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Abstract

The discrete nature of the toroidal magnetic field coils spoils the symmetry of the tokamak and creates small modulations of the toroidal field called ripples. Particles trapped in the ripple well drift off the flux surface and enhance both particle and heat fluxes. In the usual ripple transport calculations, the derivative of the perturbed particle distribution function in velocity space is discontinuous at the boundary between ripple trapped and untrapped regions. To smooth out the particle distribution across the boundary, the particle distribution in the boundary layer is calculated by using the Wiener-Hopf technique. The correction to the ripple-trapped particle distribution function away from the boundary layer and the ripple transport associated with it are obtained. It is found that the particle diffusion and heat conduction coefficients are increased by about a factor of 2 and scale like $\nu^{-1/2}$ as $(\nu_{eff}/\omega_{b\delta})^{1/2} \rightarrow 1$. Even for very collisionless plasmas, $\nu_{eff}/\omega_{b\delta} = 10^{-2}$, the corrections are still non-negligible and are about 15%.

I. Introduction

The discrete nature of toroidal magnetic field coils spoils the symmetry of a tokamak and produces small modulations of the toriodal field called ripples. Particles trapped in the magnetic mirror of a ripple well drift off the magnetic flux surface and enhance both the particle and heat flux. This type of transport has been studied by several authors.[1-3] However, they did not treat the particle distribution in detail in the boundary layer between ripple trapped and untrapped particles. In the boundary layer collisional effects are no longer negligible due to the large gradient of the particle distribution function in phase space. Collisional effects smooth the transition between the trapped and untrapped particle distribution functions. The net result is to increase the particle and heat transport, since this smoothing process tends to increase the magnitude of the particle distribution function that contributes to the transport processes.

In Sec. II, we discuss the drift velocity of ripple trapped particles from two different approaches. The magnetic field model used in the boundary layer analysis is also presented. The relation between banana drift transport and ripple transport is discussed in Sec. III. The boundary layer equations for transport due to ripple and banana drift effects are derived. In Sec. IV, we solve the ripple boundary layer equation using the Wiener-Hopf technique, and calculate the corrected particle and heat fluxes. Concluding remarks are given in Sec. V.

II. Ripple-Trapped Particle Drifts

To simplify the boundary layer analysis, we use a sinusoidal model for the ripple well. The magnetic field for a tokamak with N toroidal field coils can be written approximately as

$$B = B_0(1 - \varepsilon \cos\theta - \delta \cos N\phi) \quad , \tag{1}$$

where B_0 is the magnetic field on the axis, r, θ, ϕ are the usual toroidal coordinates, $\varepsilon \equiv r/R$, R is the major radius, and δ is the ripple depth. The minimum (maximum) magnetic field along the field line can be found from the equation $\partial B/\partial\theta = 0$, which leads to

$$\varepsilon \sin\theta + Nq\delta \sin(N\phi_0 + Nq\theta) = 0 \quad , \tag{2}$$

where q is the safety factor, and $\phi_0 \equiv \phi - q\theta$ is the angular variable label for a particular field line. Equation (2) can be satisfied only if $\alpha^* = \varepsilon|\sin\theta|/Nq\delta < 1$, which is the criterion for the existence of a local magnetic well due to the ripple. One set of consecutive maxima and minima obtained from Eq. (2) is

$$N\phi_0 + Nq\theta_m = -\sin^{-1}\alpha^* \quad , \tag{3}$$

and

$$N\phi_0 + Nq\theta_\pm = \pm\pi + \sin^{-1}\alpha^* \quad , \tag{4}$$

where $\theta_\pm$ are two consecutive maxima and θ_m is the minimum in-between. To obtain Eqs. (3) and (4), we have assumed α^* does not vary to lowest order in 1/Nq across one ripple well and is evaluated at $\theta = \theta_m$. The effective ripple well depth $\delta_{eff} \equiv [B(\theta_-) - B(\theta_m)]/B_0$ is thus

$$\delta_{eff} = \delta\sqrt{1 - \alpha^{*2}} - \delta\alpha^*(\frac{\pi}{2} - \sin^{-1}\alpha^*) \tag{5}$$

and the corresponding sinusoidal ripple well with well depth δ_{eff} and length $2(\theta_m - \theta_-)$ is

$$B = \overline{B}_0[1 - \delta_{eff} \cos (\pi \frac{N\phi_0 + Nq\theta + \sin^{-1} \alpha^*}{\pi - 2 \sin^{-1} \alpha^*})] , \quad (6)$$

where $\overline{B}_0 = B_0 [1 - \varepsilon \cos \theta_m - \delta\alpha^*(\pi/2 - \sin^{-1} \alpha^*)]$, is shown in Fig. 1.

With the sinusoidal ripple well approximation given by Eq. (6), we can calculate the bounce averaged drift velocity for the ripple trapped particles. The magnetic field in flux coordinates can be expressed as

$$\vec{B} = \vec{\nabla}\phi_0 \times \vec{\nabla}\psi \quad , \quad (7)$$

where ψ is the poloidal flux function. The bounce averaged drift velocity in the $\vec{\nabla}\psi$ direction is

$$\langle\vec{v}_d \cdot \vec{\nabla}\psi\rangle = - \frac{c}{e} \frac{\partial J/\partial\phi_0}{\partial J/\partial E} \quad , \quad (8)$$

where $\vec{v}_d$ is the particle drift velocity across the magnetic field, c is the speed of light, e is the electric charge, $E = mv^2/2$ is the energy of a particle with mass m and speed v, and J is the second adiabatic invariant defined as

$$J = \oint v_\parallel \frac{B d\theta}{\vec{B} \cdot \vec{\nabla}\theta} \quad .$$

Using Eq. (6), $\langle\vec{v}_d \cdot \vec{\nabla}\psi\rangle$ is found to be

$$\langle\vec{v}_d \cdot \vec{\nabla}\psi\rangle = - \frac{c}{e} \frac{\mu B_0}{q} \varepsilon \sin \theta_m \quad , \quad (9)$$

where $\mu = mv_\perp^2/2B$, and $v_\perp$ is the particle's perpendicular (to the magnetic field line) speed. For a tokamak with poloidal magnetic field B_p, $\vec{\nabla}\psi = RB_p\hat{r}$,

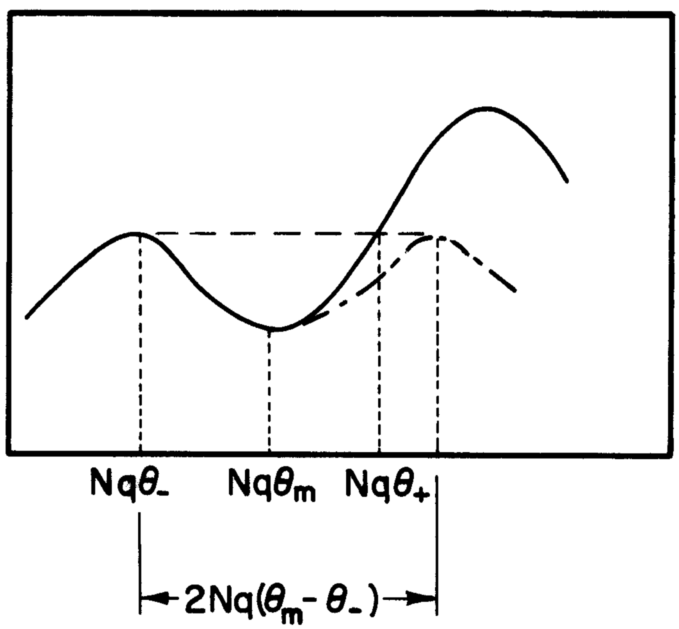

Fig. 1. Sinusoidal ripple well model.

