Alpha Particle Loss and Energy Deposition in Tandem Mirrors

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Abstract

The Fokker-Planck equation is solved for alpha particles slowing down in a background plasma of electrons and ions in the central cell of a tandem mirror. The magnetic field is assumed to be a square well of arbitrary mirror ratio, and a substantial number of angular harmonics of the Fokker-Planck equation are retained. The resulting loss of alpha particles by scattering into the magnetic loss cone and the distribution of alpha particle energy to ions and electrons are given. For typical tandem mirror reactor parameters ($T_e \sim 30$ keV, $\phi \sim 100$ keV, $R_m \sim 4$), more than half of the alpha particles are born in or scatter into the loss cones, but less than 20% of the total alpha particle energy is lost by these mechanisms.

I. Introduction

When energetic ions slow down in a background plasma and their velocity space contains a loss region, the loss of particles into that region must be calculated accurately since it impacts both the particle and energy balance of the plasma. The problem was originally addressed for neutral beam injection into tokamaks with the loss region due to drift orbits of large radial extent(1), and for neutral beam slowing down in a tokamak with no loss region.(2) Work has also been done for the slowing of alphas in a homogeneous plasma, using an ad hoc loss time to model alpha scattering into the loss region.(3) Fokker-Planck codes have been used to find alpha particle loss and energy deposition rates for a tandem mirror machine by Rensink,(4) resulting in approximate analytic estimates.[5] An excellent, general study of alpha loss and heating in a mirror machine with no end plug potential has been carried out by Steinhauer.[6] Included were alpha particle, non-zero transient response time, finite Larmor radius effects, slowing down, collisional and non-adiabatic angle scattering, work done on the magnetic field by plasma expansion and contraction, and alpha particle pressure. However, the angular dependence of the Fokker-Planck equation was solved either by assuming a special value for the magnetic mirror ratio or by an approximation to the eigenfunction and eignenvalue. Only one term in the expansion of angular eigenfunctions was retained.

Here, only collisional scattering and slowing down are included, but multiple terms in the angular dependence expansion are kept, and an arbitrary mirror ratio is allowed. This allows a test of the assumption in Ref. 6 that the first term in the expansion is dominant. Arbitrary mirror ratio allows the method to be used in a computer code where it can serve as an adjunct to other reactor calculations. For example, the code has been used in conjunction with the tandem mirror power balance code POWBAL for the TASKA study.[7]

Section II contains the derivation of the necessary equations. Section III specializes to the case of a tandem mirror. Section IV discusses the numerical methods used to solve the problem. Section V gives results.

II. Analysis

The analysis will be carried out for a general, monotonic magnetic field with arbitrary loss regions. Specialization to a tandem mirror with a square well magnetic field will occur where necessary.

Gyrophase averaging the Fokker-Planck equation and neglecting small derivatives of the distribution function f in energy and magnetic moment gives the drift kinetic equation (DKE)[8]

$$\frac{\partial f}{\partial t} + (\underline{v}_{\parallel} + \underline{v}_D) \cdot \nabla f = Cf + S \tag{1}$$

where $\underline{v}_{\parallel}$ is velocity along $\underline{B}$, $\underline{v}_D$ is the perpendicular drift velocity, C is the collision operator, and S is a source term.

Three characteristic times are important: 1) τ_s, the slowing down time for an alpha particle; 2) τ_D, the perpendicular drift time; and 3) τ_b, the bounce time for a trapped particle. The ordering chosen here is $\tau_b << \tau_s << \tau_D$. Drift motion is thus neglected; to retain it in the limit $\tau_s >> \tau_D$, one would have to drift average the drift kinetic equation. We expect this procedure to change neither the equation nor the results significantly. Including only slowing down and pitch angle diffusion in C, the DKE is[1,9]

$$\tau_s \frac{\partial f}{\partial t} + \tau_s \overline{v}_{\parallel} \cdot \nabla f = \frac{1}{v^2} \frac{\partial}{\partial v} (v^3 + v_c^3) f + A \frac{v_c^3}{v^3} \frac{\partial}{\partial \zeta} (1 - \zeta^2) \frac{\partial f}{\partial \zeta} + S \tau_s \tag{2}$$

where $\zeta \equiv v_{\parallel}/v$, v_c is the critical velocity where slowing down and angle scattering are equally strong, and

$$A \equiv \frac{1}{2m_\alpha} \frac{\Sigma n_j z_j^2}{\Sigma n_j z_j^2/m_j} . \tag{3}$$

The following transformation is useful:

$$E = \frac{v^2}{2} \tag{4}$$

$$\lambda \equiv \frac{\mu B_{min}}{E} \equiv \frac{v_\perp^2 B_{min}}{v^2 B} \tag{5}$$

where $\mu \equiv v_\perp^2/2B$ is the magnetic moment.

The DKE then becomes

$$\tau_s \frac{\partial f}{\partial t} + \tau_s \underline{v}_\parallel \cdot \nabla f = 3f + [1 + (\frac{E_c}{E})^{3/2}] \, 2E \frac{\partial f}{\partial E}$$
$$+ 4A (\frac{E_c}{E})^{3/2} \frac{B_{min}}{B} (1 - \lambda \frac{B}{B_{min}})^{1/2} \frac{\partial}{\partial \lambda} \lambda (1 - \lambda \frac{B}{B_{min}})^{1/2} \frac{\partial f}{\partial \lambda} + S\tau_s \tag{6}$$

where E_c is the "critical energy"

$$E_c \equiv \frac{m_\alpha v_c^2}{2} \equiv 14.8 \; T_e \, (\frac{m_\alpha}{m_H})^{1/3} \; (m_\alpha \frac{\Sigma n_j z_j^2/m_j}{\Sigma n_j z_j})^{2/3} \tag{7}$$

and m_H is the mass of a proton.

The dependence of λ and E on $\phi(s)$, the electrostatic potential along a field line, will be neglected. Then, the assumption $\tau_b << \tau_s$ allows a bounce average to be done. Define the bounce average of a quantity G(s) by

$$\langle G(s) \rangle \equiv \frac{1}{\oint \frac{ds}{L_s v_\parallel}} \oint \frac{ds}{L_s v_\parallel} G(s) = \frac{1}{\oint \frac{ds}{L_s v_\parallel}} \oint \frac{G(s) \; ds/L_s}{\sigma(2E)^{1/2} \; (1-\lambda \frac{B}{B_{min}})^{1/2}} \tag{8}$$

where $\sigma \equiv v_\parallel/|v_\parallel|$ and L_s is magnetic field length. Approximating f by ⟨f⟩ and defining

$$F \equiv \langle f \rangle \, e^{-3t/\tau_s} \tag{9}$$

gives

$$\tau_s \frac{\partial F}{\partial t} - [1 + (\frac{E_c}{E})^{3/2}] \, 2E \frac{\partial F}{\partial E} = \langle S \rangle \, \tau_s \, e^{-3t/\tau_s}$$

$$+ \frac{4A}{\oint \frac{\sigma \, ds/L_s}{(1-\lambda B/B_{min})^{1/2}}} (\frac{E_c}{E})^{3/2} \frac{\partial}{\partial \lambda} \lambda \, [\oint \frac{ds}{\sigma L_s} \frac{B_{min}}{B} (1 - \lambda \frac{B}{B_{min}})^{1/2}] \frac{\partial F}{\partial \lambda} \, . \qquad (10)$$

The left hand side is a total derivative if

$$\tau_s \frac{dF}{dt} = \tau_s \frac{\partial f}{\partial t} + \tau_s \frac{dE}{dt} \frac{\partial F}{\partial E} \qquad (11)$$

which is true if

$$\tau_s \frac{dE}{dt} = - 2E \, [1 + (\frac{E_c}{E})^{3/2}] \qquad (12)$$

or

$$\frac{t}{\tau_s} = \frac{1}{3} \ln (\frac{E_o^{3/2} + E_c^{3/2}}{E^{3/2} + E_c^{3/2}}) \qquad (13)$$

where E_o is the initial alpha particle energy. This equation defines a slowing-down orbit.

