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## Abstract

A model for the ion distribution function in the central cell and barrier of a tandem mirror with an inboard thermal barrier is presented. The model is based on collisional distribution functions obtained from the Fokker-Planck equation with a pitch angle scattering operator. The rate of ion trapping in the barrier and the trapped ion density can be estimated; numerical results are given for a particular example.

## I. Introduction

The thermal barrier<sup>(1)</sup> is a region of reduced magnetic field strength and density between the end-plug and central cell of a tandem mirror. The thermal barrier allows the electrons in the plug to be heated to a higher temperature than those in the central cell and consequently permits one to obtain an electrostatic potential peak in the plugs even when the plug density is less than the central cell density. This greatly improves the  $Q$  (ratio of fusion power to total injected power) and reduces the need for high magnetic fields in the end-plugs. These improvements are a significant advance in the tandem mirror confinement scheme.

Successful operation of the thermal barrier requires that the density of trapped ions in the barrier be kept small by some pumping mechanism which either removes them from the plasma, as in drift-orbit pumping,<sup>(2)</sup> or converts them back to passing ions, as in neutral beam<sup>(3)</sup> or RF pumping.<sup>(4)</sup> Questions of interest include the resulting trapped ion density and the rate of ion trapping in the barrier, which affects the required pumping power and the barrier potential.

In this paper we consider a simple (and somewhat phenomenological) model of barrier pumping and calculate the ion distribution function in the barrier and central cell using the square well approximation and a pitch angle scattering collision operator in the Fokker-Planck equation. The results of the calculation are in reasonable agreement with Monte-Carlo calculations of Rognlien<sup>(5)</sup> for charge exchange pumped barriers.

## II. The Model

The magnetic field strength and electrostatic potential profile along a field line from the central cell through the barrier and into the plug are shown schematically in Fig. 1. The potential and magnetic field determine a