$\hat{r} = \vec{r}/r$, and we have

$$\langle v_{dr} \rangle = \langle \vec{v}_d \cdot \hat{r} \rangle = -\frac{1}{2}\frac{v_\perp^2}{\Omega R}\sin\theta_m \quad , \tag{10}$$

which is just the usual ∇B drift, with $\Omega \equiv eB/mc$.

Though the result of Eq. (10) is not surprising, it is still worthwhile to explore it from another approach. The drift velocity can be written as

$$\vec{v}_d = \frac{v_\parallel}{B}\vec{\nabla}x\left(\frac{v_\parallel}{\Omega}\vec{B}\right) \quad , \tag{11}$$

and

$$\begin{aligned} \vec{v}_d \cdot \vec{\nabla}\psi = {} & v_\parallel \frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\frac{\partial}{\partial\theta}\left[\frac{v_\parallel}{\Omega}\frac{\vec{B}\cdot(\vec{\nabla}\psi \times \vec{\nabla}\theta)}{\vec{B}\cdot\vec{\nabla}\theta}\right] \\ & - v_\parallel \frac{\vec{B}\cdot\vec{\nabla}\theta}{B}\frac{\partial}{\partial\phi_o}\left(\frac{v_\parallel}{\Omega}\frac{B^2}{\vec{B}\cdot\vec{\nabla}\theta}\right) \quad , \end{aligned} \tag{12}$$

where $v_\parallel$ is the particle speed along the magnetic field line. Using Eq. (1), and neglecting the curvature drift, we obtain

$$\begin{aligned} \vec{v}_d \cdot \vec{\nabla}\psi = {} & -\frac{c}{e}\mu B_o\left[\frac{\varepsilon}{q}\sin\theta + N\delta\sin(N\phi_o + Nq\theta)\right] \\ & + \frac{c}{e}\mu B_o N\delta\sin(N\phi_o + Nq\theta) \quad . \end{aligned} \tag{13}$$

The first term on the right side of Eq. (13) is the "usual" axisymmetric $\vec{\nabla}B$ drift, which comes from the $\partial/\partial\theta$ term in Eq. (12), and has an extra term due to the ripple field. The bounce average of this axisymmetric $\vec{\nabla}B$ drift is zero as can be seen from Eq. (12), which means this drift does not contribute to the ripple transport and can be written as

$$\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta} \frac{c}{e} \mu B_0 \left[\frac{\varepsilon}{q} \sin\theta + N\delta \sin(N\phi_0 + Nq\theta)\right] = 0 \quad . \tag{14}$$

The second term on the right side of Eq. (13) comes from the $\partial/\partial\phi_0$ term in Eq. (12), and represents the new non-axisymmetric $\vec{\nabla}B$ drift. The bounce average of this term is not zero which means this term causes a net drift away from the original flux surface and thus a contribution to ripple transport. However, from Eq. (14), we have

$$\begin{aligned} &\left[\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta} \frac{c}{e} \mu B_0 N\delta \sin(N\phi_0 + Nq\theta)\right]\left(\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta}\right)^{-1} \\ &= \left[-\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta} \frac{c}{e} \mu B_0 \frac{\varepsilon}{q} \sin\theta\right]\left(\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta}\right)^{-1} \quad . \end{aligned} \tag{15}$$

Since the θ variation is small across the ripple well $(\Delta\theta = |\theta - \theta_m| \lesssim \pi/Nq \ll 1)$, we obtain

$$\begin{aligned} &\left[\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta} \frac{c}{e} \mu B_0 N\delta \sin(N\phi_0 + Nq\theta)\right]\left(\oint \frac{Bd\theta}{v_{\parallel}\vec{B}\cdot\vec{\nabla}\theta}\right)^{-1} \\ &\simeq -\frac{c}{e} \frac{\mu B_0}{q} \varepsilon \sin\theta_m \quad . \end{aligned} \tag{16}$$

Thus, we can see that although $\langle \vec{v}_d \cdot \vec{\nabla}\psi \rangle$ has the same form as usual $\vec{\nabla}B$ drift, its origin is different.

III. Non-Axisymmetric Transport Analysis

Ripple transport can be studied by separating the linear drift kinetic equation into axisymmetric and non-axisymmetric parts.* This technique was first pointed out by Boozer[4] and was used in his banana drift transport calculations. The separated drift kinetic equations can be written as

$$v_{\parallel} \frac{\vec{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial F}{\partial \theta} + \vec{v}_{nc} \cdot \vec{\nabla}\phi_o \frac{\partial F}{\partial \phi_o} + \vec{v}_{nc} \cdot \vec{\nabla}\psi \frac{\partial f_M}{\partial \psi} = C(F) \quad , \tag{17}$$

$$v_{\parallel} \frac{\hat{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial f}{\partial \theta} + \vec{v}_{r} \cdot \vec{\nabla}\phi_o \frac{\partial f}{\partial \phi_o} + \vec{v}_{r} \cdot \vec{\nabla}\psi \frac{\partial f_M}{\partial \psi} = C(f) \quad , \tag{18}$$

and

$$\vec{v}_{nc} \cdot \vec{\nabla}\phi_o = -v_{\parallel} \frac{\hat{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega} \frac{\vec{B} \cdot (\vec{\nabla}\theta \times \vec{\nabla}\phi_o)}{\vec{B} \cdot \vec{\nabla}\theta}\right) \quad , \tag{19}$$

$$\vec{v}_{nc} \cdot \vec{\nabla}\psi = v_{\parallel} \frac{\vec{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial}{\partial \theta} \left(\frac{v_{\parallel}}{\Omega} \frac{\vec{B} \cdot (\vec{\nabla}\psi \times \vec{\nabla}\theta)}{\vec{B} \cdot \vec{\nabla}\theta}\right) \quad , \tag{20}$$

$$\vec{v}_{r} \cdot \vec{\nabla}\phi_o = v_{\parallel} \frac{\vec{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial}{\partial \psi} \left(\frac{v_{\parallel}}{\Omega} \frac{B^2}{\vec{B} \cdot \vec{\nabla}\theta}\right) \quad , \tag{21}$$

$$\vec{v}_{r} \cdot \vec{\nabla}\psi = -v_{\parallel} \frac{\vec{B} \cdot \vec{\nabla}\theta}{B} \frac{\partial}{\partial \phi_o} \left(\frac{v_{\parallel}}{\Omega} \frac{B^2}{\vec{B} \cdot \vec{\nabla}\theta}\right) \quad , \tag{22}$$

where F (f) is the perturbed particle distribution function due to the axisymmetric (F) and non-axisymmetric (f) particle drifts, and f_M is the local

*An alternative and perhaps more rigorous procedure is to separate the drift kinetic equation into fluctuating and bounce-averaged parts. The results of such a procedure are equivalent to Eqs. (17) and (18) for applications considered in this paper.

Maxwellian distribution. The bounce-averaged drift velocites in Eq. (7) vanish. This means there is no net drift off the flux surface for the axisymmetric case and gives rise to the usual neoclassical transport.[5] However, the bounce-averaged drift velocities in Eq. (18) are not zero; thus, particles drift away from the original flux surface and give rise to ripple and banana drift transport in a non-axisymmetric system.