Define

$$I \equiv \oint \frac{ds}{\sigma L_s} \frac{B_{min}}{B} (1 - \lambda \frac{B}{B_{min}})^{1/2} \qquad (14)$$

and

$$I_b \equiv \oint \frac{\sigma ds/L_s}{(1-\lambda \frac{B}{B_{min}})^{1/2}} \, . \tag{15}$$

Then, the DKE becomes

$$\tau_s \frac{dF}{dt} = \langle S\rangle \, \tau_s \, e^{-\frac{3t}{\tau_s}} + \frac{4A}{I_b} \left(\frac{E_c}{E}\right)^{3/2} \frac{\partial}{\partial\lambda} \lambda \, I \, \frac{\partial F}{\partial\lambda} \, . \tag{16}$$

To make headway, F is expanded in a set of eigenfunctions, details of which will be determined later:

$$F = \sum_n a_n(E) \, F_n(\lambda) \, . \tag{17}$$

The $F_n(\lambda)$ will be chosen such that

$$I_b^{-1}(\lambda) \frac{d}{d\lambda} \lambda I(\lambda) \frac{dF_n(\lambda)}{d\lambda} = - k_n^2 \, F_n(\lambda) \tag{18}$$

which, under conditions applying here, is of Sturm-Liouville form so eigenfunctions of unequal eigenvalues will be orthogonal. The DKE is now

$$\sum_n \left[\left(\frac{E}{E_c}\right)^{3/2} \tau_s \frac{da_n(E)}{dt} + 4A \, k_n^2 \, a_n(E)\right] F_n(\lambda) = \langle S\rangle \left(\frac{E}{E_c}\right)^{3/2} \tau_s \, e^{-3t/\tau_s} \, . \tag{19}$$

The F_n's are defined over some region in λ space which may consist of segments. The symbol $\int_{\lambda_1}^{\lambda_2}$ will denote this region, even if more than two end-points are involved. The orthogonality condition is

$$\int_{\lambda_1}^{\lambda_2} d\lambda \, F_n(\lambda) \, F_m(\lambda) \, I_b(\lambda) = \delta_{mn} \tag{20}$$

where δ_{mn} is the Kronecker delta function. Writing dt in terms of dE and multiplying the DKE by $\int_{\lambda_1}^{\lambda_2} d\lambda\, F_n(\lambda)\, I_b(\lambda)$ gives

$$\frac{-2(E^{3/2}+E_c^{3/2})}{E^{1/2}} \left(\frac{E}{E_c}\right)^{3/2} \frac{da_n(E)}{dE} + 4Ak_n^2\, a_n(E) = b_n(E) \qquad (21)$$

where

$$b_n(E) \equiv \left(\frac{E}{E_c}\right)^{3/2} e^{-3t/\tau_s} \int_{\lambda_1}^{\lambda_2} d\lambda\, F_n(\lambda)\, I_b(\lambda)\, \langle S\rangle\ . \qquad (22)$$

The solution is

$$a_n(E) = \left[\frac{1+\left(\frac{E_c}{E_o}\right)^{3/2}}{1+\left(\frac{E_c}{E}\right)^{3/2}}\right]^{4Ak_n^2/3} \int_{E_o}^{E} dE\, \frac{-E_c^{3/2} b_n(E)}{2E(E^{3/2}+E_c^{3/2})} \left[\frac{1+\left(\frac{E_c}{E}\right)^{3/2}}{1+\left(\frac{E_c}{E_o}\right)^{3/2}}\right]^{4Ak_n^2/3} \qquad (23)$$

where the constant of integration is zero since F vanishes for $E > E_o$.

A reasonable assumption for the source function S is that alpha particles are born isotropically in λ. That is

$$S(E,\lambda) = \frac{\dot{n}_\alpha}{4\pi} \frac{\delta(E - E_o)}{(2E_o)^{1/2}} \qquad (24)$$

where $\dot{n}_\alpha$ is the alpha particle production rate. Then

$$b_n(E) = \frac{\dot{n}_\alpha}{4\pi} \frac{\tau_s}{(2E_o)^{1/2}} \left(\frac{E_o}{E_c}\right)^{3/2} C_n \delta(E - E_o) \qquad (25)$$

where E has been set to E_0 because of the delta function and

$$C_n \equiv \int_{\lambda_1}^{\lambda_2} d\lambda \; F_n(\lambda) I_b(\lambda).$$

Then

$$a_n(E) = \frac{\dot{n}_\alpha \tau_s c_n}{4\pi (2E_0)^{3/2}} \frac{[1+(\frac{E_c}{E_0})^{3/2}]^{(4Ak_n^2/3)-1}}{[1+(\frac{E_c}{E})^{3/2}]^{4Ak_n^2/3}} . \tag{26}$$

Therefore,

$$F(E,\lambda) = \frac{\dot{n}_\alpha \tau_s}{4\pi (2E_0)^{3/2}} \sum_n C_n \; F_n(\lambda) \frac{[1+(\frac{E_c}{E_0})^{3/2}]^{(4Ak_n^2/3)-1}}{[1+(\frac{E_c}{E})^{3/2}]^{4Ak_n^2/3}} , \tag{27}$$

so

$$f(E,\lambda) = \frac{\dot{n}_\alpha \tau_s}{4\pi (2E)^{3/2}} \sum_n C_n \; F_n(\lambda) \frac{[1+(\frac{E_c}{E_0})^{3/2}]^{4Ak_n^2/3}}{[1+(\frac{E_c}{E})^{3/2}]^{(4Ak_n^2/3)+1}} . \tag{28}$$

Particle and energy loss rates then come from taking moments of the DKE. The velocity-space integral is

$$\int d^3v = \pi \frac{B}{B_{min}} \sum_\sigma \int_0^\infty dE \; (2E)^{1/2} \int_0^{B_{min}/B} \frac{d\lambda}{(1-\lambda \frac{B}{B_{min}})^{1/2}} . \tag{29}$$

To find total rates, a flux surface average must also be taken; the operator is

$$\frac{1}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \oint \frac{ds}{L_s} \frac{B_{min}}{B} \, . \tag{30}$$

For particle loss rates, we start with the bounce-averaged form of Eq. (6), then take the first moment of the DKE followed by a flux surface average.

$$\begin{aligned}
&\frac{\tau_s}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \oint \frac{ds}{L_s} \frac{B_{min}}{B} \frac{\partial n}{\partial t} = \frac{\pi}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \oint \frac{ds}{L_s} \sum_\sigma \int_0^{B_{min}/B} \frac{d\lambda}{(1-\lambda \frac{B}{B_{min}})^{1/2}} \\
&\int_0^{E^*} dE \; (2E)^{1/2} \; Cf + \frac{\pi}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \oint \frac{ds}{L_s} \sum_\sigma \int_0^{B_{min}/B} \frac{d\lambda}{(1 - \lambda \frac{B}{B_{min}})^{1/2}} \int_{E^*}^{\infty} dE \frac{\partial}{\partial E} \\
&[(2E)^{3/2} + (2E_c)^{3/2}] \, f + \frac{\pi}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_\sigma \int_{E^*}^{\infty} dE \; (2E)^{1/2} \left(\frac{E_c}{E}\right)^{3/2} \oint \frac{ds}{L_s} \\
&\int_0^{B_{min}/B} \frac{d\lambda}{(1 - \lambda \frac{B}{B_{min}})^{1/2}} \frac{4A}{I_b} \frac{\partial}{\partial \lambda} \lambda I \frac{\partial f}{\partial \lambda} + \dot{n}_\alpha \, \tau_s \, (1 - \gamma_{LC})
\end{aligned} \tag{31}$$

where γ_{LC} is the fraction of alphas that are born in the loss cone. The left hand side equals zero by particle number conservation. The first right hand side term is zero since alphas are assumed to be in equilibrium below E = E* and thus collisions will not change the distribution. The second term gives the number of alphas slowing to E* by energy scattering. The third term gives the number of alphas scattering into the loss cone and out of the system; these are assumed to be immediately lost. The last is the source term.