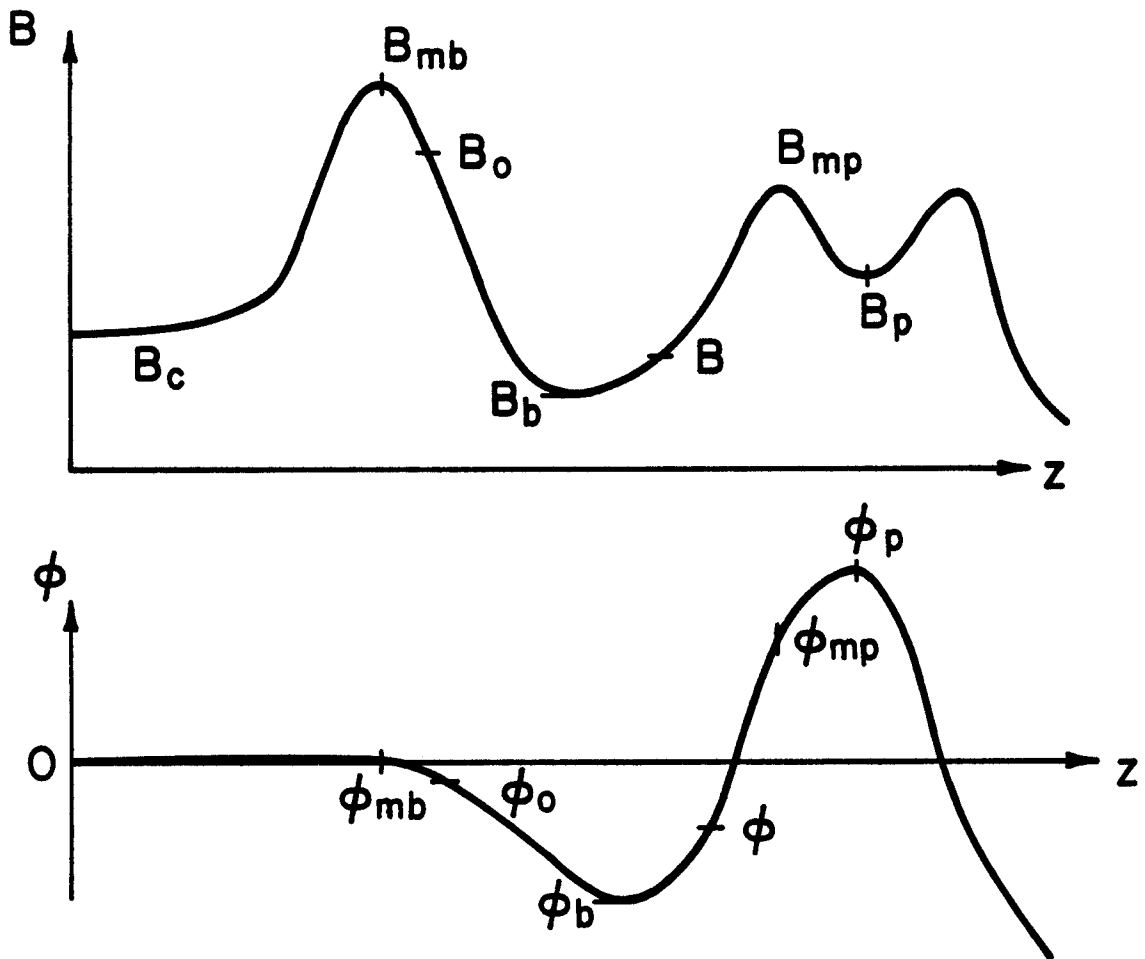


Fig. 1. The electrostatic potential and magnetic field strength in an inboard thermal barrier.

pitch angle  $\theta_{mb}^b$  in the barrier such that ions with pitch angle  $\theta > \theta_{mb}^b$  are trapped and  $\theta < \theta_{mb}^b$  are passing. Using conservation of total energy,  $\epsilon$ , and adiabatic invariance of the magnetic moment,  $\mu$ ,  $\theta_{mb}^b$  maps into a pitch angle  $\theta_{mb}^c$  in the central cell, where ions with pitch angle  $\theta > \theta_{mb}^c$  reflect off the barrier peak field and with  $\theta < \theta_{mb}^c$  penetrate into the barrier as passing ions.

The ion distribution function,  $F$ , is determined by the solution to the Fokker-Planck equation, which may be approximately written as<sup>(6)</sup>:

$$\frac{\partial}{\partial \theta} (\sin \theta \frac{\partial F}{\partial \theta}) = - \frac{qv^2}{D_{\perp}} \sin \theta \quad , \quad (2.1)$$

where  $v$  is the speed,  $D_{\perp}$  is the transverse diffusion coefficient in velocity space, and  $q$  is the source function in velocity space. The source is introduced in order to make  $F$  stationary in the presence of losses, and is taken to be isotropic in the central cell and zero in the barrier. We assume that the bounce frequency for trapped (or passing) particles in the thermal barrier is much greater than the collision frequency. Consequently, the particle motion is assumed collisionless. The parallel motion is described using guiding center theory with the magnetic moment,  $\mu$ , treated as a constant of the motion (i.e. an adiabatic invariant) along with the total particle energy,  $E$ . The ion distribution function is assumed to have relaxed to a collisional distribution since the equilibrium lasts for times long compared with the mean collision time. The distribution function  $F$  is therefore taken to be constant along a particle orbit, which is determined by  $E$  and  $\mu$ . Hence for passing ions:

$$F_c(\theta^c) = F_b(\theta^b) \quad , \quad (2.2)$$



where  $\theta^c$  is the pitch angle in the central cell of an ion which has pitch angle  $\theta^b$  in the barrier;  $F_c$  and  $F_b$  are the ion distribution functions in the central cell and barrier, respectively.

Equation (2.1) in conjunction with boundary conditions determines the pitch angle dependence of  $F$ ; the energy dependence in the central cell is taken to be Maxwellian:

$$F_c(\theta^c, E) = F_c(\theta^c) e^{-E/T_c} \quad (2.3)$$

for all  $0 \leq \theta^c \leq \pi/2$ . The assumption that  $F_c(\theta^c, E)$  can be separated as shown in Eq. (2.3) is made for convenience and has not been justified; it seems reasonable, however, since one expects the ions to be roughly Maxwellian except in the tail near the loss-boundary for ions escaping through the end-plug.

