Effective Time and Resonance Width in Cyclotron Resonance Heating

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May 1980

UWFDM-346

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ABSTRACT

The concepts of effective time and interaction time are examined in a unified context that includes electron cyclotron resonance heating as well as fast and slow wave ion cyclotron resonance heating. Here effective time t_e is defined by $\Delta v_{\perp} = (eE_{+}/m)t_e$, while interaction time t_i is the period over which the phase between the particle and the wave is slowly changing. Elliptically polarized travelling waves are assumed obliquely incident upon an arbitrary magnetic mirror field. Arbitrary harmonics and Doppler shift are included in the analysis.

It is shown that, for ECRH and slow wave ICRH, $t_e \simeq 0.7\ t_i$ for particular resonance locations, but both t_e and t_i become ill-defined in between, owing to overlapping resonances. A new expression is derived relating t_e and t_i for fast wave ICRH. A cubic equation, valid for $t_i \ll t_b$, the bounce time, is derived for a general magnetic well and solved numerically for a parabolic well. A simple condition for resonance overlap is given by the vanishing of the discriminant of the cubic. The results are applied to ECRH with $k_{\parallel} \to 0$ and to fast wave midplane heating, for which $k_{\parallel} v_{\parallel} \ll \omega$. In general, resonances are found to be much broader in typical ICRH experiments than for ECRH.

1. INTRODUCTION

The notion of effective time plays a central role in single particle theories of electron and ion cyclotron resonance heating (SPROTT and EDMONDS, 1971; JAEGER et al, 1972; STIX, 1975). This time (t_e) is defined in such a way that the maximum perpendicular velocity gained or lost by an ion or electron passing through resonance is $\Delta v_\perp = (eE_\pm/m)t_e$, where $E_\pm$ is the left or right hand circularly polarized component of $\vec{E}$. GUEST (1968) and CANOBBIO (1969) defined a "correlation" or "transit time" such that $E_\pm$ and $v_\perp$ undergo a relative phase slip of ±90°; we shall employ the more descriptive terms "interaction time" or "resonance width" (t_i). It has often been assumed, or shown indirectly in special cases (GRAWE, 1969) that t_e and t_i are roughly equal. However, it is not a priori obvious why these two concepts should be equivalent.

In this paper we unify previous derivations of t_e by means of a rigorous asymptotic expansion of a phase integral common to most cases of interest. An elliptically polarized travelling wave is assumed obliquely incident upon an arbitrary magnetic well possessing a symmetry plane; arbitrary harmonics and Dopper shift are allowed. This treatment encompasses the fast magnetosonic and slow Alfvén waves, as well as nonrelativistic electron cyclotron waves. By expanding the phase about the resonance point, t_i is obtained as the solution of a cubic equation valid when $t_i \ll t_b$. We find that $t_e \simeq 0.7t_i$ for midplane, turning point resonance and a region in between where the phase is locally quadratic in time, but that both t_e and t_i become ill-defined for intermediate points where resonances overlap. Midplane and turning point resonances are described by the same formula, owing to the fact that the particle sees a slowly changing B-field in each case. For slow wave heating t_e must include a Doppler term, while for the fast wave t_e

acquires a Bessel function $J_1(k_\perp \rho)$ and is no longer equivalent to t_i.

The behavior of t_e in the nebulous overlap regions is quite complicated and has been reported elsewhere (HOWARD and KESNER, 1979). Fortunately, it is possible to assign a meaning to t_i in these regions with the aid of an overlap condition, which is simply that the discriminant of the cubic vanish. In the next section this condition is exhibited for a parabolic well in the limit of zero Doppler shift and shown to be itself a cubic. The original cubic is solved numerically for a parabolic well in order to bridge the gap between the analytically tractable limits.

One motivation for this study is to determine the true resonance width in practical RF heating situations in order to see whether or not an impulsive heating model is justified. The numerical results for $k_{//} v_{//} << \omega$ show that under typical ECRH conditions ($\omega_{ce} = 200\omega_b$), midplane heating is well localized, but that off-midplane heating is likely to be more evenly distributed unless a rather extreme clustering of turning points occurs at the resonance position. In general, resonance widths are substantially larger in ICRH, where typically $\omega_{ci} = 20\ \omega_b$. Since the resonance width is roughly proportional to $v_\perp$, it is not unusual to reach a point where heating occurs evenly throughout most orbits. Under these conditions the basic idea of a resonance is lost and heating falls off dramatically. A concomitant mathematical snare is the failure of asymptotic expansions based on localized resonances. Again, this is much more likely to be a problem in ICRH than in ECRH.