Equation (31) may be rewritten as

$$\dot{n}_\alpha \tau_s (1 - \gamma_{LC}) = \dot{n}_\alpha \tau_s (\gamma_s + \gamma_\lambda) \tag{32}$$

where γ_s is the fraction of alphas surviving to E* and γ_λ is the fraction lost via angle scattering.

Substituting for f gives

$$\gamma_s = \frac{1}{2 \oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \frac{[1 + (\frac{E_c}{E_o})^{3/2}]^{4Ak_n^2/3}}{[1 + (\frac{E_c}{E^*})^{3/2}]^{4Ak_n^2/3}} \oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda}{(1 - \lambda \frac{B}{B_{min}})^{1/2}} F_n(\lambda) \tag{33}$$

and

$$\gamma_{LC} = \frac{-A}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \int_{E^*}^{E_o} \frac{dE}{E} \left(\frac{E_c}{E}\right)^{3/2} \frac{[1 + (\frac{E_c}{E_o})^{3/2}]^{4Ak_n^2/3}}{[1 + (\frac{E_c}{E})^{3/2}]^{(4Ak_n^2/3)+1}}$$

$$\oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda/I_b}{(1 - \lambda \frac{B}{B_{min}})^{1/2}} \frac{\partial}{\partial\lambda} \lambda I \frac{\partial F_n}{\partial\lambda} . \tag{34}$$

Similarly, the E moment gives

$$\dot{n}_\alpha \tau_s \frac{\partial E}{\partial t} = 0 = \dot{n}_\alpha \tau_s E_o (1 - \varepsilon_{LC} - \varepsilon_R - \varepsilon_e - \varepsilon_i - \varepsilon_\lambda) \tag{35}$$

where ε_{LC} is the fraction of energy lost immediately by alphas being born in the loss cone; ε_R is the residual energy fraction after the alphas have slowed to E*; ε_e is the energy fraction given to electrons; ε_i is the energy fraction carried out by alphas scattering into the loss cone.

Explicitly,

$$\varepsilon_R = \frac{-\pi/\dot{n}_\alpha \tau_s}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \oint \frac{ds}{L_s} \sum_\sigma \int_0^{B_{min}/B} \frac{d\lambda}{(1 - \lambda \frac{B}{B_{min}})^{1/2}} \int_0^{E*} dE\, (2E)^{1/2} \frac{E}{E_o} Cf \tag{36}$$

which will be found from the condition

$$\varepsilon_R = 1 - \varepsilon_{LC} - \varepsilon_e - \varepsilon_i - \varepsilon_\lambda \tag{37}$$

$$\varepsilon_e = \frac{1}{2 \oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \frac{E*}{E_o} \frac{[1 + (\frac{E_c}{E_o})^{3/2}]^{4Ak_n^2/3}}{[1 + (\frac{E_c}{E*})^{3/2}]^{(4Ak_n^2/3)+1}} \oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda}{(1-\lambda \frac{B}{B_{min}})^{1/2}} F_n(\lambda)$$

$$+ \frac{1}{2 \oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \int_{E*}^{E_o} \frac{dE}{E_o} \frac{[1 + (\frac{E_c}{E_o})^{3/2}]^{4Ak_n^2/3}}{[1 + (\frac{E_c}{E})^{3/2}]^{(4Ak_n^2/3)+1}} \oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda}{(1-\lambda \frac{B}{B_{min}})^{1/2}} F_n(\lambda) \tag{38}$$

$$\varepsilon_i = \frac{1}{2 \oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \frac{E^*}{E_o} \left(\frac{E_c}{E^*}\right)^{3/2} \frac{\left[1 + \left(\frac{E_c}{E_o}\right)^{3/2}\right]^{4Ak_n^2/3}}{\left[1 + \left(\frac{E_c}{E^*}\right)^{3/2}\right]^{(4Ak_n^2/3)+1}} \tag{39}$$

$$\oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda}{\left(1 - \lambda \frac{B}{B_{min}}\right)^{1/2}} F_n(\lambda) + \frac{1}{2 \oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \int_{E^*}^{E_o} \frac{dE}{E_o} \left(\frac{E_c}{E}\right)^{3/2}$$

$$\frac{\left[1 + \left(\frac{E_c}{E_o}\right)^{3/2}\right]^{4Ak_n^2/3}}{\left[1 + \left(\frac{E_c}{E_o}\right)^{3/2}\right]^{(4Ak_n^2/3)+1}} \oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda}{\left(1 - \lambda \frac{B}{B_{min}}\right)^{1/2}} F_n(\lambda)$$

$$\varepsilon_\lambda = \frac{-A}{\oint \frac{ds}{L_s} \frac{B_{min}}{B}} \sum_n C_n \int_{E^*}^{E_o} \frac{dE}{E_o} \left(\frac{E_c}{E}\right)^{3/2} \frac{\left[1 + \left(\frac{E_c}{E_o}\right)^{3/2}\right]^{4Ak_n^2/3}}{\left[1 + \left(\frac{E_c}{E}\right)^{3/2}\right]^{(4Ak_n^2/3)+1}} \tag{40}$$

$$\oint \frac{ds}{L_s} \int_0^{B_{min}/B} \frac{d\lambda \; I_b^{-1}}{\left(1 - \lambda \frac{B}{B_{min}}\right)^{1/2}} \frac{\partial}{\partial\lambda} \lambda I \frac{\partial F_n}{\partial\lambda} .$$

III. Tandem Mirror Special Case

We now specialize to a tandem mirror with a square-well magnetic field. Alpha loss is governed by the plug potential ϕ and by the mirror ratio

$R = B_{max}/B_{min}$. The loss boundary is then

$$E = \frac{\phi}{1 - \lambda \frac{B_{max}}{B_{min}}} , \tag{41}$$

which is shown in Fig. 1.

Since, for this case, $E_o \sim 30\ \phi$, the deviation of the loss boundary from $\lambda = .2$ does not significantly alter the results. Therefore, we identify $E^* = \phi$ and model the loss boundary by

Lost particles: $0 \leqslant \lambda \leqslant \frac{B_{min}}{B_{max}}$, $E > E^*$

Trapped particles: $\frac{B_{min}}{B_{max}} < \lambda \leqslant 1$, $E > E^*$

$0 \leqslant \lambda \leqslant 1$, $E \leqslant E^*$.

Solutions are now needed for the λ eigenvalue equation (18). Boundary conditions are taken to be

$$F_n \left(\frac{B_{min}}{B_{max}}\right) = 0 \tag{42}$$

$$I(\lambda) \frac{dF_n}{d\lambda} \Big|_{\lambda=1} = 0 \tag{43}$$

for $E > E^*$. For $E \leqslant E^*$, F is not explicitly needed.