We consider first the central cell and let:

$$F_c(\theta^c) = \begin{cases} F_0 + F_1(\theta^c) , & \theta_{mb}^c < \theta^c < \pi/2 \\ F_0 + F_2(\theta^c) , & 0 < \theta^c < \theta_{mb}^c \end{cases} \quad (2.4)$$

The functions  $F_1(\theta^c)$  and  $F_2(\theta^c)$  satisfy the boundary conditions:

$$F_1(\theta_{mb}^c) = F_2(\theta_{mb}^c) = 0 \quad , \quad (2.5)$$

$$\left. \frac{\partial F_1}{\partial \theta^c} \right|_{\theta^c = \pi/2} = 0 \quad , \quad (2.6)$$

$$\left. \frac{\partial F_2}{\partial \theta^c} \right|_{\theta^c=0} = 0 \quad . \quad (2.7)$$

The solution to Eq. (2.1) in terms of  $F_1$  and  $F_2$  is then

$$F_1(\theta^c) = c_1 \ln \left( \frac{\sin \theta^c}{\sin \theta_{mb}^c} \right) \quad (2.8)$$

$$F_2(\theta^c) = c_2 \ln \left[ \frac{\cos (\theta^c/2)}{\cos (\theta_{mb}^c/2)} \right] \quad . \quad (2.9)$$

In the barrier region, we take the ion source,  $q$ , to be zero for  $\theta^b > \theta_{mb}^b$ .

The solution to Eq. (2.1) is then:

$$F_b(\theta^b) = \begin{cases} c_t \ln \left[ \frac{\text{tg} (\theta_0^b/2)}{\text{tg} (\theta^b/2)} \right] , & \theta_{mb}^b < \theta^b < \theta_0^b \\ 0 & , \quad \theta_0^b < \theta^b < \pi/2 \end{cases} \quad (2.10)$$

where the boundary condition

$$F_b(\theta_0^b) = 0$$

has been imposed. This represents a crude modeling of the effect of barrier pumping. The velocity space diffusion in the central cell is towards  $\theta_{mb}^c$ ; ion motion parallel to  $\vec{B}$  carries this into the barrier where it appears as a "delta-function source" at  $\theta_{mb}^b$  (see Fig. 2). The subsequent velocity space diffusion from  $\theta_{mb}^b$  to  $\theta_0^b$  causes ions to be lost at  $\theta_0^b$ .