Neglecting $\vec{v}_r \cdot \vec{\nabla}\phi_0$ and assuming $v_\perp^2 \simeq v^2$, we can write Eq. (18) explicitly as

$$\frac{v_\parallel}{Rq}\frac{\partial f}{\partial \theta} + \frac{mc}{2e} v^2 N\delta \sin N\phi \frac{\partial f_M}{\partial \psi} = C(f) \quad . \tag{23}$$

In the case where pitch angle scattering is dominant, the Coulomb collision operator can be written as

$$C(f) = \nu \frac{v_\parallel}{B}\frac{\partial}{\partial \mu}\left(m v_\parallel \mu \frac{\partial f}{\partial \mu}\right) \quad , \tag{24}$$

where ν is the collision frequency. Ripple transport will appear if $\nu_{eff} = \nu/\delta < \omega_{b\delta} = vN\sqrt{\delta}/R$. In this collision frequency regime the ripple trapped particle orbit is well defined. Since $\langle \vec{v}_d \cdot \vec{\nabla}\psi \rangle \neq 0$, we have to use the Ansatz[1,3]

$$f = \frac{\omega_{b\delta}}{\nu_{eff}} f^{-1} + f^0 + \frac{\nu_{eff}}{\omega_{b\delta}} f^1 + \ldots \quad . \tag{25}$$

To the lowest order, Eq. (23) can be written as

$$\frac{v_\parallel}{Rq}\frac{\partial f^{-1}}{\partial \theta} = 0 \quad . \tag{26}$$

To the next order, we have

$$\frac{v_\parallel}{Rq}\frac{\partial f^0}{\partial\theta} + \frac{mc}{2e} v^2 N\delta \sin N\phi \frac{\partial f_M}{\partial\psi} = C(f^{-1}) \quad . \tag{27}$$

Applying the bounce-averaging operator $\oint Rqd\theta/v_\parallel$ to Eq. (27) and using Eq. (16) we obtain

$$-\frac{v^2}{2\Omega R}\sin\theta_m(\oint\frac{d\theta}{v_\parallel})\frac{\partial f_M}{\partial r} = \frac{\nu}{B}\frac{\partial}{\partial\mu}(m\mu\oint v_\parallel d\theta\frac{\partial f^{-1}}{\partial\mu}) \quad . \tag{28}$$

In Eq. (28), we have rewritten $\partial f_M/\partial\psi$ in terms of $\partial f_M/\partial r$. Notice that Eq. (28) is the same as that obtained by Connor and Hastie[3], and can be solved for $\partial f^{-1}/\partial\mu$ subject to the boundary condition that $\partial f^{-1}/\partial\mu$ should be well behaved at the bottom of the ripple well where $\oint v_\parallel d\theta = 0$. Using the relation that[3]

$$\frac{\partial}{\partial\mu}(\mu\oint v_\parallel d\theta) \simeq -\oint\frac{v^2}{2v_\parallel}d\theta + O(\delta) \quad , \tag{29}$$

we have

$$\frac{\partial f^{-1}}{\partial\mu} = \frac{\partial f_M}{\partial r}\frac{c}{e}\frac{1}{R}\sin\theta_m\frac{1}{\nu} \quad , \tag{30}$$

and hence

$$f^{-1} = \frac{\partial f_M}{\partial r}\frac{c}{e}\frac{\sin\theta_m}{R\nu}(\mu - \frac{E}{B_{max}}) \quad , \tag{31}$$

where $B_{max} = B(\theta_-)$. The usual ripple transport coefficients[1,3] can be calculated from Eq. (31).

Although the next order solution of f, namely f^0, is not important for the ripple transport calculations, it is worthwhile to calculate it for reasons which will be shown later. To solve for f^0, we formally integrate Eq. (27) and obtain

$$f_\sigma^0 = \sigma\left[\frac{mc}{2e} v^2 N\delta \frac{\partial f_M}{\partial \psi} Rq \int_\theta^{\theta_t} \frac{d\theta}{|v_\parallel|} \sin(N\phi_0 + Nq\theta) - \int_\theta^{\theta_t} \frac{Rqd\theta}{|v_\parallel|} c(f^{-1})\right] + g \quad , \tag{32}$$

with $\partial g/\partial\theta = 0$, $\sigma = \text{sgn}(v_\parallel)$ and θ_t is the turning point with $\theta_t > \theta_m$. To determine g, we have to go to the next higher order equation:

$$\frac{v_\parallel}{Rq}\frac{\partial f^1}{\partial \theta} = C(f^0) \quad . \tag{33}$$

After some algebra, we obtain

$$\frac{|v_\parallel|}{Rq}\frac{\partial}{\partial\theta}\left(f_+^1 - f_-^1\right) = 2C(g) \quad . \tag{34}$$

Applying the bounce-averaging operator $\oint Rqd\theta/|v_\parallel|$ to this equation, we have $\partial g/\partial\mu = 0$ since $\oint d\theta v_\parallel = 0$ at the bottom of the ripple well. Thus, g must be independent of θ (or s) and μ and it will not contribute to transport; hence we may take $g = 0$.

When $\nu_{eff} < \omega_{b\delta}$, the distortion of the distribution function due to the ripple trapped particles is shown in Eqs. (31) and (32). However, when $\nu_{eff} < \omega_{b\delta}$, the banana trapped particles are in the collisionless ripple plateau regime. The particles are in this regime since for $\alpha^* < 1$, the pitch angle variation due to motion along the magnetic field line is on the order of $\sqrt{\delta}$, and the transition between the collisional and collisionless plateau regimes occurs at the critical pitch angle[4] where $\lambda_c \equiv (\nu R/vN)^{1/3}$ is equal to $\sqrt{\delta}$, which gives $\nu_{eff} = \omega_{b\delta}$. The perturbed distribution function over the last ripple phase in the collisionless ripple plateau regime can be written as

$$f_\sigma^o = \sigma \frac{mc}{2e} v^2 N\delta \frac{\partial f_M}{\partial \psi} Rq \int_\theta^{\theta_t} \frac{d\theta}{|v_\parallel|} \sin (N\phi_o + Nq\theta)$$
$$- \frac{mc}{2e} v^2 N\delta \frac{\partial f_M}{\partial \psi} Rq \int_{-\infty}^{\theta_t} \frac{d\theta}{|v_\parallel|} \sin (N\phi_o + Nq\theta) \quad , \tag{35}$$

and diminishes away from the ripple well. Notice that there is no $1/\nu$ type solution in the collisionless ripple plateau regime because particles make only a finite radial excursion from a flux surface in going over the last ripple well.

From Eqs. (31), (32), and (35), we see that $\partial f/\partial \mu$ is discontinuous across the boundary between ripple trapped and banana trapped particles. Thus, the collisional effect which is proportional to $\partial^2 f/\partial \mu^2$ is no longer negligible around this boundary and should be taken into account. As we approach the boundary, Eqs. (26) and (27) are no longer valid and should be replaced by

$$\frac{v_\parallel}{Rq} \frac{\partial f^{-1}}{\partial \theta} = C(f^{-1}) \quad , \tag{36}$$

$$\frac{v_\parallel}{Rq} \frac{\partial f^o}{\partial \theta} + \frac{mc}{2e} v^2 N\delta \sin N\phi \frac{\partial f_M}{\partial \psi} = C(f^o) \quad . \tag{37}$$

Equation (36) is used to join the $O(1/\nu)$ solution which is given by Eq. (31) inside the ripple well and zero outside the well. Equation (37) is used to join the $O(\nu^0)$ solutions which are given in Eqs. (32) and (35). Equation (37) can be simplified by defining

$$f_\sigma^0 = h_\sigma + \sigma \frac{mc}{2e} v^2 N\delta \frac{\partial f_M}{\partial \psi} Rq \int_\theta^{\theta_t} \frac{d\theta}{|v_\parallel|} \sin (N\phi_0 + Nq\theta) \quad , \tag{38}$$

so that h_σ satisfies

$$\sigma \frac{|v_\parallel|}{Rq} \frac{\partial h_\sigma}{\partial \theta} = C(h_\sigma) \quad . \tag{39}$$

To obtain Eq. (39), we have neglected $O(\nu)$ corrections in Eq. (38). The function h_σ is now used to connect the noncommon parts of the solutions Eqs. (32) and (35) on each side of the boundary. Notice that Eqs. (36) and (39) have the same form, but their boundary conditions are different. In this paper, we will concentrate on the boundary layer effect on ripple transport. The boundary layer correction to banana drift transport will be discussed in a subsequent paper.