Substituting the square well B field into Eqs. (14) and (15) gives

$$I(\lambda) = 2(1 - \lambda)^{1/2} \tag{44}$$

$$I_b(\lambda) = \frac{2}{(1 - \lambda)^{1/2}} , \tag{45}$$

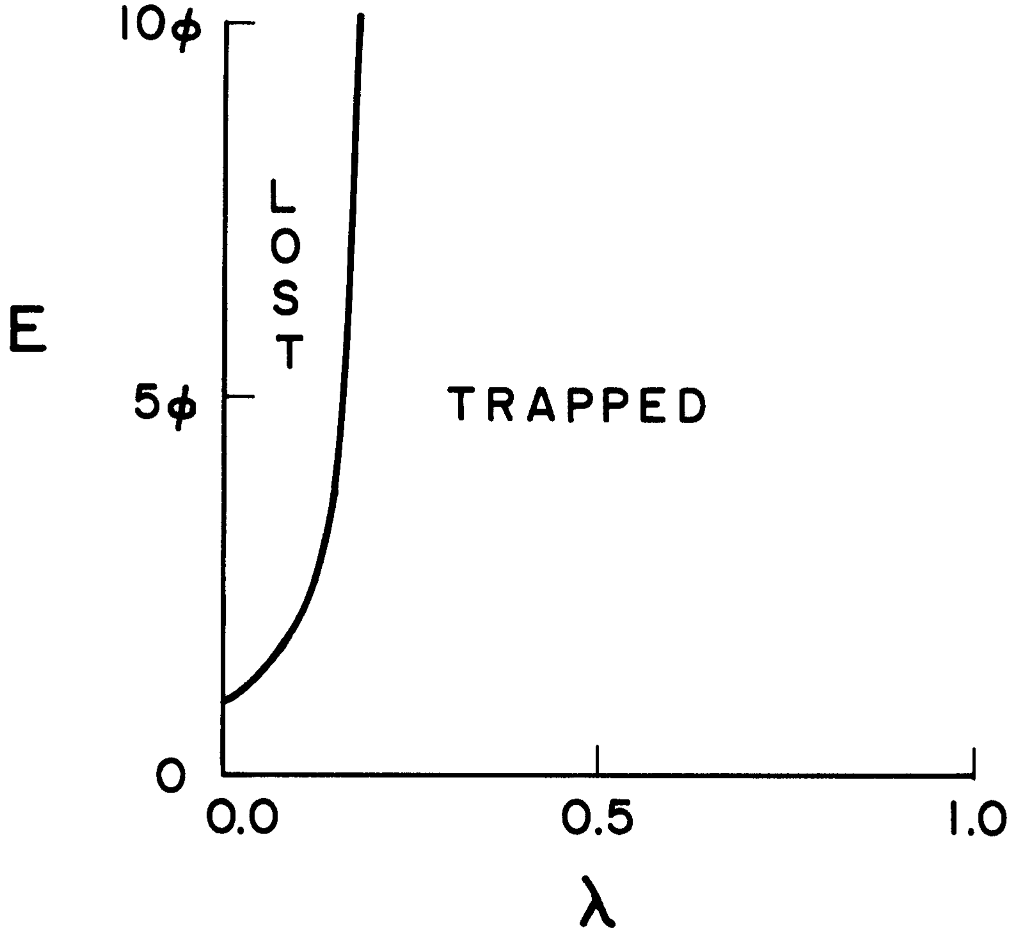

Fig. 1. Tandem mirror particle loss boundary.

so Eq. (28) becomes

$$(1 - \lambda)^{1/2} \frac{d}{d\lambda} \lambda (1 - \lambda)^{1/2} \frac{dF_n}{d\lambda} = - k_n^2 F_n . \tag{46}$$

Changing variables to $x \equiv (1 - \lambda)^{1/2}$ gives

$$\frac{d}{dx} (1 - x^2) \frac{dF_n}{dx} + 4 k_n^2 F_n = 0 \tag{47}$$

which is Legendre's equation. Solutions are $P_{\nu_n}(x)$ and $Q_{\mu_n}(x)$ which are Legendre functions of the first and second kind. ν_n and μ_n are found from the condition that $P_{\nu_n} = 0$ and $Q_{\mu_n} = 0$ at $x = (1 - B_{min}/B_{max})^{1/2}$.

For certain special values of R, the lowest ν_n or μ_n assumes an integral value. For example, R = 3.277 gives $F_1(x) = Q_1(x)$, as first pointed out in Ref. 10.

The γ's and ε's simplify in an obvious way for a square well B field. Note that $\oint \frac{ds}{L_s} \frac{B_{min}}{B} = 2$.

IV. Numerical Solution

The F_n eigenfunction equation is solved numerically in either the x or λ variable. Similar finite difference schemes are used; these will be illustrated by showing the x-variable solution in detail.

The range of x is $0 \leq x \leq (1 - \frac{1}{R})^{1/2}$. For an equally spaced mesh of N points, the interval between points is

$$h = \frac{(1 - \frac{1}{R})^{1/2}}{N-1} . \tag{48}$$

Let i run from 1 to N, where x = (i-1)h.

The boundary conditions, Eqs. (42) and (43), are then

$$F(x_N) = 0 \tag{49}$$

$$\frac{dF(x_1)}{dx} = 0 \ . \tag{50}$$

Centered finite differences for i=2, ..., N give for the differential equation

$$\frac{1}{4h^2} \{(1-x^2_{i+1/2}) [F(x_{i+1}) - F(x_i)] - (1 - x^2_{i-1/2}) [F(x_i) - F(x_{i-1})]\}$$
$$+ k^2 F(x_i) = 0 \tag{51}$$

To implement boundary condition (50), write the eigenfunction equation at $x = x_{5/4}$:

$$\frac{1}{h/2} [(1 - x^2_{3/2}) \frac{dF(x_{3/2})}{dx} - (1 - x^2_1) \frac{dF(x_1)}{dx}] + 4 \ k^2 \ F(x_{5/4}) \ . \tag{52}$$

Substituting Eq. (50), writing $F(x_{5/4}) = 3/4 \ F(x_1) + 1/4 \ F(x_2)$, and substituting for $k^2 \ F(x_2)$ from the i=2 equation gives

$$\frac{2}{3} \frac{(1-x^2_{3/2})}{h^2} [F(x_2) - F(x_1)] - \frac{1}{12 \ h^2} \{(1 - x^2_{5/2}) [F(x_3) - F(x_2)]$$
$$- (1 - x^2_{3/2}) [F(x_2) - F(x_1)]\} + k^2 \ F(x_1) = 0 \ . \tag{53}$$

Equations (51) and (53) may now be written in matrix form and solved by standard techniques. The required energy integrals are also amenable to standard routines.

V. Results

The case R = 3.277 can be solved exactly for F_1 as mentioned in Sec. IV. The first four eigenfunctions in x are shown in Fig. 2 and those in λ are shown in Fig. 3. Eigenvalues are ordered from smallest to largest, with eigenfunctions corresponding. Both cases had A = .36, E_0 = 3.52 MeV, E_C = 1.07 MeV, and E* = 0.204 MeV. $F_1(x)$ is easily seen to be $Q_1(x)$, and $k_1^2 = 2$ as required.

In general, the γ's and ε's tend to be more accurate when the numerical solution is found in x rather than in λ. This occurs because the condition dF/dx = 0 is easier to numerically fit than is $I(\lambda)\, dF/d\lambda = 0$, as is obvious from Figs. 2 and 3.

The number N of mesh points does not significantly impact the results unless the mesh is very coarse, $N \lesssim 40$, as shown in Fig. 4. However, if a very large number of eigenfunctions is desired, more points will be required because of the oscillatory nature of the eigenfunctions. Usually, 120 mesh points are used in this study.

