{nw}$  are given in parenthesis.
- Figure 10 - Cost/Power vs. Magnetic Field with core constraint for  $q = 1.75$ ,  $\tau_b = 2.5m$ ,  $H = 2.0m$  and  $P = 1000 \text{ Mw}$ ,  $2000 \text{ Mw}$ ,  $3000 \text{ Mw}$ ,  $5000 \text{ Mw}$ ,  $10,000 \text{ Mw}$ . Neutron wall loadings  $P_{nw}$  are given in parenthesis.
- Figure 11 - Cost/Power vs. Magnetic Field with core constraint for  $q = 1.5$ ,  $H = 2.0$ ,  $P = 2000 \text{ Mw}$ ,  $5000 \text{ Mw}$  for two values of  $\tau_b$ , 1.5m and 2.5m.
- Figure 12 - Cost/Power vs. Magnetic Field with core constraint for  $q = 1.5$ ,  $\tau_b = 1.5m$ . at  $P = 1000 \text{ Mw}$ ,  $5000 \text{ Mw}$  for  $H = 1.0m$ ,  $2.0m$ ,  $3.0m$ .

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# TOKAMAK GEOMETRY

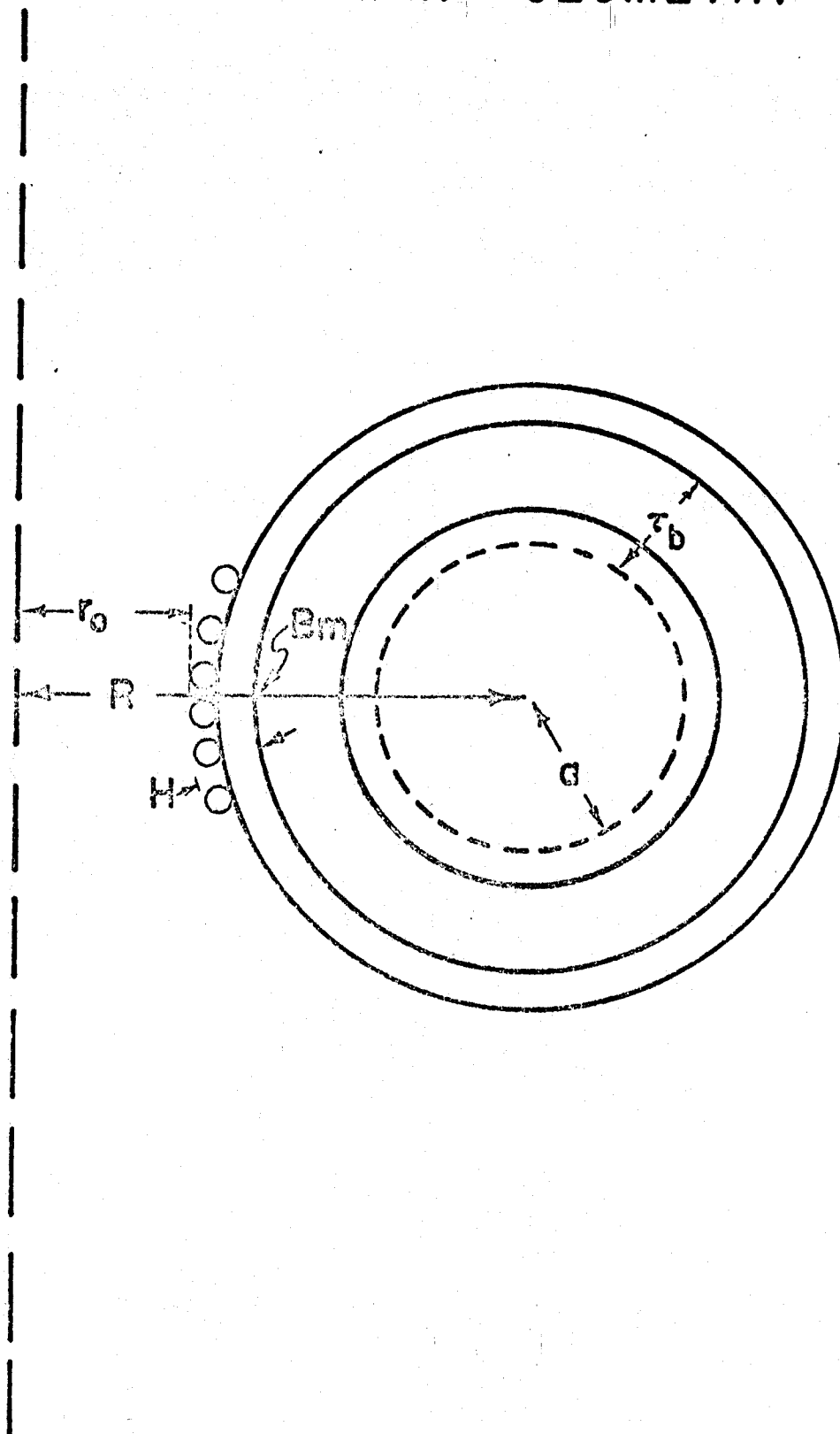


FIGURE 1



POWER AND COST / POWER VS. PLASMA RADIUS

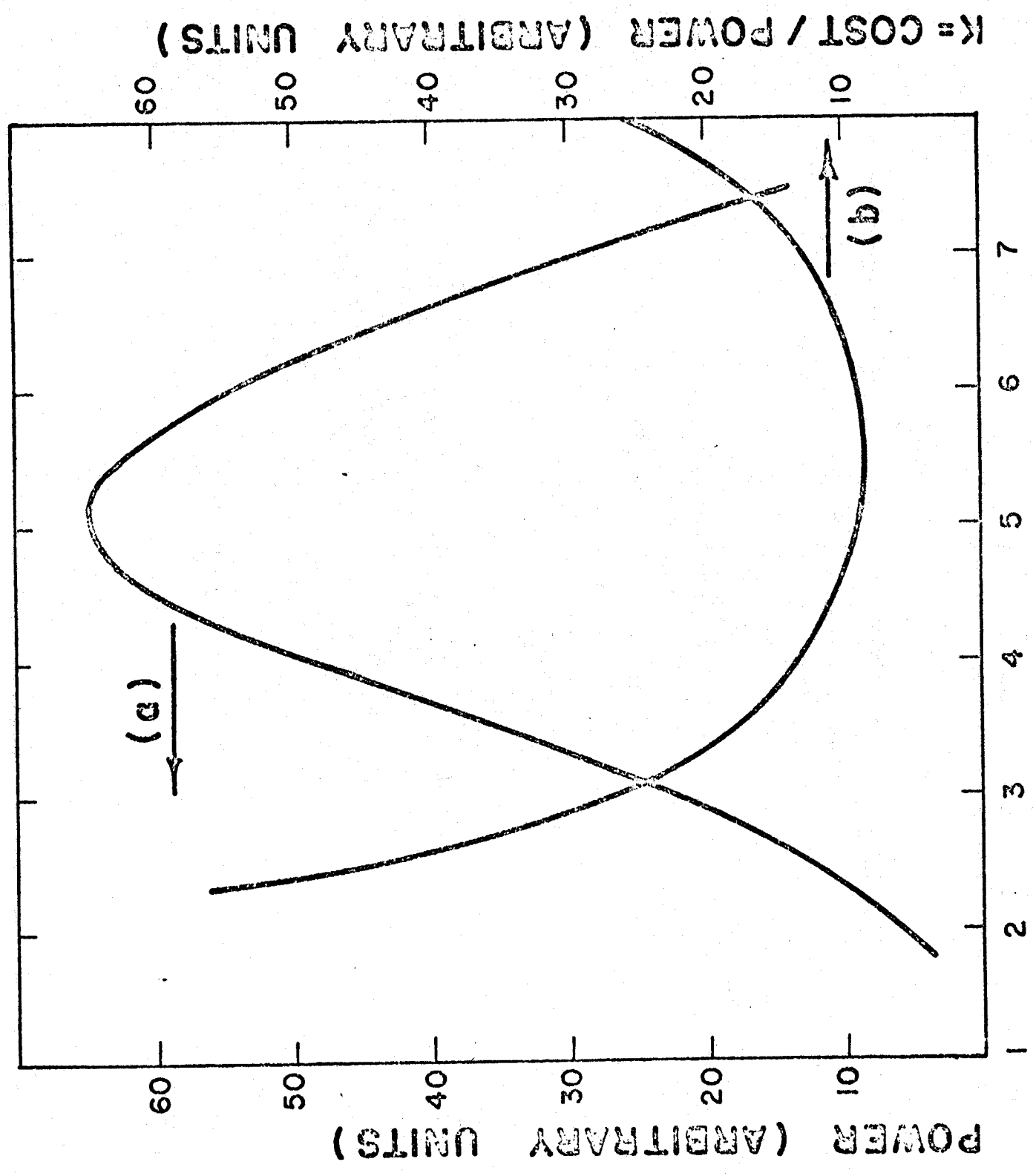


FIGURE 2

a (METERS)