IV. Boundary Layer Analysis

The boundary layer equation for the ripple trapped particles is given in Eq. (36) and can be written explicitly as

$$\frac{v_\parallel}{Rq} \frac{\partial f}{\partial \theta} = \nu \frac{v_\parallel}{B} \frac{\partial}{\partial \mu} m v_\parallel \mu \frac{\partial f}{\partial \mu} \quad . \tag{40}$$

The superscript -1 in f will be dropped from now on. Defining

$$\lambda = \mu \bar{B}/E \quad ,$$

$$|v_\parallel| = \sqrt{\frac{2E}{m}} (1 - \lambda/h)^{1/2} = \sqrt{\frac{2E}{m}} \bar{v}_\parallel \quad ,$$

$$h = 1 + \delta_{eff} \cos \left(\pi \frac{N\phi_o + Nq\theta + \sin^{-1} \alpha^*}{\pi - 2 \sin^{-1} \alpha^*}\right) ,$$

and $y = N\phi_o + Nq\theta$,

we can simplify Eq. (40) to

$$\frac{\partial f_\pm}{\partial y} = \pm \bar{\nu} \frac{\partial}{\partial \lambda} \bar{v}_\parallel \lambda \frac{\partial f_\pm}{\partial \lambda} , \tag{41}$$

where $\bar{\nu} = 2\nu/\omega_{bu}$ with $\omega_{bu} = vN/R$. Particles with $1 - \delta_{eff} < \lambda < 1 + \delta_{eff}$ will be trapped in the ripple well and particles with $0 < \lambda < 1 - \delta_{eff}$ will not be trapped in the ripple well. Since we are only interested in the region where $\lambda \cong 1 - \delta_{eff}$, we can define the boundary layer parameter x as

$$\lambda = 1 - \delta_{eff} - \bar{\nu}^{1/2} \left[2\sqrt{2\delta_{eff}} (\pi - 2 \sin^{-1} \alpha^*)/\pi^2\right]^{1/2} x . \tag{42}$$

Thus, ripple trapped (untrapped) particles are categorized by $x < 0$ ($x > 0$). Notice that the parameter in front of x is of the order of $\sqrt{\nu_{eff}/\omega_{b\delta}}\ \delta$ and is the thickness of the boundary layer in phase space. Assuming $\sqrt{\bar{\nu}/\delta_{eff}^{3/2}} << 1$, we obtain

$$\bar{v}_\parallel = \sqrt{2\delta_{eff}} \cos \left(\frac{\pi}{2} \frac{y + \sin^{-1} \alpha^*}{\pi - 2 \sin^{-1} \alpha^*}\right) , \tag{43}$$

in the boundary layer. Defining

$$\phi = \pi \sin \left(\frac{\pi}{2} \frac{y + \sin^{-1} \alpha^*}{\pi - 2 \sin^{-1} \alpha^*}\right) , \tag{44}$$

and assuming $\sqrt{\nu/\delta_{eff}} \ll 1$ and $\sqrt{\nu/\delta_{eff}^{3/2}} \ll 1$, we reduce the boundary layer equation to

$$\frac{\partial f_\pm}{\partial \phi} = \pm \frac{\partial^2 f_\pm}{\partial x^2} \quad . \tag{45}$$

To obtain Eq. (45), we have neglected the boundary layer due to the smallness of $v_\parallel$ near the turning points, since the thickness of this boundary layer is $O(\nu)$ and is smaller than that due to the large gradient of $\partial f/\partial \mu$, which is $O(\sqrt{\nu})$. At the maxima of the ripple well $\theta = \theta_\pm$, we have $\phi = \pm\pi$, and this defines the boundary in real space. Also, we have $\overline{v}_\parallel = 0$ at $\phi = \pm\pi$. The analysis to obtain Eq. (45) is similar to that in axisymmetric neoclassical transport.[6]

The boundary conditions for ripple trapped particles ($x < 0$) which arise from mirroring on the sides of the ripple well are $f_+ = f_-$ at $\phi = \pm\pi$. For ripple untrapped particles ($x > 0$) reflection off the mostly toroidal magnetic well yields a boundary condition $f_+ = f_-$ at $\phi = \pi$. As a final boundary condition we use $f_+ = 0$ at $\phi = -\pi$, which we justify by requiring that particles with $v_\parallel > 0$ coming from the ripple boundary layer at $\theta = -\theta_m$ cannot reach the ripple boundary layer at $\theta = +\theta_m$ without scattering in pitch-angle by more than the boundary layer width. The criterion for this to occur is

$$\frac{\nu}{\left(\sqrt{\frac{\nu_{eff}}{\omega_{b\delta}}}\,\delta\right)^2} > \omega_{b\varepsilon} = \frac{v\sqrt{\varepsilon}}{Rq} \quad ,$$

or

$$1 > \frac{\sqrt{\varepsilon/\delta}}{Nq} \quad . \tag{46}$$

This condition is almost always valid for a tokamak. The boundary layer solution of Eq. (45) should match asymptotically to Eq. (31) as $x \to -\infty$ and zero as $x \to +\infty$.

The boundary layer equations will now be solved by a Wiener-Hopf procedure similar to that used by Hinton and Rosenbluth[6] for axisymmetric neoclassical transport. The Fourier transform of $f_\pm$ is defined as

$$F_\pm(\phi,k) = \int_{-\infty}^{\infty} dx \exp(ikx)\ f_\pm(\phi,x) \quad . \tag{47}$$

The integral of Eq. (47) is carried out for $\mathrm{Im}k = -\gamma$ ($\gamma > 0$) to make $F_\pm$ converge as $x \to -\infty$. The transformed Eq. (45) is

$$\frac{\partial F_\pm}{\partial \phi} = \mp k^2 F_\pm \quad , \tag{48}$$

with solutions

$$F_+(\phi,k) = F_+(\pi,k) \exp[-k^2(\phi - \pi)] \quad , \tag{49}$$

$$F_-(\phi,k) = F_-(-\pi,k) \exp[k^2(\phi + \pi)] \quad . \tag{50}$$

The one-sided Fourier transforms are defined as

$$S_\pm(\phi,k) = \int_{-\infty}^{0} dx \exp(ikx)\ f_\pm(\phi,x) \quad , \tag{51}$$

$$R_\pm(\phi,k) = \int_{0}^{\infty} dx \exp(ikx)\ f_\pm(\phi,x) \quad , \tag{52}$$

and $S_\pm$ is analytic for Imk < $-\gamma$ (lower half plane) and $R_\pm$ is analytic for Imk > $-\gamma$ (upper half plane). Along Imk = $-\gamma$, we have

$$F_\pm(\phi,k) = S_\pm(\phi,k) + R_\pm(\phi,k) \quad . \tag{53}$$

The transformed boundary conditions are

$$S_+(\pi,k) = S_-(\pi,k) \quad , \quad S_+(-\pi,k) = S_-(-\pi,k) \tag{54}$$

for x < 0, and

$$R_+(\pi,k) = R_-(\pi,k) \quad , \quad R_+(-\pi,k) = 0 \tag{55}$$

for x > 0. From Eqs. (49) and (50), we have

$$R_+(-\pi,k) + S_+(-\pi,k) = [R_+(\pi,k) + S_+(\pi,k)] \exp(2\pi k^2) \quad , \tag{56}$$

$$R_-(\pi,k) + S_-(\pi,k) = [R_-(-\pi,k) + S_-(-\pi,k)] \exp(2\pi k^2) \quad . \tag{57}$$