The number of eigenfunctions retained primarily affects the angular scattering terms γ_λ and ε_λ. This is because the x or λ derivative may be large even if the energy bracket is small. The accuracy of γ_λ may be maintained by using $\gamma_\lambda = 1 - \gamma_{LC} - \gamma_R$, but Fig. 5 for R = 3.277 and Fig. 6 for R = 10 show that some accuracy will be lost in the ε's also.

The temperature dependence of the γ's and ε's is exhibited in Figs. 7 and 8 for the cases R = 3.277 and R = 10. The dotted lines of Fig. 8 are the

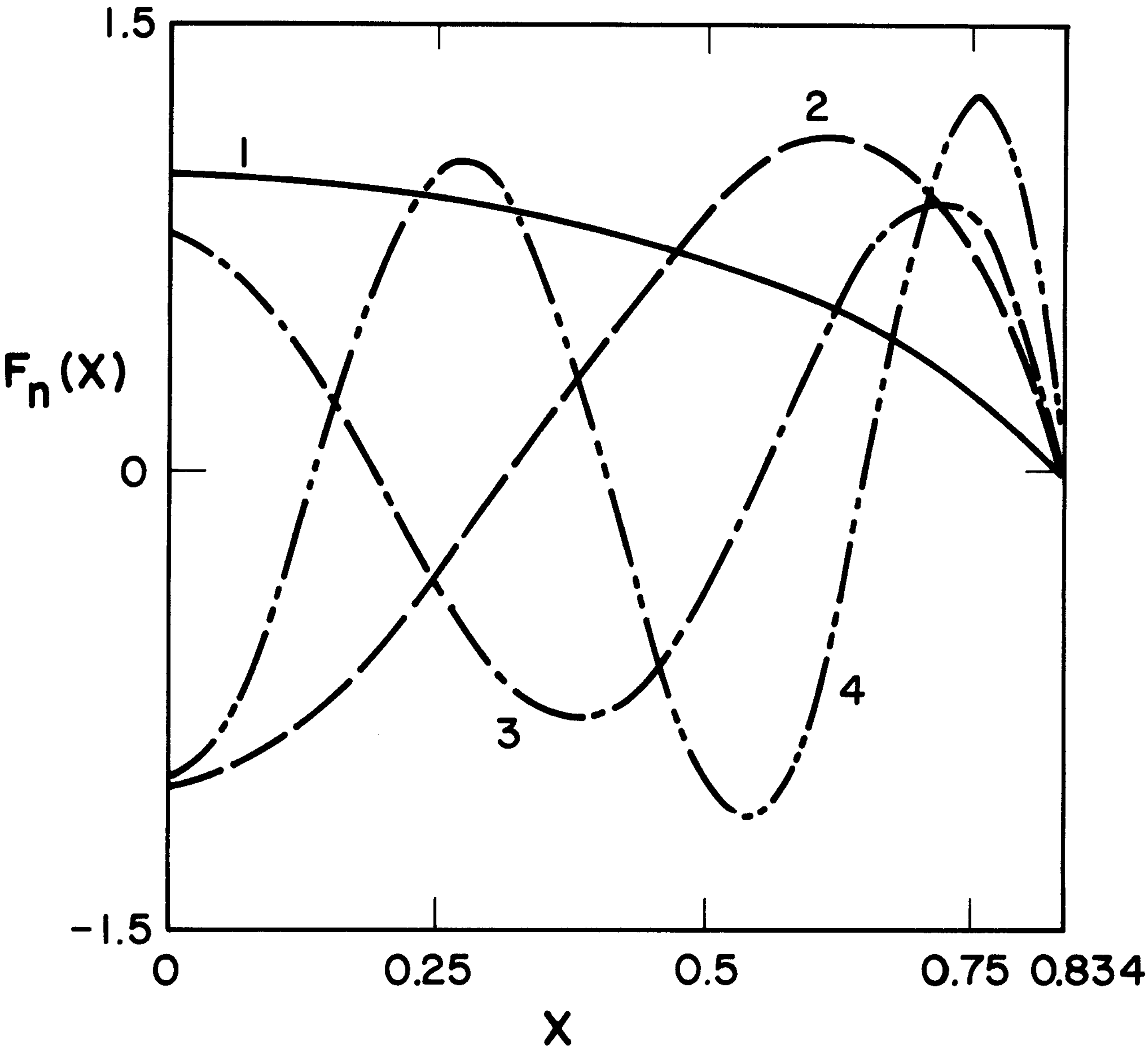

Fig. 2. Eigenfunctions in x. R = 3.277, A = .36, E_o = 3.52 MeV, E_c = 1.07 MeV, E* = 0.204 MeV.

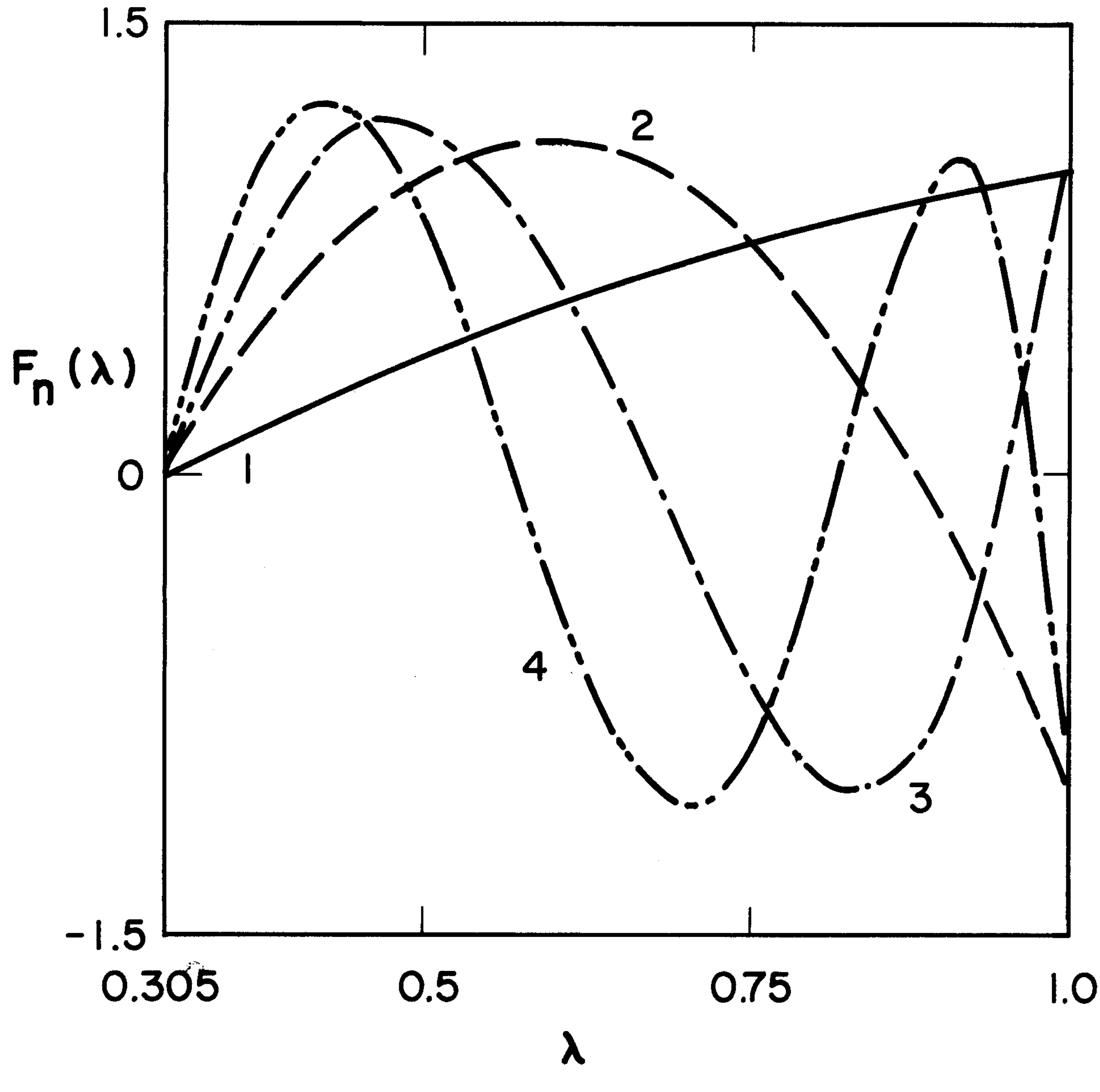

Fig. 3. Eigenfunctions in λ. $R = 3.277$, $A = .36$, $E_o = 3.52$ MeV, $E_c = 1.07$ MeV, $E^* = 0.204$ MeV.