POWER AND COST / POWER vs.  
MAJOR RADIUS

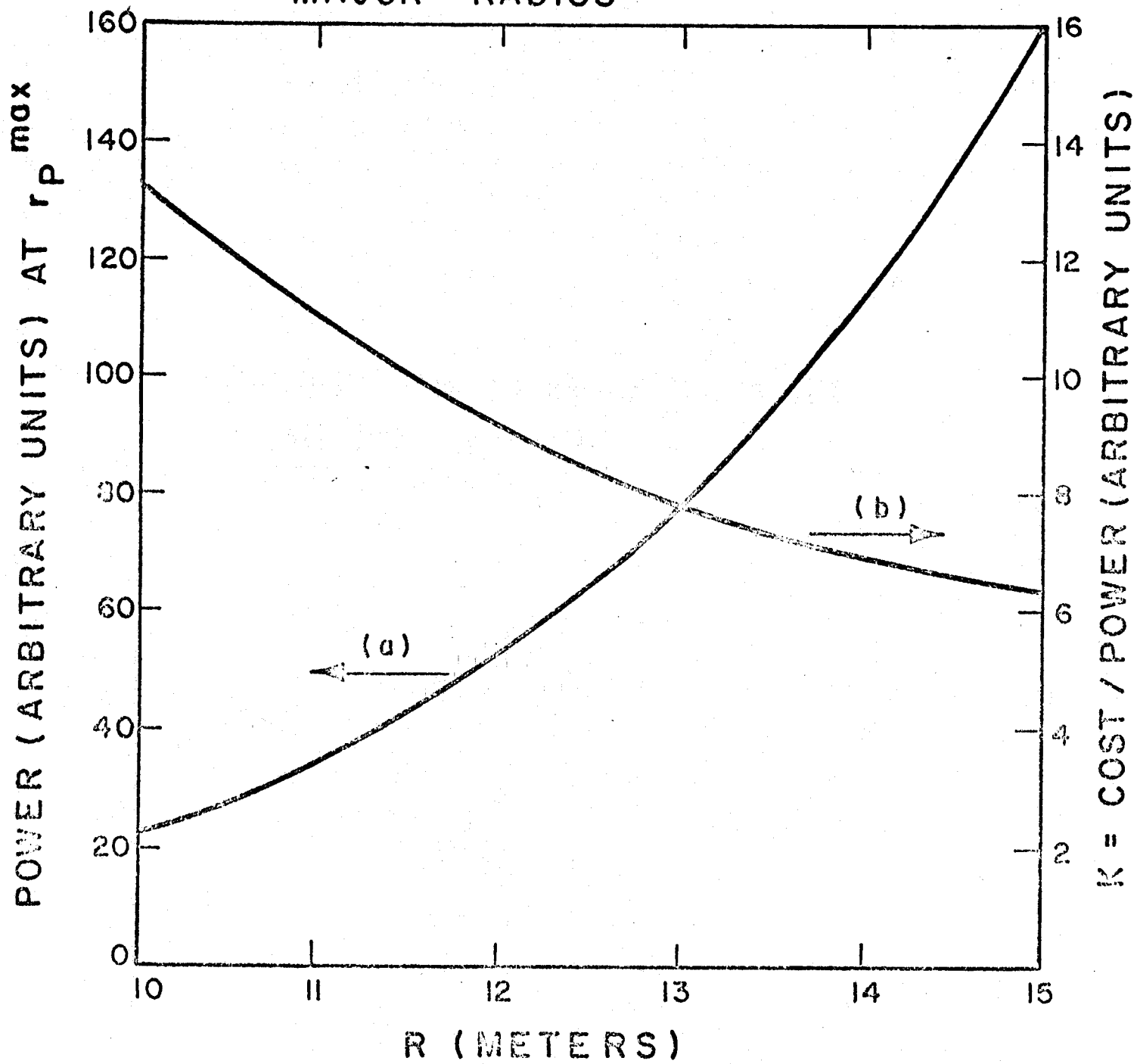


FIGURE 3

# COST / POWER vs. ASPECT RATIO

K = COST / POWER (ARBITRARY UNITS)