Using the boundary conditions Eqs. (54) and (55), we obtain

$$S_- = (R_+ + S_+) \exp(2\pi k^2) \quad , \tag{58}$$

$$R_+ + S_+ = (R_- + S_-) \exp(2\pi k^2) \quad , \tag{59}$$

where $S_\pm \equiv S_\pm(\pm\pi,k)$ and $R_\pm \equiv R_\pm(\pm\pi,k)$. Substituting Eq. (59) into Eq. (58), we have

$$-S_-[1 - \exp(-4\pi k^2)] = R_- \quad . \tag{60}$$

We factorize

$$1 - \exp(-4\pi k^2) = \frac{L(k)}{U(k)} \quad , \tag{61}$$

where L(k) (U(k)) is analytic and has no zeros in the lower (upper) half plane. Following the analysis by Baldwin et al.[7], we define

$$q(k) = k^{-1} \ln [1 - \exp(-4\pi k^2)] \quad , \tag{62}$$

which is analytic in the region $-1/2 < \mathrm{Im}k < 0$, and can be separated into the form

$$q(k) = \frac{1}{2\pi i} \int_{-\infty - i/2^+}^{\infty - i/2^+} \frac{d\eta q(\eta)}{\eta - k} - \frac{1}{2\pi i} \int_{-\infty + io^-}^{\infty + io^-} \frac{d\eta q(\eta)}{\eta - k} \equiv q_+(k) - q_-(k) \quad , \tag{63}$$

where $1/2^+ = 1/2 - \Delta$, and $0^- = -\Delta$ with Δ a positive infinitesimal. The function q_+ (q_-) is analytic in the upper (lower) half plane. However, the definition of q(k) in Eq. (61) is by no means unique. In Appendix A, we give another definition of q(k), and we show that it gives the same results. Comparing Eq. (62) with (61) we have

$$L(k) = \exp(-kq_-) \quad , \tag{64}$$

$$U(k) = \exp(-kq_+) \quad , \tag{65}$$

and we can write Eq. (61) as

$$-S_- L(k) = R_- U(k) \quad . \tag{66}$$

The left side of Eq. (66) is analytic in the lower half plane and the right side is analytic in the lower half plane, and they are analytic continuations of each other across $-1/2 < \mathrm{Im}k = -\gamma < 0$. Thus, the function $\psi(k)$ defined by

$$\psi(k) = -S_-L(k) \qquad (\mathrm{Im}k \leq -\gamma)$$

$$= R_-U(k) \qquad (\mathrm{Im}k \geq -\gamma) \quad , \tag{67}$$

must be an entire function. As $|k| \to \infty$, $S_- \sim O(1/k)$ and $L(k) \sim k$ and so $\psi \sim$ constant as $|k| \to \infty$. We thus have

$$S_- = -A/L \quad , \tag{68}$$

$$R_- = A/U \quad , \tag{69}$$

and

$$R_+ + S_+ = -\frac{A}{L}\exp(-2\pi k^2) \quad . \tag{70}$$

The constant A is to be determined by matching the solution to the slope of Eq. (31) as $x \to -\infty$. The particle distribution $f_\pm$ can be obtained by inverse transforming Eqs. (49) and (50)

$$f_+(\phi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ \left(-\frac{A}{L}\right)\exp[-ikx - 2\pi k^2 - k^2(\phi - \pi)] \quad , \tag{71}$$

$$f_-(\phi,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ (-A)\left(\frac{1}{L} - \frac{1}{U}\right)\exp[-ikx + k^2(\phi + \pi)] \quad . \tag{72}$$

For $x < 0$, the contour of the integrals Eq. (71) and (72) is from $(-\infty + i\mathrm{Im}k)$ to $(+\infty + i\mathrm{Im}k)$ along $-1/2 < \mathrm{Im}k = -\gamma < 0$ and goes around the upper half plane. The analytic configuration of q_- to the upper half plane can be obtained from Eqs. (62) and (63)

$$q_-(k) = -\frac{1}{k}\ln[1 - \exp(-4\pi k^2)] + q_+(k) \quad , \tag{73}$$

As $x \to -\infty$, only the pole at $k = 0$ contributes to the integrals. We thus obtain

$$f_{\pm}(\phi, x\to-\infty) = -\frac{A}{4\pi}(x - 2.92) = C(x - 2.92) = f(\phi, x\to-\infty) \quad . \tag{74}$$

The constant C (or A) can be determined from

$$\frac{\partial f^{-1}}{\partial \mu}\frac{\partial \mu}{\partial \lambda}\frac{\partial \lambda}{\partial x} = \frac{\partial f}{\partial x}$$

and hence

$$C = -\bar{\nu}^{1/2}\left[2\sqrt{2\delta_{eff}}\,(\pi - 2\sin^{-1}\alpha^{*})/\pi^{2}\right]^{1/2}\frac{E}{\bar{B}}\frac{c}{e}\frac{\sin\theta_m}{R\nu}\frac{\partial f_M}{\partial r} \quad . \tag{75}$$

The distorted ripple trapped particle distribution with the boundary layer correction is thus finally

$$f^{-1} = \frac{\partial f_M}{\partial \gamma}\frac{c}{e}\frac{\sin\theta_m}{R\nu}\left\{\left(\mu - \frac{E}{B_{max}}\right) + 2.92\,\bar{\nu}^{1/2}\frac{E}{\bar{B}}\left[2\sqrt{2\delta_{eff}}\,(\pi - 2\sin^{-1}\alpha^{*})/\pi^{2}\right]^{1/2}\right\} \tag{76}$$

$$\equiv f_{ch} + f_{bl} \quad ,$$

where $f_{ch} = f^{-1}$ is the first term in the braces of Eq. (76) and f_{bl} is the second term.

The corrected particle flux is the flux surface average of the velocity space integral of this distribution function times the radial drift velocity:

$$\Gamma_{b1} = \int_0^{2\pi} \frac{d\theta}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\infty} \frac{4\pi dE}{m^2} \int_{E/B_{max}}^{E/B} \frac{Bd\mu}{|v_\parallel|} f_{b1} \left(\frac{\vec{v}_d \cdot \vec{\nabla}\psi}{|\vec{\nabla}\psi|}\right)$$

$$= \int_0^{2\pi} \frac{d\theta_m}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} \int_0^{\infty} \frac{4\pi dE}{m^2} \int_{E/B_{max}}^{E/B} \frac{Bd\mu}{|v_\parallel|} f_{b1} \frac{\langle \vec{v}_d \cdot \vec{\nabla}\psi \rangle}{|\vec{\nabla}\psi|} \quad . \tag{77}$$

After carrying out the μ integration, we have

$$\Gamma_{b1} = -2.92\ A_{b1} \int_0^{\infty} dE\ \frac{\partial f_M}{\partial r} \sqrt{m}\ \frac{E^{5/2}}{\nu}\ \bar{\nu}^{1/2} \left(\frac{c}{eBR}\right)^2 \quad , \tag{78}$$

where

$$A_{b1} = 2\sqrt{2} \int_0^{2\pi} \frac{d\theta_m}{2\pi} \int_0^{2\pi} d\phi\ \sin^2\theta_m \sqrt{1 - \frac{B}{B_{max}}}\ [2\sqrt{2\delta_{eff}}\ (\pi - 2\sin^{-1}\alpha^*)/\pi^2]^{1/2} \quad . \tag{79}$$