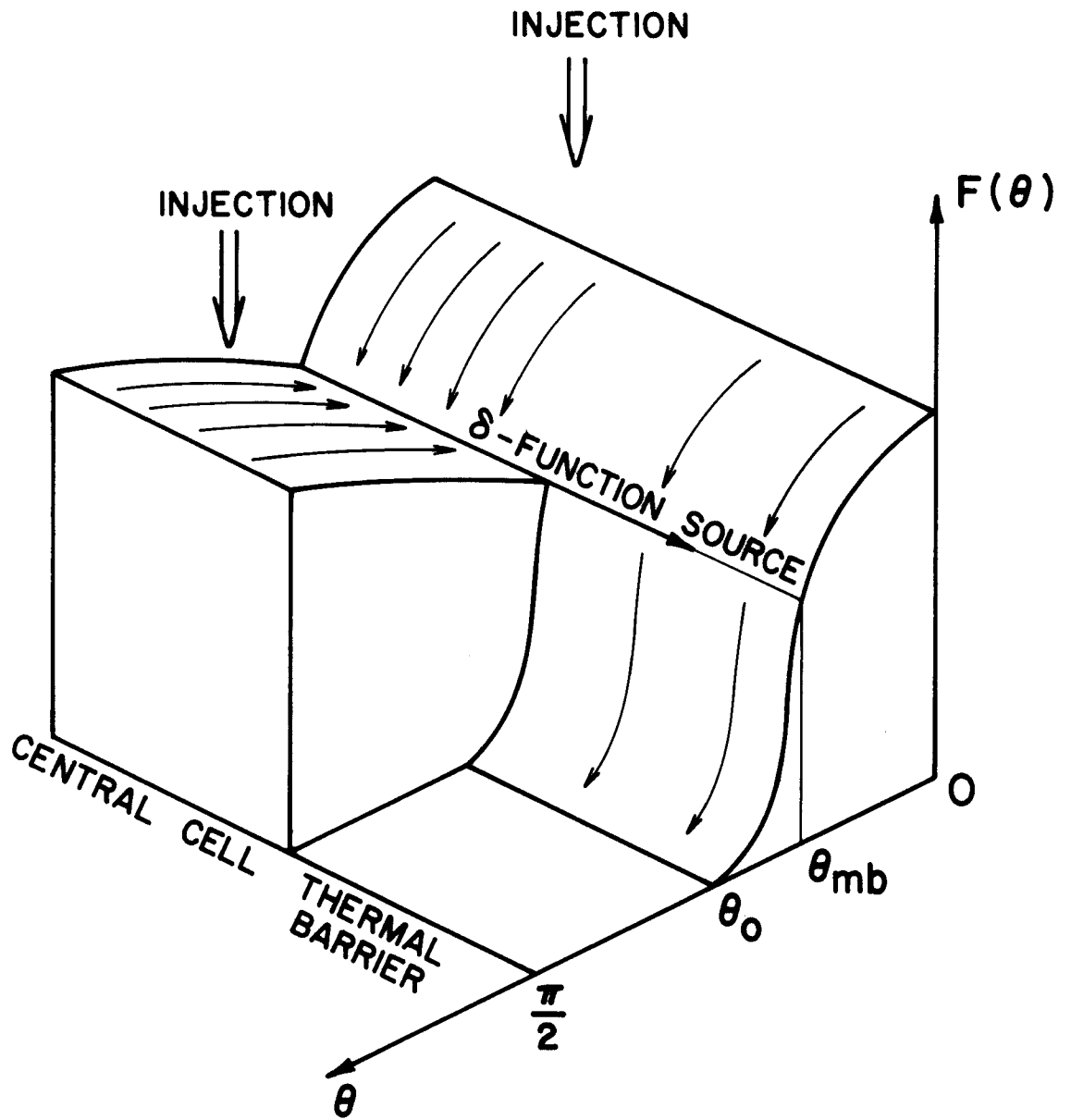


Fig. 2. Schematic representation of the ion pitch angle distribution function in the central cell and thermal barrier.

### III. Particle Current Balance

In the previous section, three coefficients ( $c_1, c_2, c_t$ ) are used to express the ion distribution functions in different regions. These coefficients are related through the requirement of a balance between the input and pumpout particle currents. The particle diffusion current from trapped in the central cell to trapped in the barrier can be written as:

$$J_1 = n_1^1 n_c H_3 \quad (3.1)$$

and the diffusion current from passing in the central cell to trapped in the barrier can be written as:

$$J_2 = n_2^1 n_c H_4 \quad (3.2)$$

Here  $n_c$  is the central cell ion density,  $n_1^1$  is the density of the anisotropic part of the trapped ions in the central cell, and  $n_2^1$  is the density of the anisotropic part of the central cell passing ions (see Fig. 3). The coefficients  $H_3$  and  $H_4$  are:

$$H_3 = \sqrt{\frac{2}{\pi}} \frac{V_c}{T_c^{3/2}} \left[ \frac{\cos \theta_{mb}^c}{\ln \left( \text{ctn} \frac{\theta_{mb}^c}{2} \right) - \cos \theta_{mb}^c} \right] \int_0^{\infty} \frac{D_{\perp}^c(u_c)}{n_c} e^{-E/T_c} du_c \quad (3.3)$$

$$H_4 = -\frac{1}{2} \sqrt{\frac{2}{\pi}} \frac{V_c}{T_c^{3/2}} \left[ \frac{1 - \cos \theta_{mb}^c}{\ln \cos^2 \frac{\theta_{mb}^c}{2} + \sin^2 \frac{\theta_{mb}^c}{2}} \right] \int_0^{\infty} \frac{D_{\perp}^c(u_c)}{n_c} e^{-E/T_c} du_c \quad (3.4)$$

using the distribution functions in section II. Here  $V_c$  is the volume of the