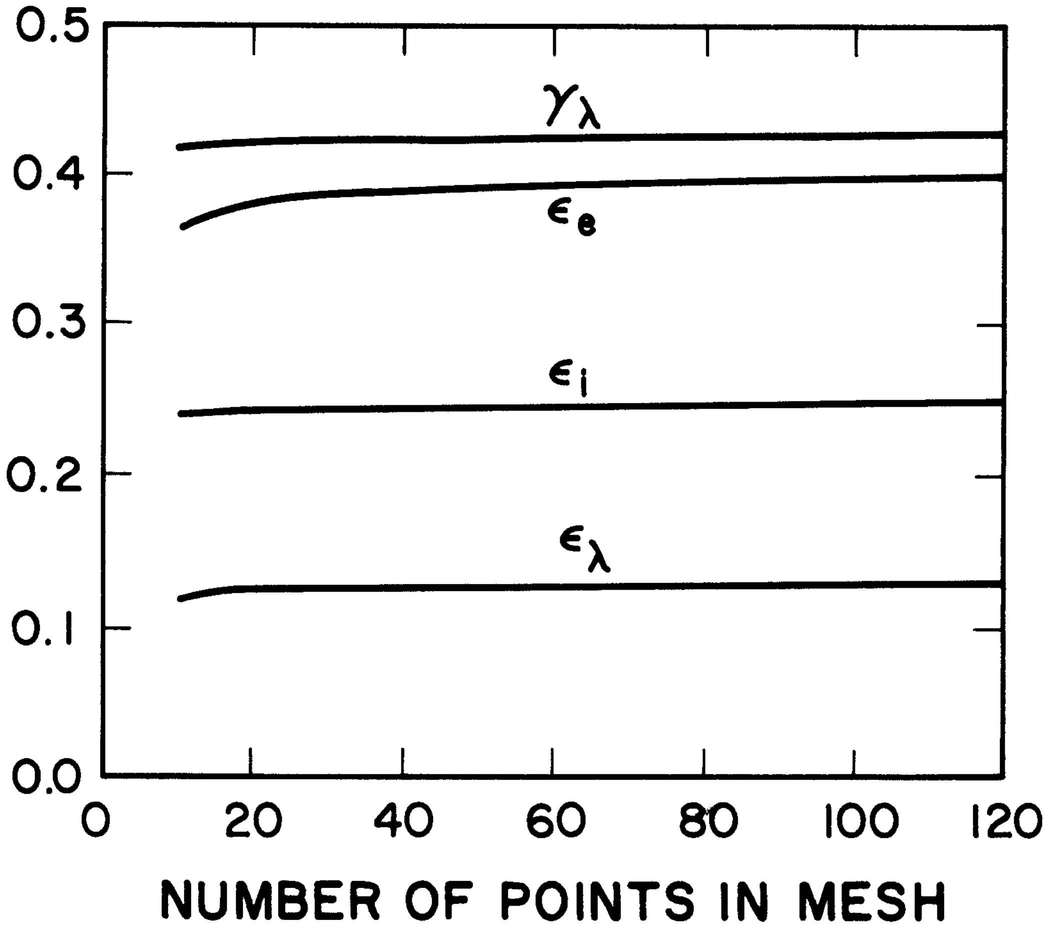

Fig. 4. Dependence of ε's and γ's on the number of mesh points used.

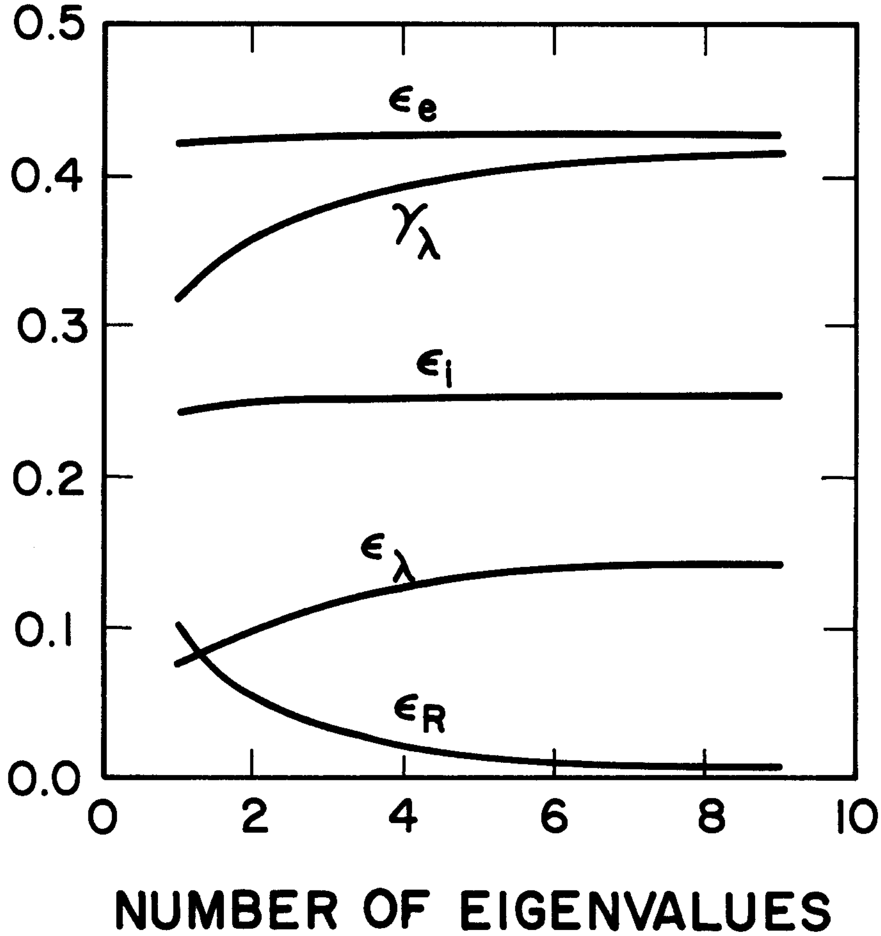

Fig. 5. Dependence of ε's and γ's on the number of eigenfunctions retained. $R = 3.277$, $A = .36$, $E_0 = 3.52$ MeV, $E_c = 1.07$ MeV, $E^* = 0.204$ MeV.