If we use the field model given by Eq. (6), we obtain

$$A_{b1} = \frac{8}{\sqrt{\pi}} (2\delta)^{3/4} \quad , \tag{80}$$

in the limit of $\alpha = 0$, and in general,

$$A_{b1} = \frac{8}{\sqrt{\pi}} (2\delta)^{3/4}\ \bar{G}(\alpha) \quad , \tag{81}$$

where

$$\overline{G}(\alpha) = \frac{1}{\sqrt{2}\,\pi^{3/2}\alpha^2} \int_0^c dX \frac{\cos X \sin^2 X}{\sqrt{\alpha^2 - \sin^2 X}} [\cos X - \frac{\sin X}{2} (\pi - 2X)]^{1/4} (\pi - 2X)^{1/2}$$

$$\times \int_X^{Y_1(X)} dY\, [\cos X - \cos Y + (X - Y) \sin X]^{1/2} \quad , \tag{82}$$

where $Y = Y_1(X)$ is the zero of the inner integrand and $c = \pi/2$ for $\alpha > 1$ and $c = \sin^{-1} \alpha$ for $\alpha < 1$. The numerical calculation of $\overline{G}(\alpha)$ is shown in Fig. 2. After doing the energy integral with $\nu = \nu(T)(E/T)^{-3/2}$, we obtain

$$\Gamma_{bl} = -(\frac{1}{2\pi})^{3/2} \delta^{3/2} \frac{\overline{G}(\alpha)}{\nu} \times 2.92 \times 2^{3/4} \times 48 \sqrt{\frac{2}{\pi}} (\frac{\nu}{\omega_{bu}\delta^{3/2}})^{1/2} (\frac{CT}{eBR})^2$$

$$\times (\frac{1}{n}\frac{dn}{dr} + \frac{e}{T}\frac{d\Phi}{dr} + \frac{5}{2}\frac{1}{T}\frac{dT}{dr}) \quad , \tag{83}$$

where $\nu = \nu(T)$ is ν_{ii} for ions and is $\nu_{ee} + \nu_{ei}$ for electrons, in which

$$\nu_{ij} = \frac{\sqrt{2}\,\pi n e^4 \ln \Lambda}{\sqrt{m_i}\, T_j^{3/2}} \quad .$$

The boundary layer corrected ion and electron particle fluxes are

$$\Gamma_i = -\frac{64}{9} \frac{\delta^{3/2}}{(2\pi)^{3/2}} (\frac{CT_i}{eBR})^2 \frac{24}{\nu_{ii}^*} (\frac{3}{4}\sqrt{\pi})^{-1} n_i G(\alpha) [(\frac{1}{n_i}\frac{dn_i}{dr} + \frac{e}{T_i}\frac{d\Phi}{dr}$$

$$+ 3.5 \frac{1}{T_i}\frac{dT_i}{dr}) + 1.27 \frac{\overline{G}}{G} \sqrt{\nu_{eff}/\omega_{b\delta}} (\frac{1}{n}\frac{dn}{dr} + \frac{e}{T_i}\frac{d\Phi}{dr} + 2.5 \frac{1}{T_i}\frac{dT_i}{dr})] \quad , \tag{84}$$

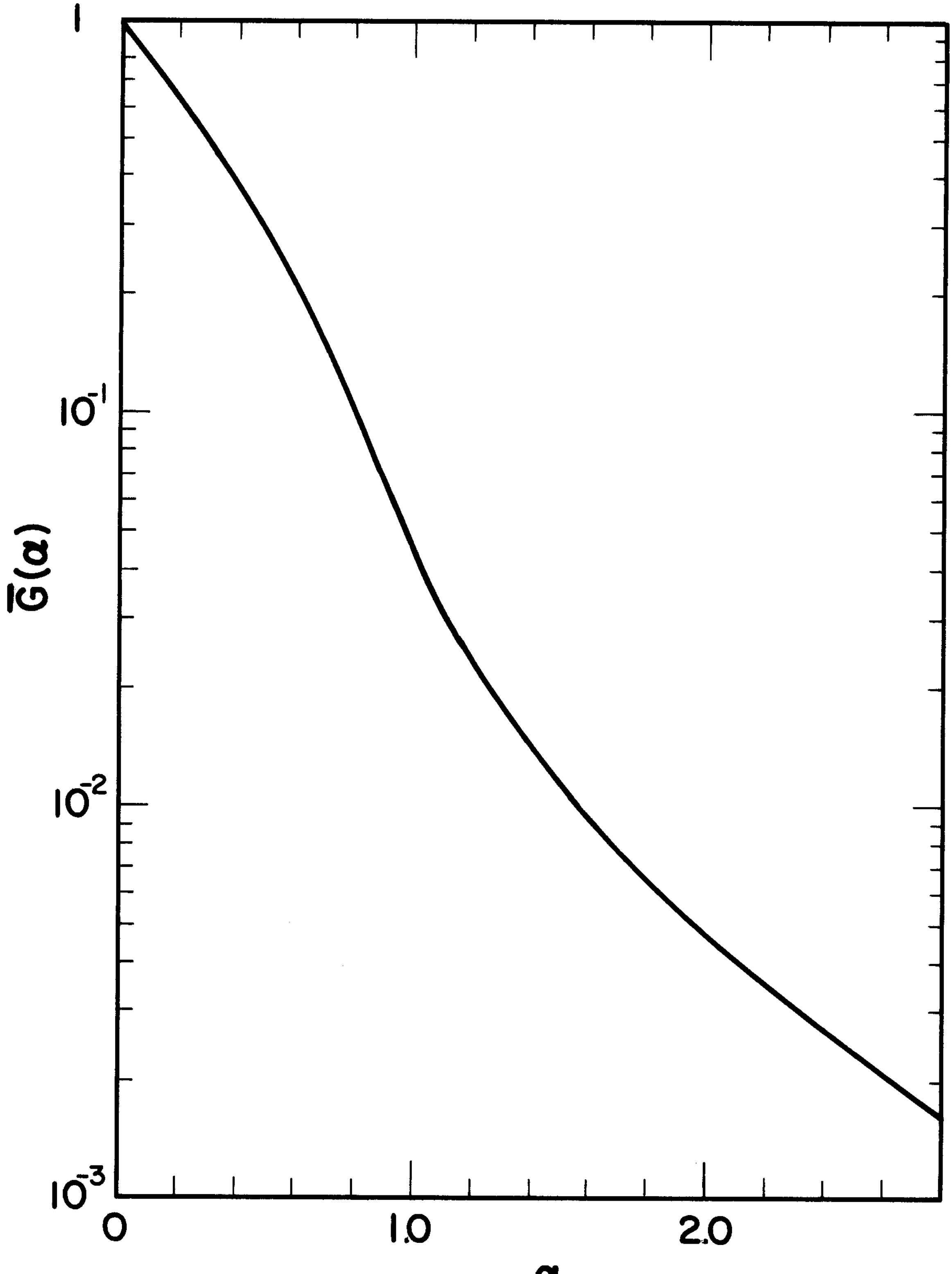

Fig. 2. $\overline{G}(\alpha)$ versus α.

$$\Gamma_e = -\frac{64}{9}\frac{\delta^{3/2}}{(2\pi)^{3/2}}\left(\frac{CT_e}{eBR}\right)^2 \frac{12}{\overset{*}{\nu}_{ei}}\left(\frac{3}{4}\sqrt{\pi}\right)^{-1} n_e G(\alpha)\left[\left(\frac{1}{n_e}\frac{dn_e}{dr} - \frac{e}{T_e}\frac{d\Phi}{dr}\right.\right.$$

$$\left.\left. + 3.5\frac{1}{T_e}\frac{dT_e}{dr}\right) + 1.80\frac{\overline{G}}{G}\sqrt{\nu_{eff}/\omega_{b\delta}}\left(\frac{1}{n}\frac{dn}{dr} - \frac{e}{T_e}\frac{d\Phi}{dr} + 2.5\frac{1}{T_e}\frac{dT_e}{dr}\right)\right] \quad , \tag{85}$$

where $\omega_{b\delta} = \omega_{bu}\sqrt{\delta}$ and

$$\overset{*}{\nu}_{ij} = \frac{4}{3}\frac{\sqrt{2\pi}\, n_e^4 \ln\Lambda}{\sqrt{m_i}\, T_j^{3/2}} \quad .$$