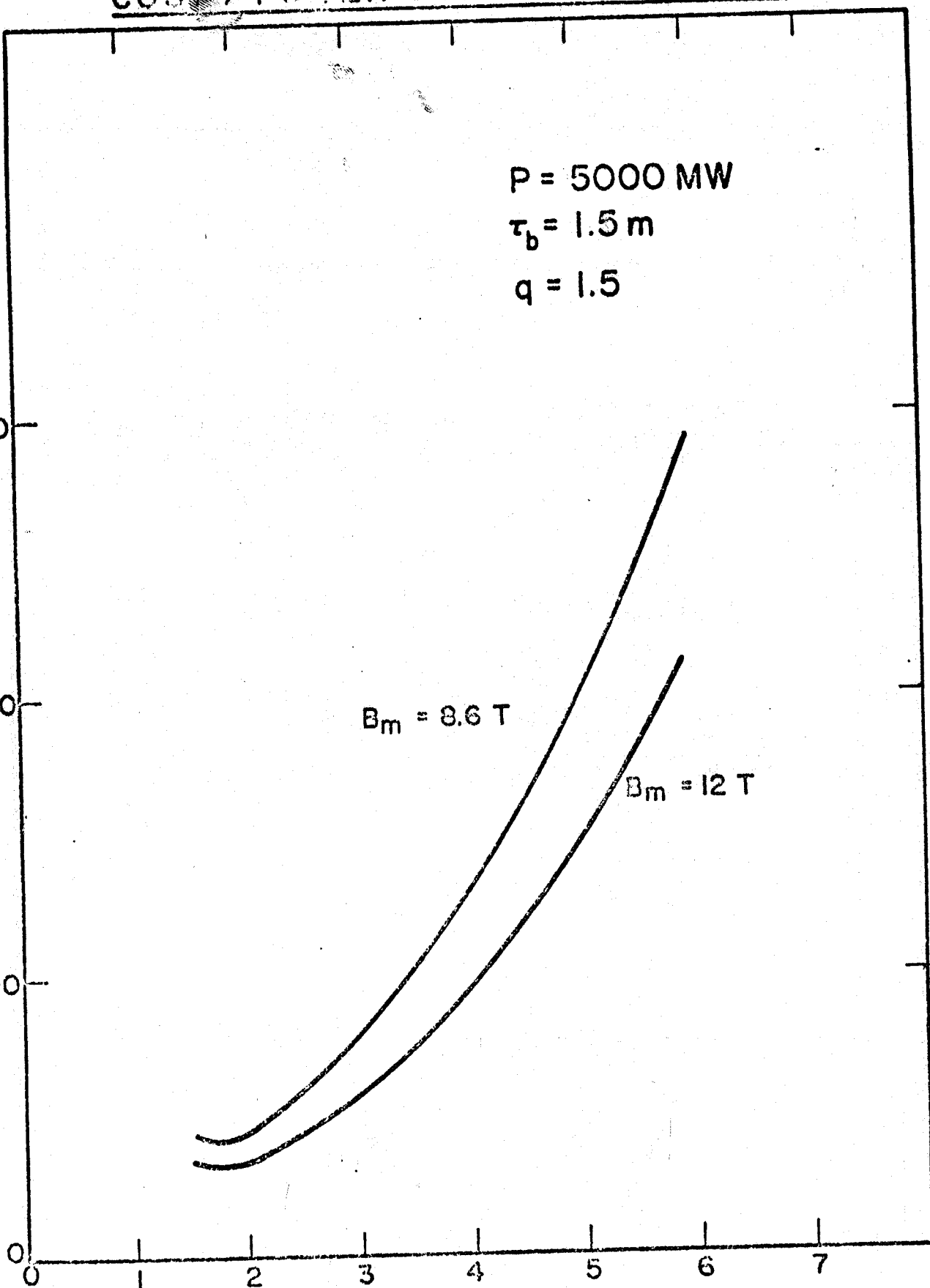
P = 5000 MW

$\tau_b = 1.5$  m

q = 1.5

$B_m = 8.6$  T

$B_m = 12$  T



ASPECT RATIO

FIGURE 4

$K = \text{COST} / \text{POWER} \quad (\text{ARBITRARY UNITS})$

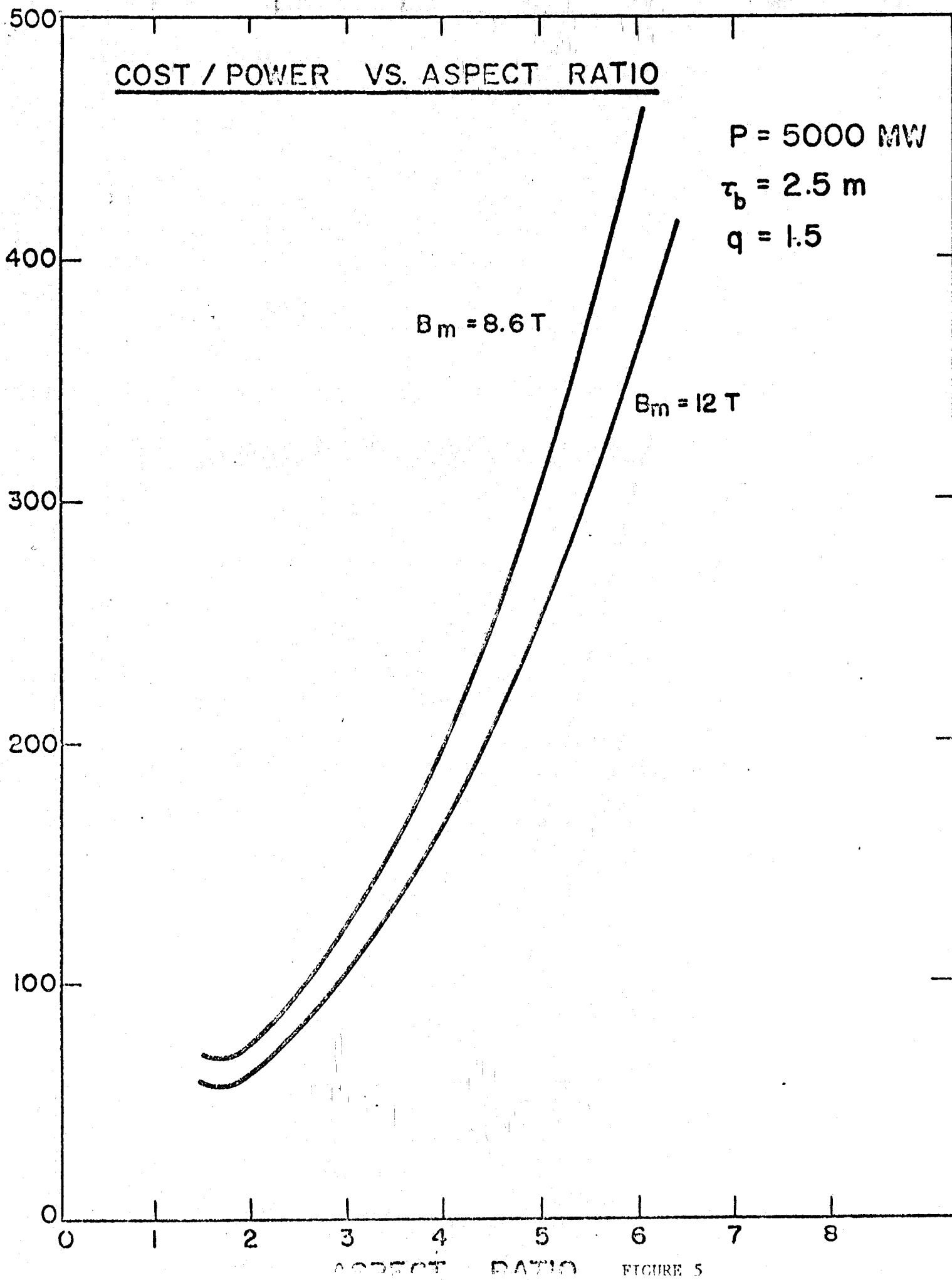


FIGURE 5

K = COST / POWER (ARBITRARY UNITS)