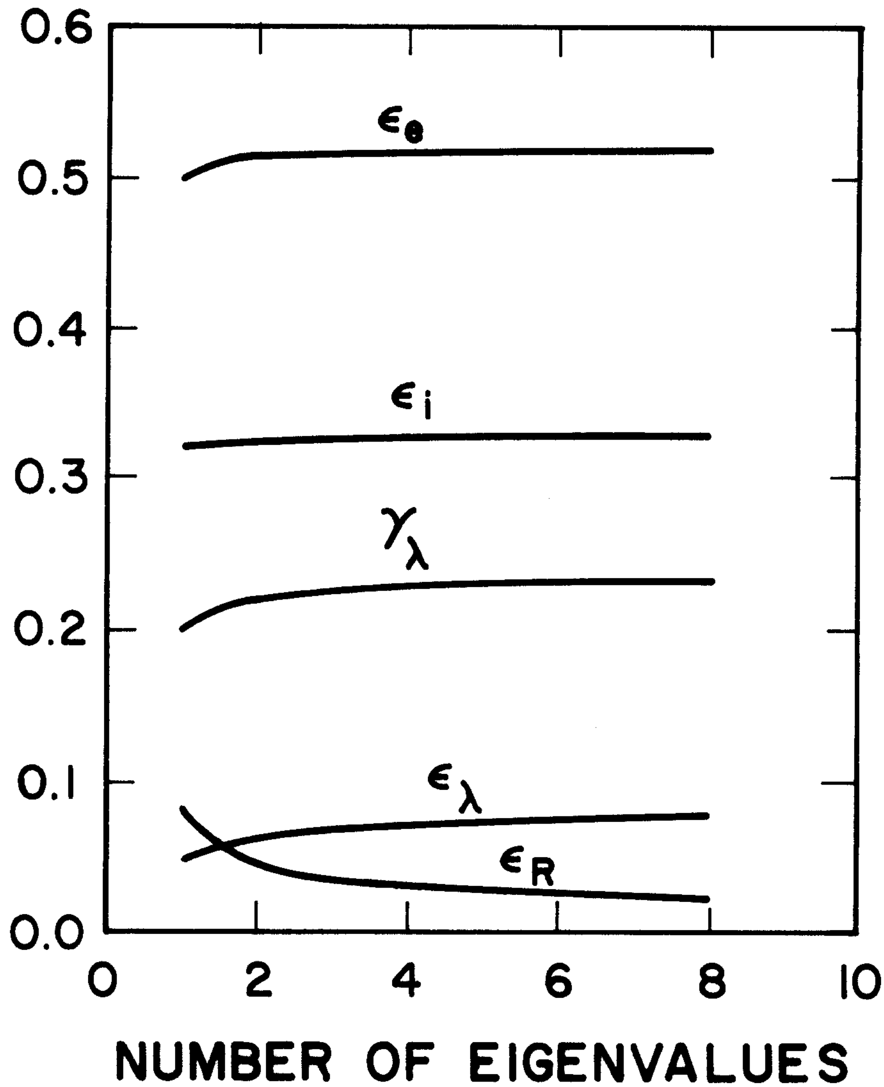

Fig. 6. Dependence of ε's and γ's on the number of eigenfunctions retained. R = 10, A = .36, E_o = 3.52 MeV, E_c = 1.07 MeV, E* = 0.204 MeV.

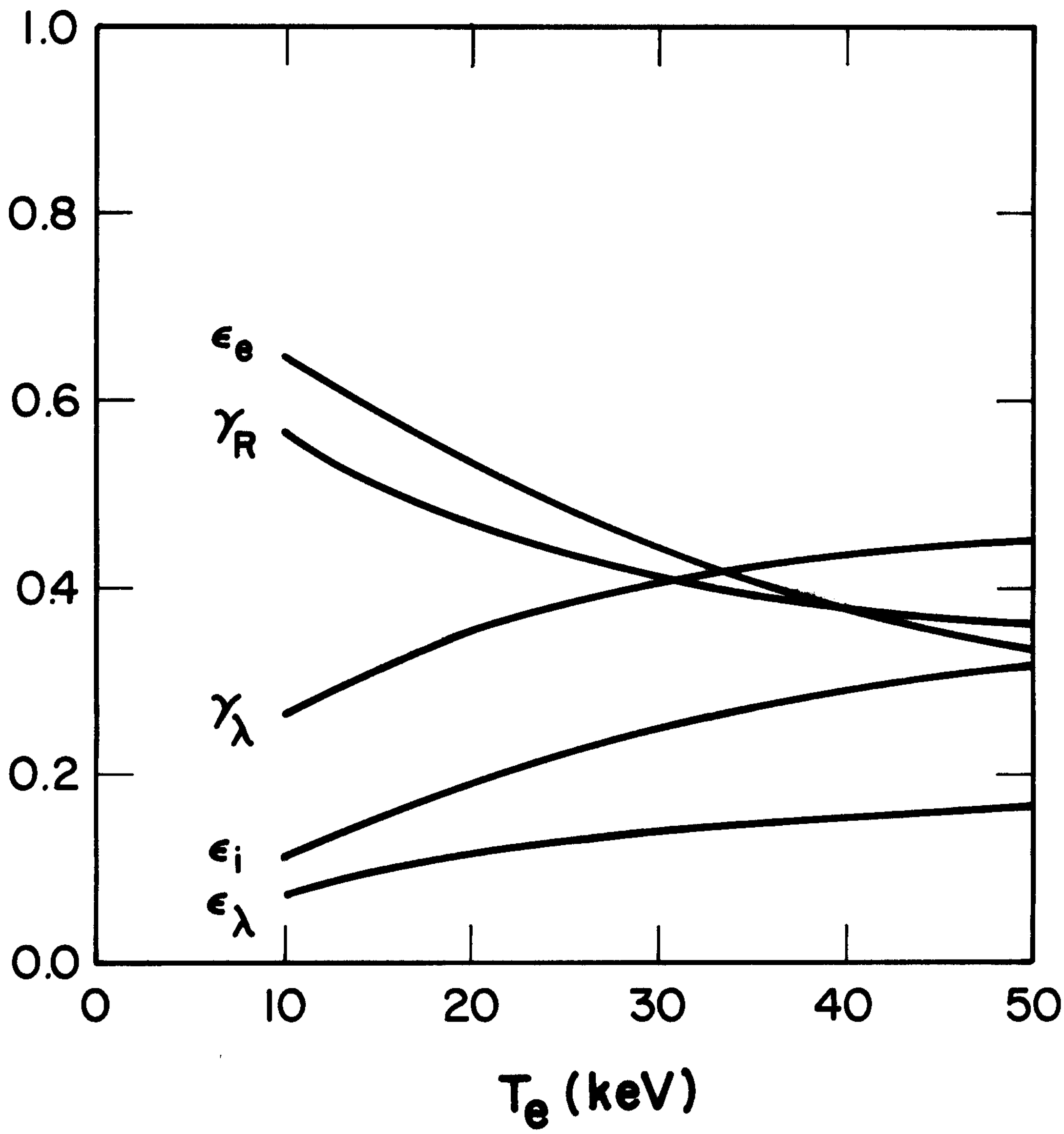

Fig. 7. Dependence of ε's and γ's on electron temperature for R = 3.277.