Note that $(\nu_{eff}/\omega_{b\delta})_e = (\nu_{eff}/\omega_{b\delta})_i$ if $T_i = T_e$. Since the ion flux is much larger than that of electrons, an ambipolar potential Φ should be developed to reduce the ion flux to maintain quasi-neutrality. To lowest order in $\sqrt{m_e/m_i}$, we thus have

$$\frac{e}{T_i}\frac{d\Phi}{dr} = -\frac{1}{n}\frac{dn}{dr} - \frac{3.5\ G(\alpha) + 2.5 \times 1.7\sqrt{\nu_{eff}/\omega_{b\delta}}\ \overline{G}(\alpha)}{G(\alpha) + 1.27\sqrt{\nu_{eff}/\omega_{b\delta}}\ \overline{G}(\alpha)}\frac{1}{T_i}\frac{dT_i}{dr} \quad . \tag{86}$$

Substituting Eq. (86) into Eq. (85), we obtain the ambipolar particle flux

$$\Gamma_a = -4.1\delta^{3/2}\left(\frac{CT_e}{eBR}\right)^2 \frac{n}{\overset{*}{\nu}_{ei}}\left\{G(\alpha)\left(1 + 1.80\frac{\overline{G}}{G}\sqrt{\nu_{eff}/\omega_{b\delta}}\right)\left[\frac{1}{n}\frac{dn}{dr}\left(1 + \frac{T_i}{T_e}\right)\right.\right.$$

$$\left.\left. + 3.5\left(\frac{T_e'}{T_e} + \frac{T_i'}{T_e}\right)\right] - 1.27\sqrt{\nu_{eff}/\omega_{b\delta}}\ \overline{G}(\alpha)\left(\frac{T_i'}{T_e} + 1.42\frac{T_e'}{T_e}\right)\right\} \quad .$$

To obtain Eq. (87), we have neglected terms of order $\nu_{eff}/\omega_{b\delta}$, but kept the order $\sqrt{\nu_{eff}/\omega_{b\delta}}$ boundary layer correction terms. Similarly, for the ion heat flux, we obtain

$$Q_i = -41 \frac{\delta^{3/2}}{\nu_{ii}^*} \left(\frac{CT_i}{eBR}\right)^2 G(\alpha)\, n \frac{dT_i}{dr} \left(1 + 1.27 \frac{\overline{G}}{G} \sqrt{\nu_{eff}/\omega_{b\delta}}\right) \quad . \tag{88}$$

For comparison purposes, we show the numerical calculation of the $G(\alpha)$ function in Fig. 3. We see that $\overline{G}$ is about a factor of 2 larger than G for $\alpha > 1$ and comparable to $G(\alpha)$ for $\alpha < 1$. The total ion heat conductivity χ_i versus collisional frequency ν with and without boundary layer correction is shown in Fig. 4.

V. Concluding Remarks

Boundary layer equations have been derived by a systematic expansion of the drift kinetic equation in the small parameter ν/ω_b for both ripple and banana drift transport. The corrected particle distribution for ripple trapped particles has been obtained by solving the boundary layer equation with Wiener-Hopf techniques. It is found that the boundary layer effect will increase both the particle and heat fluxes. The effect is important even at low collisionalities ($\nu_{eff}/\omega_{b\delta} \sim 10^{-2}$). The corrections to both the heat conductivity and diffusion coefficient are about a factor of 2 as $\nu_{eff}/\omega_{b\delta} \to 1$, and gradually decrease to about 15% for $\nu_{eff}/\omega_{b\delta} \sim 10^{-2}$.

Acknowledgment

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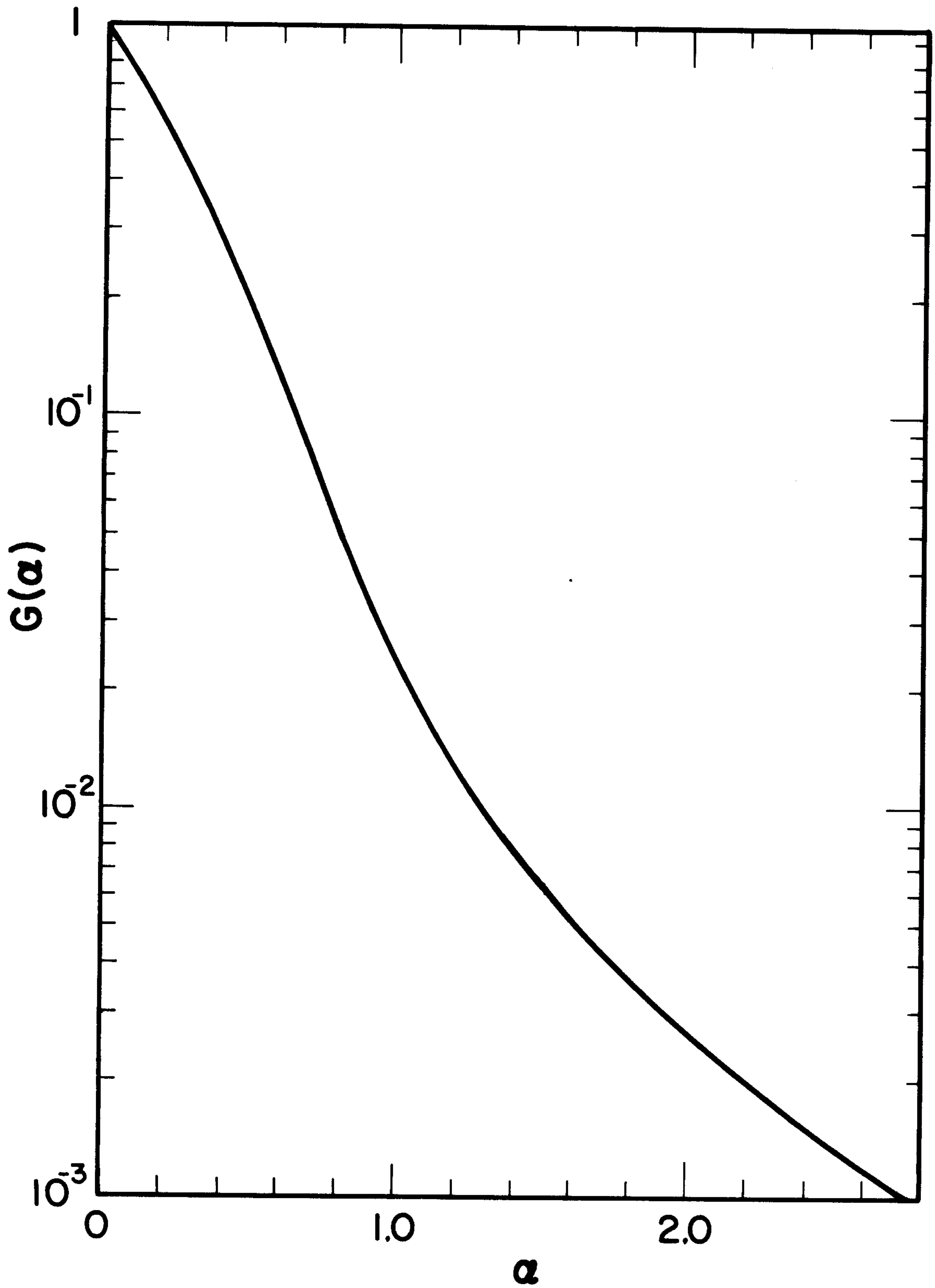

Fig. 3. G(α) versus α.