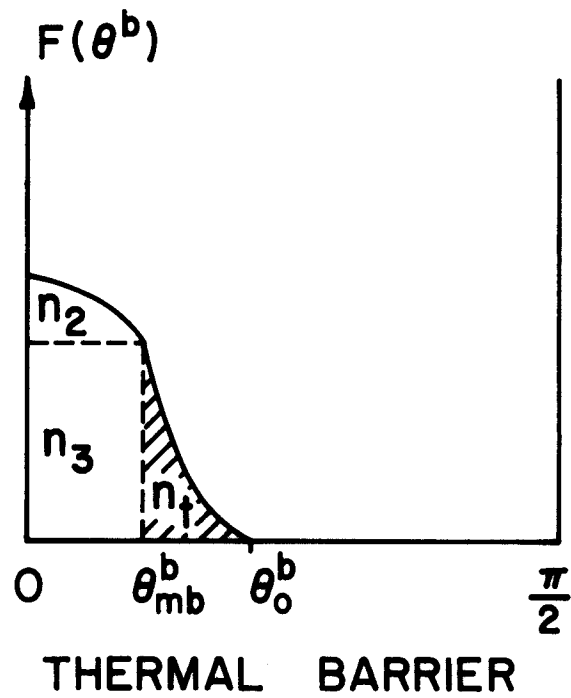
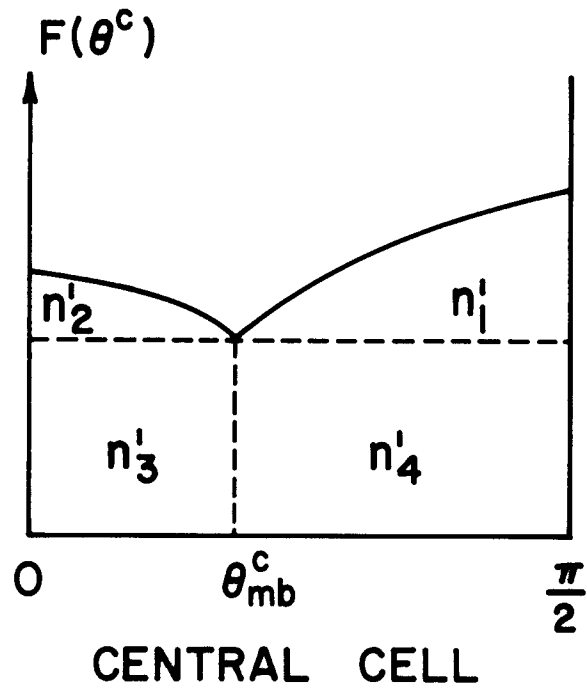


Fig. 3. Ion distribution function in the thermal barrier and central cell.

central cell,  $D_{\perp}^C(u_c)$  is the transverse diffusion coefficient in velocity space, which is proportional to the density  $n_c$ , and  $T_c$  is the central cell ion temperature.

Next we define a current  $J_L$  which is the end loss current that would result if there is no end-plug. In this case the end loss rate is determined by scattering from trapped in the central cell to passing, at which point the ions are immediately lost. Thus:

$$J_L = n_c^2 H_3 \quad . \quad (3.5)$$

We use  $J_L$  as a normalization current and write the barrier pumpout current,  $J_t$ , as:

$$J_t = J_L / I_r \quad . \quad (3.6)$$

Equation (3.6) defines  $I_r$ , which is a figure of merit for the barrier.

Clearly, we want  $I_r$  large. Since

$$J_t = J_1 + J_2 \quad (3.7)$$

then

$$J_1 = \frac{\cos \theta_{mb}^C}{I_r} J_L \quad (3.8)$$

$$J_2 = \frac{1 - \cos \theta_{mb}^C}{I_r} J_L \quad .$$

Equation (3.8) assumes that the central cell ion source is isotropic in velocity space. Combining Eqs. (3.8), (3.5), (3.1), and (3.2), we can express  $n_1^1$  and  $n_2^1$  in terms of  $n_c$ :

$$n_1^1 = \frac{\cos \theta_{mb}^c}{I_r} n_c \quad (3.9)$$

$$n_2^1 = \frac{1 - \cos \theta_{mb}^c}{I_r} n_c \left( \frac{H_3}{H_4} \right) \quad (3.10)$$

The total central cell density,  $n_c$ , however, satisfies

$$n_c = n_1^1 + n_2^1 + n_3^1 + n_4^1 \quad (3.11)$$

where  $n_3^1$  and  $n_4^1$  are isotropic parts of the ion density in the central cell.