COST / POWER vs. ASPECT RATIO

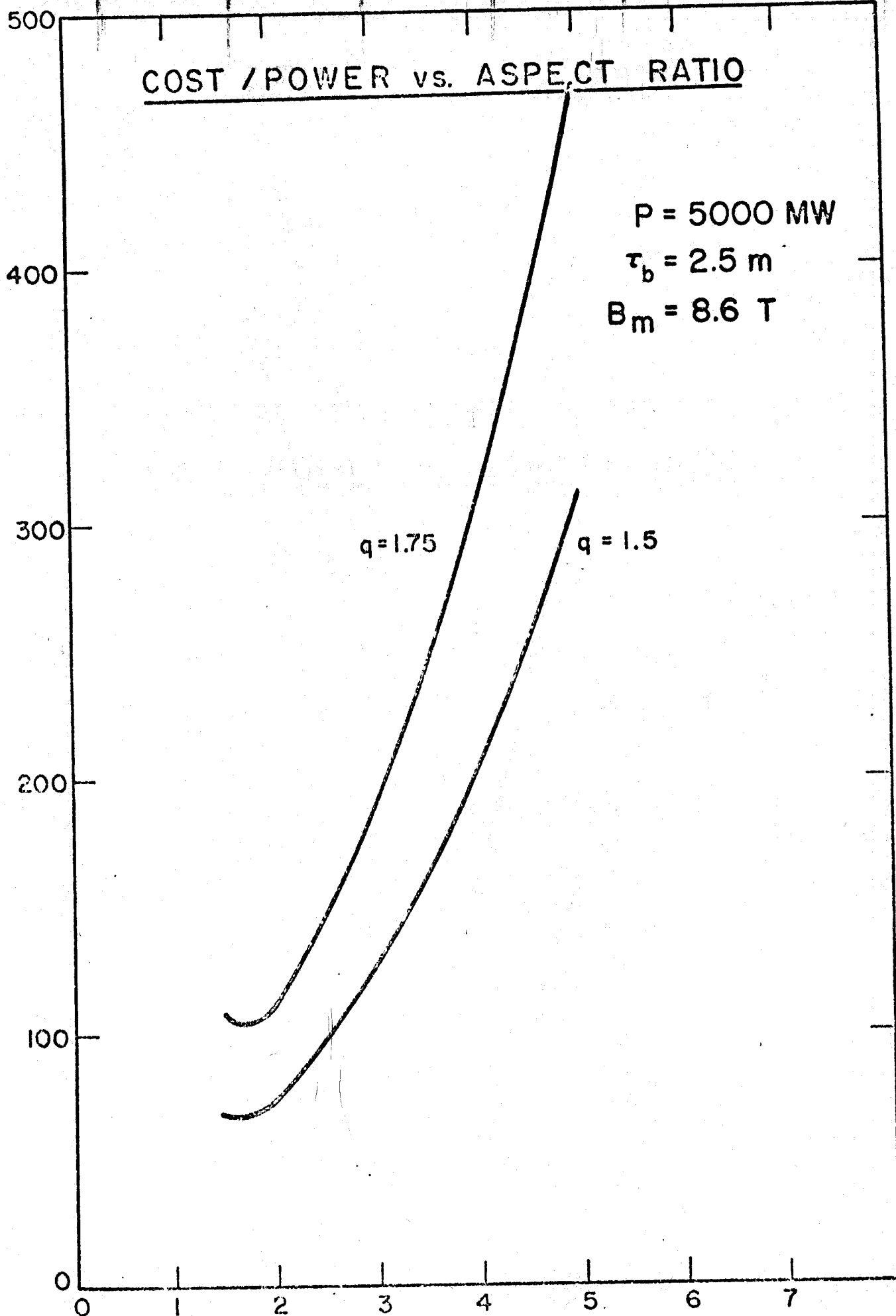
$P = 5000 \text{ MW}$   
 $\tau_b = 2.5 \text{ m}$   
 $B_m = 8.6 \text{ T}$

$q = 1.75$

$q = 1.5$

ASPECT RATIO

FIGURE 6



NEUTRON WALL LOADING  
VS.  
ASPECT RATIO

$P = 5000 \text{ MW}$   
 $B_m = 8.6 \text{ T}$   
 $\tau_b = 2.5 \text{ m}$

$q = 1.5$

$q = 1.75$

$P_{NW} (\text{MW/M}^2)$

1.4  
1.3  
1.2  
1.1  
1.0  
0.9  
0.8  
0.7  
0.6  
0.5  
0.4  
0.3

1

2

3

4

5

6

7

8

ASPECT RATIO

FIGURE 7

MINIMUM COST / POWER VS. MAGNETIC FIELD  
(NO CORE CONSTRAINT)

$$q = 1.5$$

$$\tau_b = 2.5 \text{ m}$$

K = COST / POWER (ARBITRARY UNITS)