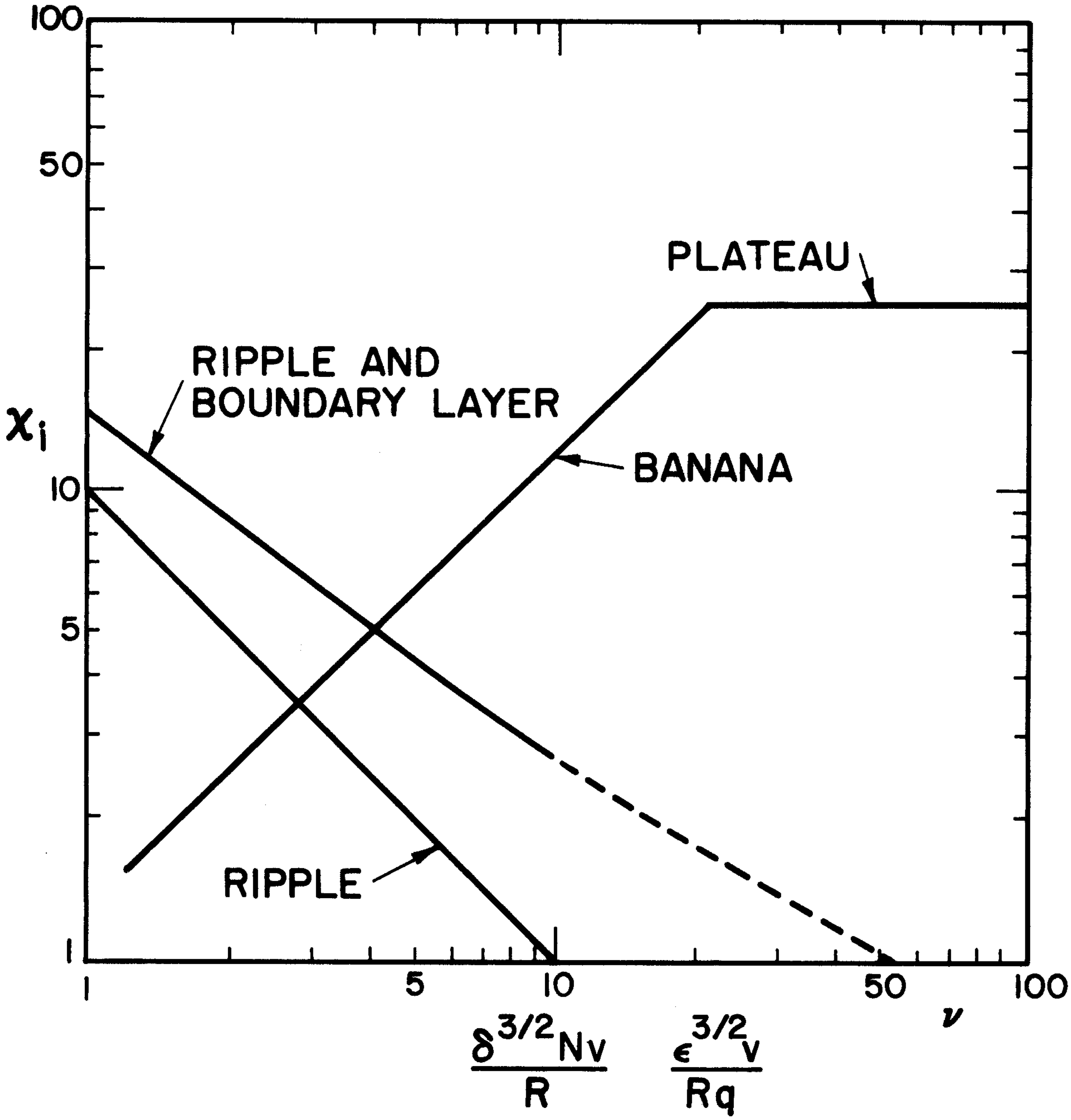

Fig. 4. Variation of boundary layer corrected χ_i with collisional frequency ν, compared with usual ripple, banana, and plateau transport.

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Appendix A

Using the general separation method in Wiener-Hopf problems[8], we can define

$$q = \ln \{\frac{(k^2 + \frac{1}{4})}{k^2} [1 - \exp(-4\pi k^2)]\} = q_+ - q_- \quad , \tag{A1}$$

where

$$q_+(k) = \int_{-\infty - i/2^+}^{\infty - i/2^+} \frac{d\eta}{\eta - k} \ln \{\frac{(\eta^2 + \frac{1}{4})}{\eta^2} [1 - \exp(-4\pi\eta^2)]\} \quad , \tag{A2}$$

$$q_-(k) = \int_{-\infty + i/2^-}^{\infty + i/2^-} \frac{d\eta}{\eta - k} \ln \{\frac{(\eta^2 + \frac{1}{4})}{\eta^2} [1 - \exp(-4\pi\eta^2)]\} \quad , \tag{A3}$$

and $1/2^+$ $(1/2^-)$ = 1/2 + (-) Δ with Δ a positive infinitesimal. The functions q_+ and q_- are analytic in the upper (Imk > -1/2) and lower (Imk < 1/2) half planes respectively, and bounded as $k \to \infty$. The function $[1 - \exp(-4\pi k^2)]$ can thus be factorized as

$$1 - \exp(-4\pi k^2) = \frac{k^2}{k^2 + \frac{1}{4}} \exp(q_+ - q_-) = \gamma_- \gamma_+ \quad , \tag{A4}$$

where

$$\gamma_- = k^2 (k - i/2)^{-1} \exp(-q_-) \quad , \tag{A5}$$

$$\gamma_+ = (k + i/2)^{-1} \exp(q_+) \quad . \tag{A6}$$

The functions γ_+ and γ_- are again analytic in the upper and lower half planes, respectively, and $\gamma_- \sim k$, and $q_- \sim 1/k$ as $k \to \infty$.

From Eq. (60), we have

$$-S_-\gamma_- = R_-/\gamma_+ \quad . \tag{A7}$$

We can define an entire function $\psi(k)$ such that

$$\psi = -S_-\gamma_- \quad , \tag{A8}$$

for $\mathrm{Im}k \leqslant -\gamma$, and

$$\psi = R_-/\gamma_+ \quad , \tag{A9}$$

for $\mathrm{Im}k \geqslant -\gamma$. Since $\psi(k) \sim 1/k \cdot k$ = constant as $k \to \infty$, $\psi(k)$ is thus a constant on the entire complex plane. We then have $S_- = -A/\gamma_-$ and $R_- = A\gamma_+$, where $A = \psi(k)$ is a constant. The functions $f_\pm(\phi,x)$ can be obtained by inverting the Fourier transform from Eqs. (71) and (72).

To obtain the asymptotic limit of $f_-(\phi,x)$ as $x \to -\infty$, we have to know γ_- as $k \to 0$ since $S_- = -A/\gamma_-$. As $k \to 0$, we have

$$S_- = i\,\frac{A}{2\sqrt{\pi}}\left(\frac{1}{k^2} - \frac{2 - iq_-'(0)}{ik}\right) \quad , \tag{A10}$$

where $q_-'(0) = [dq_-(k)/dk]_{k=0}$. To obtain Eq. (A10), we have used the fact that $q_-(k=0) = -\ln\sqrt{\pi}$, and neglected $O(k^0)$ terms. To eliminate the singularity at $\eta = 0$, we write $q_-'(k=0)$ as

$$q_-'(k=0) = \frac{1}{2\pi i}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon} \frac{d\eta}{\eta^2}\ln\left\{\frac{(\eta^2 + \frac{1}{4})}{\pi\eta^2}\left[1 - \exp(-4\pi\eta^2)\right]\right\} \quad , \tag{A11}$$

with a positive infinitesimal ε. After integrating by parts, we obtain

$$q'_-(k=0) = \frac{1}{\pi i} \int_0^\infty d\eta \left[\frac{8\pi \exp(-4\pi k^2)}{1 - \exp(-4\pi k^2)} - \frac{1}{2\eta^2(\eta^2 + \frac{1}{4})}\right] . \tag{A12}$$

The integral in Eq. (A12) can be carried out numerically, and the result is $q'_-(k=0) = 0.92i$. After inverting the Fourier transform, we obtain

$$f_-(\phi,x) \xrightarrow[x\to-\infty]{} C(x - 2.92) , \tag{A13}$$

which is the same as Eq. (74).