The density  $n_3^1 + n_4^1$  can be written as:

$$n_3^1 + n_4^1 = 4\pi \int_0^\infty \sqrt{2E} c_t \ln \left[ \frac{\text{tg} (\theta_0^b/2)}{\text{tg} (\theta_{mb}^b/2)} \right] e^{-E/T_c} dE \quad (3.12)$$

and the barrier pumpout current can be written as:

$$J_t = 4\pi c_t \int_0^\infty \frac{D_\perp^b (u_b) e^{-E/T_c} du_b V_b}{\sqrt{-\frac{2e}{m} \phi_b}} \quad (3.13)$$

where  $\phi_b$  is the barrier potential relative to the central cell,  $V_b$  is the volume of the barrier, and  $e/m$  is the ratio of charge to mass of ion. Using Eqs. (3.13), (3.5), and (3.6), we can write Eq. (3.12) as:

$$n_3^1 + n_4^1 = \frac{\int_0^\infty dE \sqrt{2E} \ln \left[ \frac{\text{tg} \theta_0^b/2}{\text{tg} \theta_{mb}^b/2} \right] e^{-E/T_c}}{I_r V_b \int_0^\infty \frac{(D_\perp^b (u_b))/n_c}{\sqrt{-(2e/m) \phi_b}} e^{-E/T_c} du_b} n_c H_3 \quad (3.14)$$

An expression for the figure of merit,  $I_r$ , can now be obtained. We substitute Eq. (3.14) for  $n_3^1 + n_4^1$ , Eq. (3.9) for  $n_1^1$ , and Eq. (3.10) for  $n_2^1$  into Eq. (3.11), and solve for  $I_r$ :

$$I_r = \cos \theta_{mb}^c + (1 - \cos \theta_{mb}^c) \frac{H_3}{H_4} + H^1 \quad (3.15)$$

where

$$H^1 = \sqrt{\frac{2}{\pi}} \frac{V_c}{V_b} \frac{\cos \theta_{mb}^c}{T_c^{3/2}} \frac{\int_0^\infty dE \sqrt{2E} \ln \left[ \frac{\text{tg}(\theta_0^b/2)}{\text{tg}(\theta_{mb}^b/2)} \right] e^{-E/T_c}}{[\ln(\text{ctn}(\theta_{mb}^c/2) - \cos \theta_{mb}^c)]} \quad (3.16)$$

$$\times \frac{\int_0^\infty D_\perp^c(v_c) e^{-E/T_c} dv_c}{\int \frac{D_\perp^b(v_b) e^{-E/T_c} dv_b}{\sqrt{-(2e/m)\phi_b}}} \cdot$$

Now we make some approximations in order to simplify Eq. (3.16). The collisional scattering from passing to trapped orbits in the barrier comes primarily from the region near the magnetic field peak  $B_{mb}$  where  $\phi_b$  is near zero. This is because the density and therefore the collision rate is larger there than near the barrier minimum. Thus in Eq. (3.13) and in Eq. (3.16) we approximate  $D_\perp^b$  by  $D_\perp^c$  and replace the lower limit on the integration by zero. A result of this approximation is that  $I_r$  is relatively independent of  $\phi_b$ , which is in reasonable agreement with the Fokker-Planck calculations<sup>(7)</sup> of the barrier. The only  $\phi_b$  dependence left in  $I_r$  is contained in the  $\ln [\text{tg}(\theta_0^b/2)/\text{tg}(\theta_{mb}^b/2)]$  term.