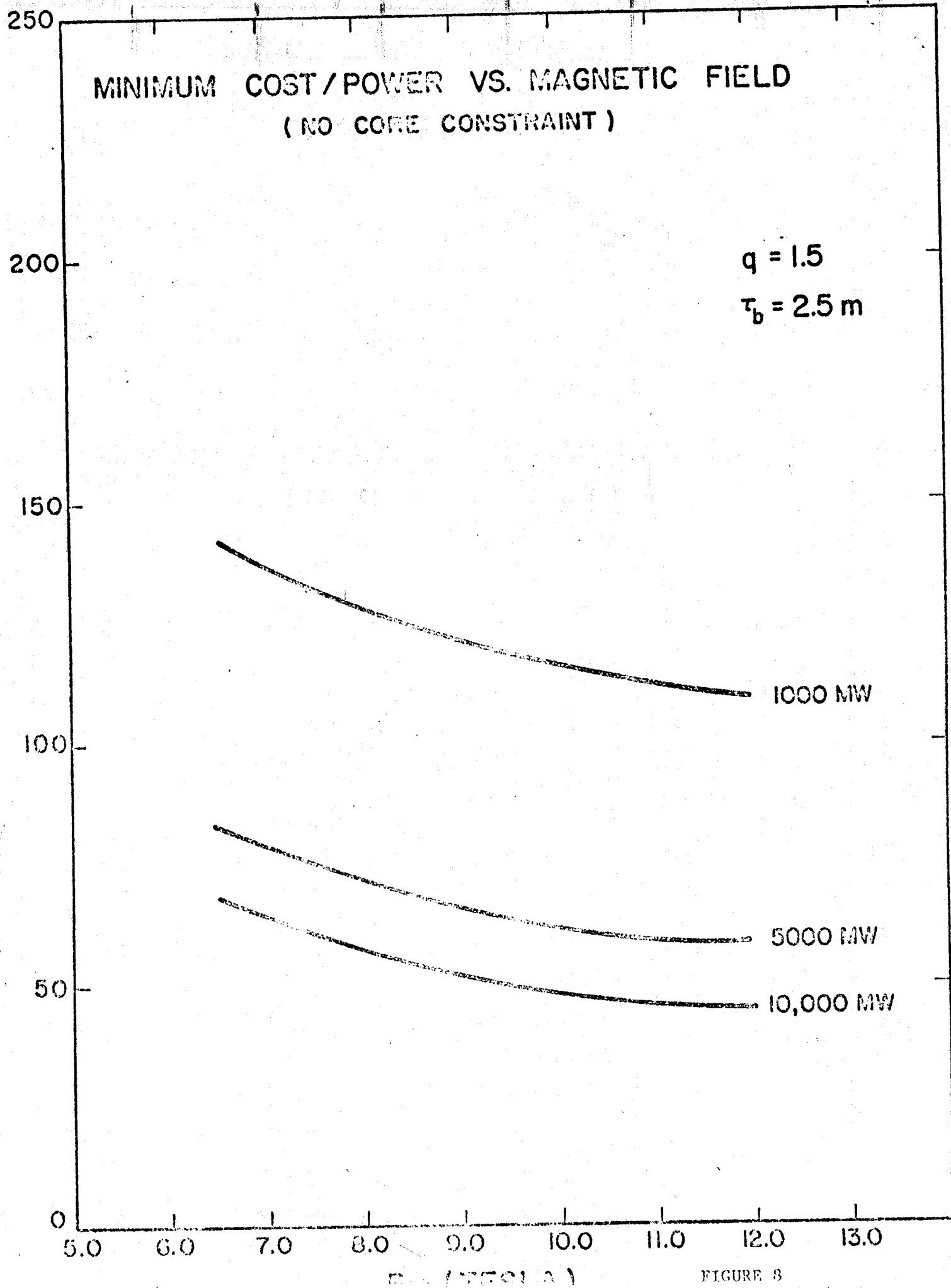


FIGURE 8

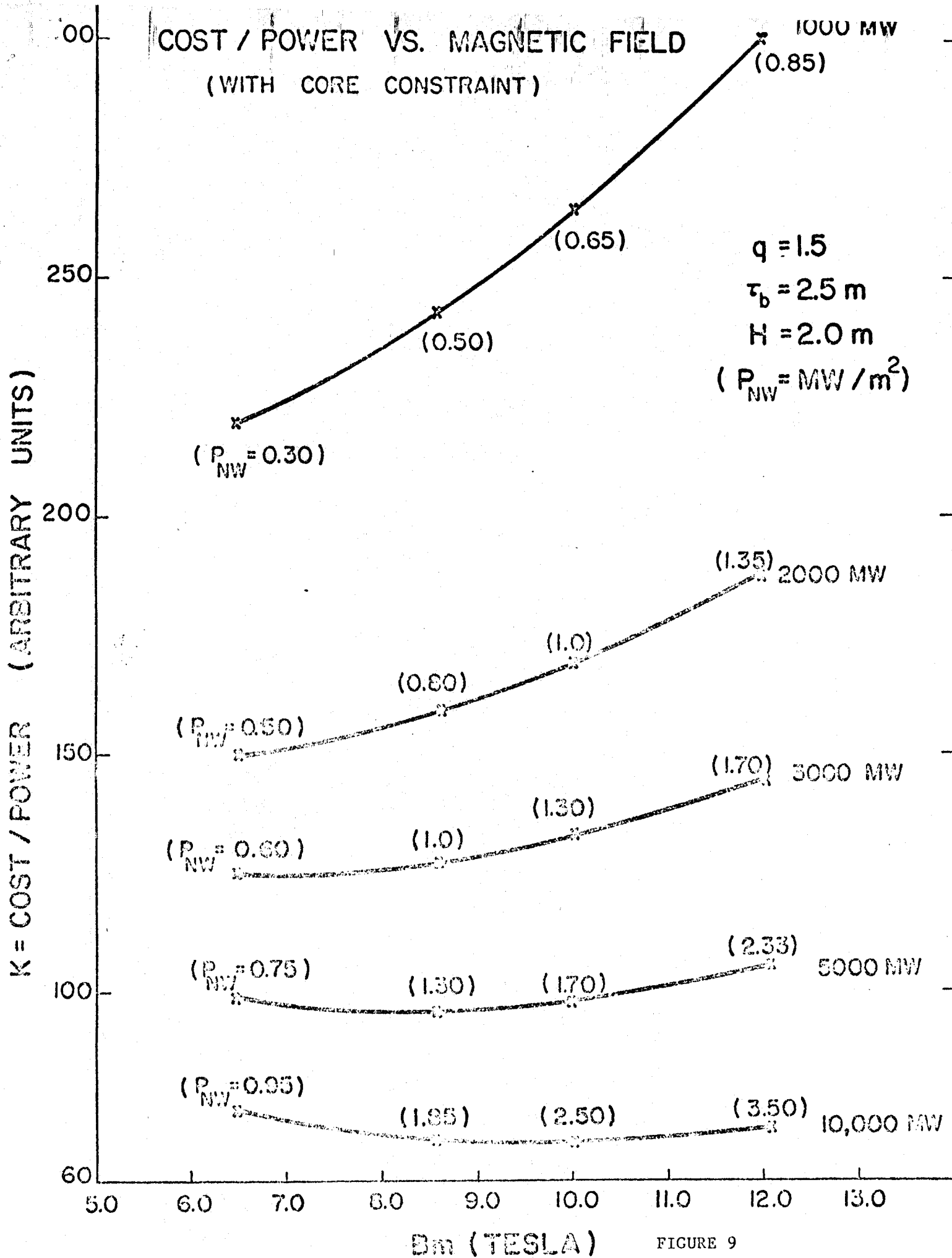


FIGURE 9



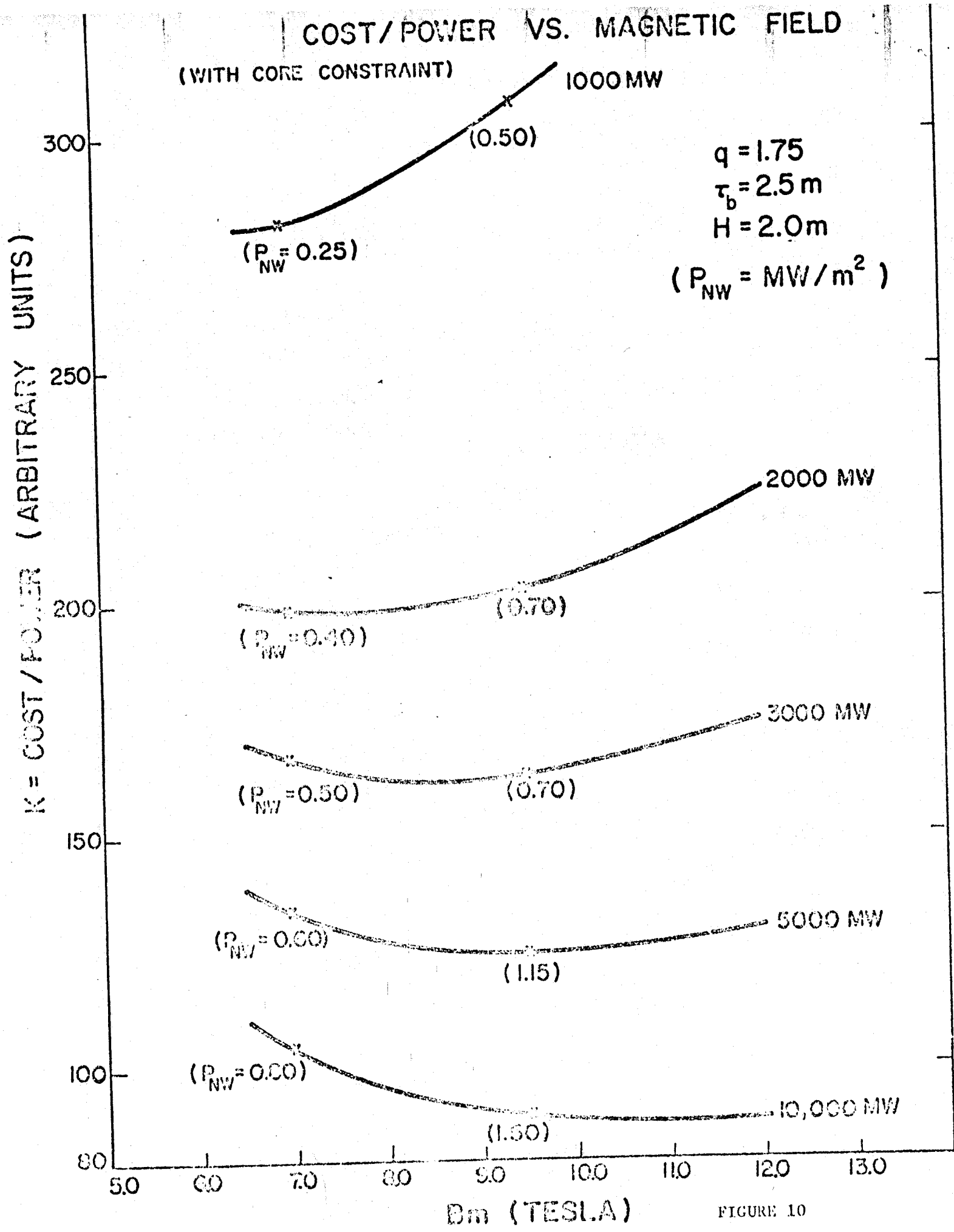


FIGURE 10

# COST/POWER VS. MAGNETIC FIELD (WITH CORE CONSTRAINT)

$q = 1.5$   
 $H = 2.0 \text{ m}$

K = COST/POWER (ARBITRARY UNITS)

200

150

100

50

0

2000 MW

$\tau_b = 2.5 \text{ m}$

$\tau_b = 2.5 \text{ m}$

5000 MW

2000 MW

$\tau_b = 1.5 \text{ m}$

$\tau_b = 1.5 \text{ m}$

5000 MW

5.0

6.0

7.0

8.0

9.0

10.0

11.0

12.0

13.0

$B_m$  (TESLA)

FIGURE 11

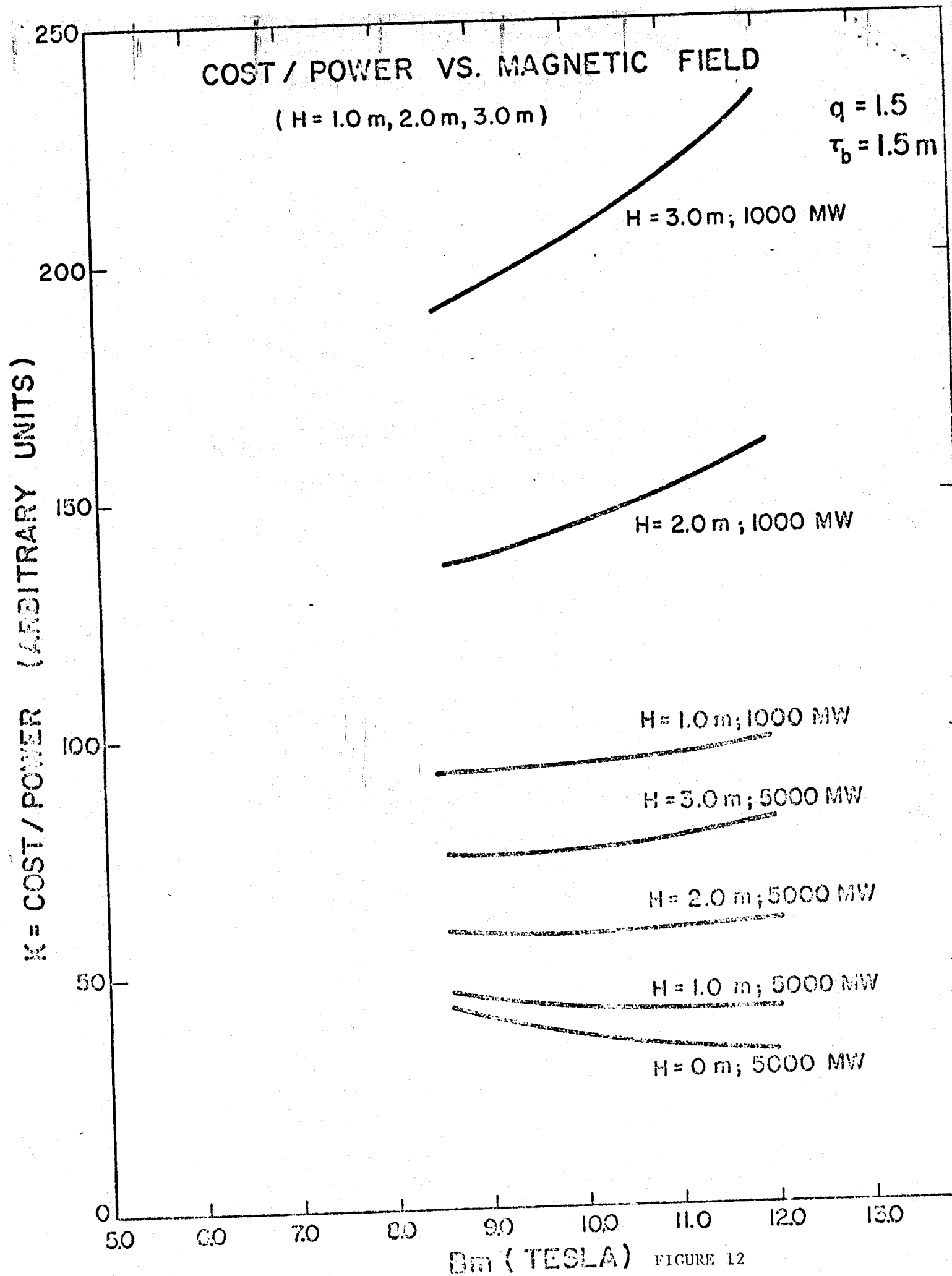


FIGURE 12