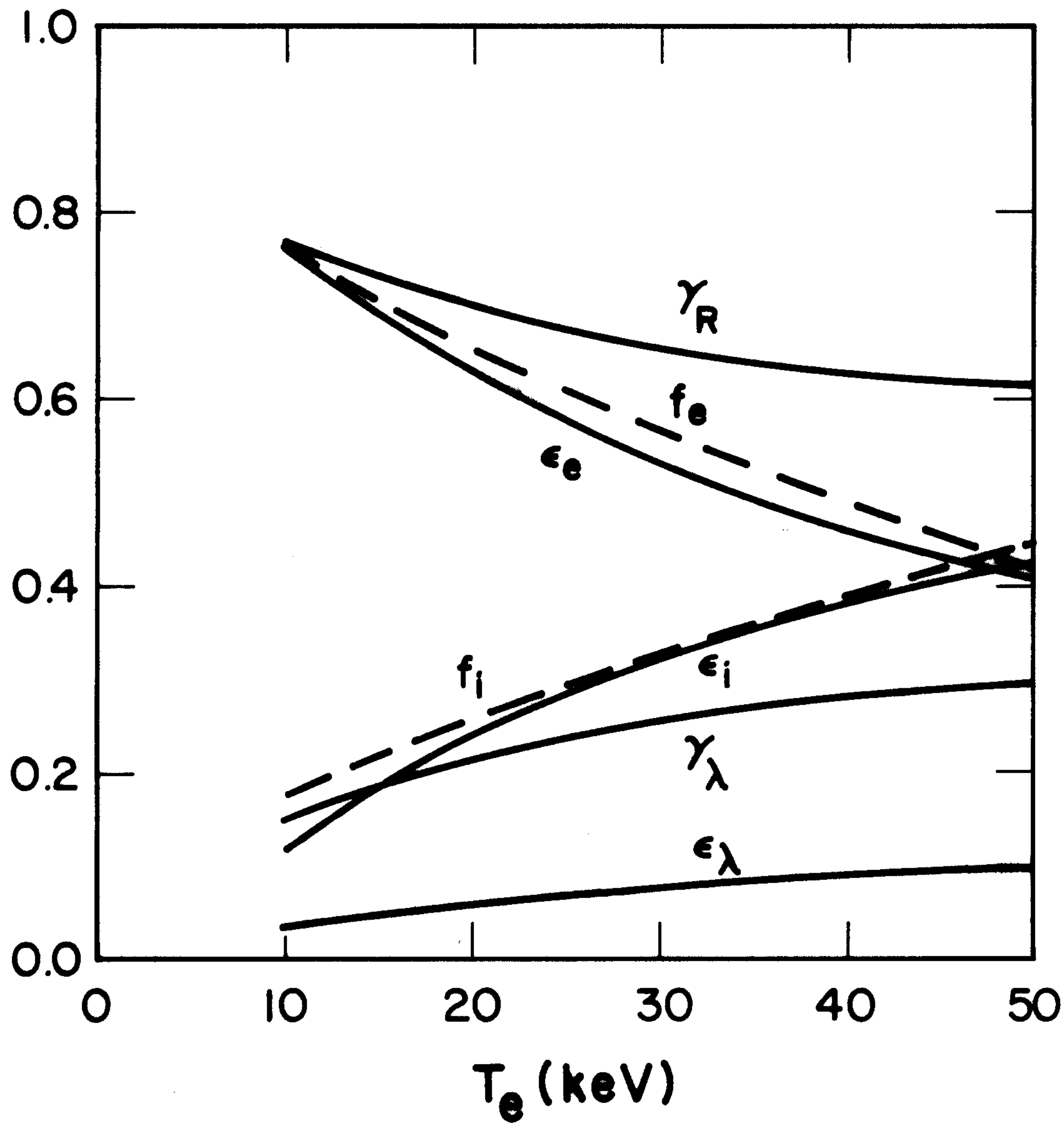

Fig. 8. Dependence of ε's and γ's on electron temperature for R = 10.

corresponding analytic formulae

$$f_e = 0.88 \exp(-T_{ec}/67.4)$$

and

$$f_i = 1.0 - 0.91 \exp(-T_{ec}/102)$$

of Ref. 5.

The dependence of the γ's and ε's on magnetic mirror ratio for a typical temperature is shown in Fig. 9. As expected, losses increase as the mirror ratio is lowered.

Solution of the full problem takes about ten seconds on the MFECC CRAY-1, allowing easy use of the program in conjunction with other studies such as of the power balance.(6) An arbitrary mirror ratio is then essential. The program is accurate with only four to six eigenfunctions retained and still remains fast with ten retained. This work supports that of Refs. 4 and 6, but emphasizes that some caution should be exercised when retaining only one eigenfunction if ε_λ and ε_R are of interest.

Future work on the problem should include the study of magnetic fields other than a square well. Also, other loss boundaries in λ are of interest, as in the case of a rippled tokamak.

Acknowledgement

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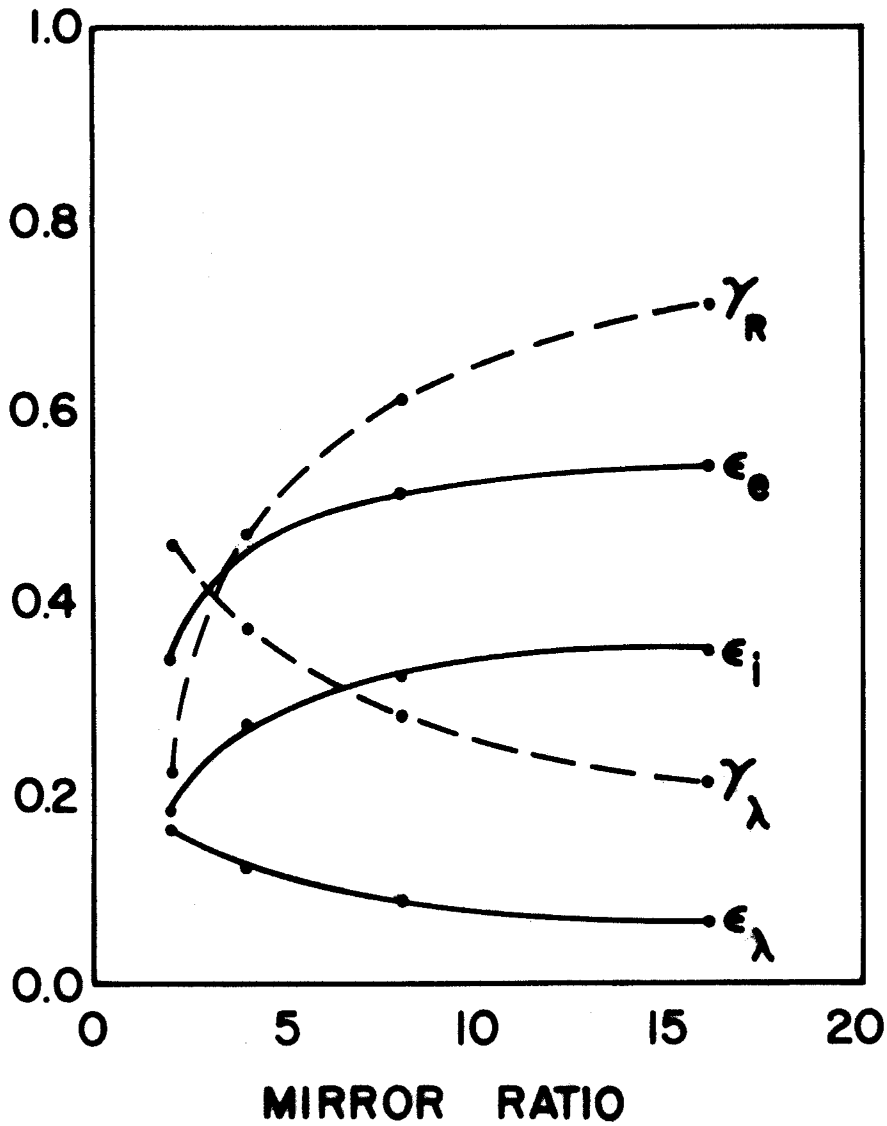

Fig. 9. Dependence of ε's and γ's on mirror ratio for E_c = 1.07 MeV.

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