The barrier potential  $\phi_b$  is an input to the above analysis. We calculate  $\phi_b$  in the usual manner<sup>(1)</sup> by assuming quasi-neutrality and a Maxwell-Boltzmann relation for the electrons. In our analysis, the ion distribution function in



the thermal barrier is modified; this has some effect on  $\phi_b$ . Thus we have

$$n_b = \int F_b d^3 v_b$$

and

$$n_b = n_c e^{e\phi_b/T_c}$$

where we have taken  $T_{ic} = T_{ec} = T_c$ , for simplicity. The result is

$$\phi_b = T_c \ln \left\{ \frac{V_c}{T_r V_b} \sqrt{\frac{2}{\pi}} \frac{1}{T_c^{3/2}} \frac{\cos \theta_{mb}^c}{[\ln(\operatorname{ctn} \frac{\theta_{mb}^c}{2}) - \cos \theta_{mb}^c]} \right. \quad (3.17)$$

$$\left. \times \left[ \int_0^\infty dE \sqrt{2(E - e\phi_b)} \ln \left[ \frac{\cos^2 \left( \frac{\theta_{mb}^b}{2} \right)}{\cos^2 \left( \frac{\theta_0^b}{2} \right)} \right] e^{-E/T_c} + \frac{V_b}{V_c} K \right] \right\} ,$$

where

$$K = \int_0^\infty dE e^{-E/T_c} \sqrt{B_b} \{ (-2) \sqrt{b-x} \ln(\sqrt{c} + \sqrt{c-x}) \quad (3.18)$$

$$+ 2 \{ \sqrt{b-x} - \sqrt{c} \ln [2(\sqrt{b-x} + \sqrt{c-x})] \}$$

$$- \sqrt{b} \ln \left\{ 2 \left[ \frac{\sqrt{b} [b - 2\sqrt{c}(\sqrt{c} + \sqrt{c-x}) + (\sqrt{c} + \sqrt{c-x})^2]}{\sqrt{c} + \sqrt{c-x}} \right] \right\}$$

$$+ \left. \left[ \frac{b}{\sqrt{c} + \sqrt{c-x}} - \sqrt{c} \right] \right\} + 2\sqrt{b-x} \ln(\sqrt{c} + \sqrt{c - \mu_{mb}}) \Big|_{x=0}^{x=\mu_{mb}} .$$

In these expressions,

$$\begin{aligned}
 b &= \frac{E - e\phi_b}{B_b} , \\
 c &= \frac{E - e\phi_c}{B_c} , \\
 \mu_{mb} &= \frac{E - e\phi_{mb}}{B_{mb}} ,
 \end{aligned} \tag{3.19}$$

where  $\phi_c$  and  $B_c$  are the electrostatic potential and magnetic field in the central cell.

The barrier parameter  $g_b$  is often defined as the ratio of the total density in the barrier to the passing ion density.<sup>(1)</sup> With our model this can be calculated. The result is:

$$g_b = 1 + \frac{\int_0^\infty \sqrt{2(E - e\phi_b)} dE [\cos \theta_{mb}^b \ln \left( \frac{\text{tg}(\theta_0^b/2)}{\text{tg}(\theta_{mb}^b/2)} \right) - \ln \left( \frac{\sin \theta_0^b}{\sin \theta_{mb}^b} \right)] e^{-E/T_c}}{\int_0^\infty (1 - \cos \theta_{mb}^b) \sqrt{2(E - e\phi_b)} dE \ln [\text{tg}(\theta_0^b/2)/\text{tg}(\theta_{mb}^b/2)] e^{-E/T_c} + \frac{V_b}{V_c} K} \tag{3.20}$$

#### IV. Numerical Results

Numerical evaluation of the integrals in Eqs. (3.16), (3.17), (3.18), and (3.20) allows us to calculate values for the figure of merit  $I_r$ , barrier potential  $\phi_b$ , and barrier parameter  $g_b$ . Table 1 shows some results for a barrier mirror ratio of 10 for various assumptions about the magnetic field  $B_0$  and potential  $\phi_0$  at which the trapped particles are pumped. Of course, one cannot choose  $B_0$  and  $\phi_0$  separately; they are related through the electrostatic

Table 1

Parameters  $I_r$ ,  $g_b$ , and  $\phi_b$  for  $B_{mb}/B_b = 10$ ,  $V_c/V_b = 5$  and various values of  $\phi_0$  and  $B_0$

$B_0/B_b$	$\phi_0$ (units of $T_c$ )	$I_r$	$g_b$	$\phi_b$ (units of $T_c$ )
9.9	- .1	1.52	1.21	-7.35
	$\phi_b/2$	5.0	2.20	-3.70
	$\phi_b + .1$	6.68	2.79	-3.00
5.5	- .1	3.87	1.48	-4.5
	$\phi_b/2$	6.90	2.60	-3.0
	$\phi_b + .1$	8.60	3.50	-2.5
1.1	-0.1	11.4	4.36	-1.55
	$\phi_b/2$	13.4	6.32	-0.9
	$\phi_b + .1$	14.6	7.33	-0.6

potential profile in the barrier. This, however, is outside the scope of this paper, but is treated in a separate paper.<sup>(8)</sup> In this work, we merely make some choices for  $\phi_0$  to determine the sensitivity of the results to  $\phi_0$ .

Inspection of Table 1 shows some interesting results. As one changes the "pumpout point"  $(B_0, \phi_0)$  from near the barrier peak  $(b_{mb}, \phi_{mb})$  to the barrier minimum  $(B_b, \phi_b)$ , the factor  $I_r$  increases, which implies a decreasing pumping current in the barrier, and the pumpout parameter  $g_b$  also increases. This suggests the gradient of the trapped ion distribution function in the barrier is becoming smaller, thereby reducing the diffusion in velocity space. A smaller gradient implies that the distribution function extends over a broader range in pitch angle for the barrier trapped ions. This effect increases the trapped ion density, which is indicated by the increasing  $g_b$ .

The barrier pumping current obtained here is greater than that obtained by Rognlien<sup>(5)</sup> from a numerical solution of the Fokker-Planck equation. The difference can be ascribed to differences in how the source was treated. In Rognlien's work the ion source was in the middle of the passing particle part of phase space in the barrier ( $v_{\perp b} = 0$ ). In our work, the source is isotropic in the central cell. The diffusion is towards the pitch angle boundary  $\theta_{mb}^C$  as shown in Fig. 2. The central cell trapped ions have to diffuse all the way to  $\theta_{mb}^C$  before they can enter the barrier; at this point they appear in the barrier at the boundary between passing and trapped ions. Consequently, they get trapped in the barrier much more easily in this model than in Rognlien's model.

For a good thermal barrier tandem mirror, one wants the actual end loss current much less than the "non-plugged" end loss current (perhaps by a factor of 50 or so) and the barrier pumping current much less than the end loss current. This says one wants  $I_r > 50$ , which is much above the values given in

Table 1. At the same time, one wants  $g_b$  small ( $\sim 2$ ) in order to get a good  $\phi_b$  and have an effective barrier. These goals can be obtained in principle, but at a much larger central cell to barrier volume ratio than the value of 5 used in Table 1.

#### V. Summary

A model for the ion distribution function in the central cell and barrier has been presented. This model assumes the distribution function is collisional with boundary conditions determined by the requirement of a particle current balance. The effect of pumping in the barrier is treated somewhat phenomenologically by the requirement that the trapped ion distribution function vanish at a pitch angle for trapped ions. This model allows the barrier pumpout current and trapped ion density to be calculated. Numerical results are given for a particular example. The results show trends which are in agreement with simple physical intuition, but the model requires more refinement before one can have some confidence in the quantitative estimates.

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References

1. D. E. Baldwin and B. G. Logan, Phys. Rev. Lett. 43 (1979), 1318.
2. J. Kesner, Comments on Plasma Physics and Controlled Fusion 5 (1979), 123.
3. D. E. Baldwin, B. G. Logan, T. C. Simonen, editors, "Physics Basis for MFTF-B", Lawrence Livermore Laboratory Report UCID-18496 IV-94 (1980).
4. Y. Matsuda, D.E. Baldwin, Bull. APS 24 (1979), 1060.
5. T. D. Rognlien in "Physics Basis for MFTF-B", D. E. Baldwin, B. G. Logan, T. C. Simonen, editors, Lawrence Livermore Laboratory Report UCID-18496, IV-177 (1980).
6. D. V. Sivukhin, "Reviews of Plasma Physics", edited by Acad. M. A. Leontovich, Vol. 4, (1966) 93.
7. L. LoDestro, "Steady-State Ion Distributions in a Potential and Magnetic Well", Mirror Theory Monthly, Lawrence Livermore Laboratory (May 15, 1980).
8. X. Z. Li and G. A. Emmert, "Potential Profile in a Tandem Mirror Thermal Barrier", UWFDM-392, University of Wisconsin (1980).