Another important consequence of a broad resonance width is the heating of non-resonant particles.

Our numerical calculations of power absorption (HOWARD and KESNER, 1979) show this to be a large effect in ICRH. A related relativistic resonance broadening in ECRH has been studied by GRAWE (1969). Finally, we apply the parabolic well results to fast and slow wave heating in Phaedrus, the University of Wisconsin tandem mirror experiment (SMITH et al., 1979). For fast wave midplane heating, the small but nonnegligible Doppler shift has a modest broadening effect on t_i, caused by the partially merged midplane resonances. A simple overlap criterion is derived for this case and shown to be independent of $v_\perp$. A curious phenomenon that arises only in midplane heating is the occurrence of virtual resonances for small Doppler shifts. The heating effect of these "ghost" resonances is very sensitive to the separation of the real resonances, so that the power absorption falls off very rapidly with increasing $k_{||}$.

2. THEORY FOR A GENERAL MAGNETIC WELL

A. Effective Time

We consider an elliptically polarized travelling wave

$$\vec{E} = E_{xo}\hat{x} \cos \xi + E_{yo}\hat{y} \sin \xi, \tag{1}$$

where

$$\xi = k_z v_z + k_x v_x - \omega t, \tag{2}$$

obliquely incident on a static magnetic field $\vec{B} = B(z)\hat{z}$. Here we have arbitrarily taken $k_y = 0$ so that $k_z = k_{||}$ and $k_x = k_\perp$, and E_z has been neglected compared to E_x and E_y. Figure 1 shows the location of the four cyclotron resonances in a magnetic well. Solving the nonrelativistic equations of motion for an ion moving in the combined fields including the RF magnetic field of the wave, it is straightforward to show (HOWARD and KESNER, 1979) that the amplitude of the velocity kick is

$$\Delta v_{\perp} = \frac{eE+}{m} J_{\ell-1} (k_{\perp}\rho) |G|, \tag{3}$$

where ℓ is the harmonic number, ρ is the midplane gyroradius, and $E_+ = \frac{1}{2}(E_{xo} + E_{yo})$ is the left hand circularly polarized component of $\vec{E}$. Here we have defined a "gain integral"

$$G = \int_{-t_{b/2}}^{t_{b/2}} (1 - k_z v_z/\omega) e^{i\phi} \, dt, \tag{4}$$

where

$$\phi(t) = \ell \int_{0}^{t} \Omega(t')dt' + k_z z - \omega t \tag{5}$$

is the phase slip between E_+ and $v_{\perp}$, and $\Omega(t)$ is the local cyclotron frequency.

JAEGER et al. (1972) define an effective time such that

$$\Delta v_{\perp} = \frac{eE_+}{m} t_e , \tag{6}$$

so that

$$t_e = J_{\ell-1} (k_{\perp}\rho) |G| . \tag{7}$$

For an isolated resonance somewhere along a trapped orbit where $\dot{\phi}(\tilde{t}) = 0$, the G-integral may be evaluated asymptotically by the method of steepest descents;

$$G \sim (1 - k_z \tilde{v}_z/\omega) \sqrt{\frac{2\pi}{\ddot{\phi}}} \exp i(\tilde{\phi} \pm \frac{\pi}{4}), \tag{8}$$

where $\ddot{\phi}$ is evaluated at $\tilde{t}$. This formula is valid for $\dddot{\phi} << (\ddot{\phi})^{3/2}$, which is usually satisfied for a resonance point about midway between the midplane and a turning point. For this case (7) and (8) give

$$t_e = (1 - k_z \tilde{v}_z/\omega) \, J_{\ell-1} \, (k_\perp \rho) \sqrt{\frac{2\pi}{\ddot{\phi}}} \, . \tag{9}$$

In fast wave ICRH and slightly nonrelativistic ECRH, the Doppler shift is so small that pairs of merged resonances occur for midplane or turning point heating. Since

$$\ddot{\phi} = \ell \Omega' v_z + k_z \dot{v}_z$$

$$\dddot{\phi} = \ell(\Omega'' v_z^2 + \Omega' \dot{v}_z) + k_z \ddot{v}_z, \tag{10}$$

where primes denote z-derivatives, we see that $\dddot{\phi} >> (\ddot{\phi})^{3/2}$ near these extremes. The standard procedure when $\ddot{\phi}$ is small is to retain the cubic term in the expansion of ϕ,

$$\phi \simeq \tilde{\phi} + \frac{1}{2}\tilde{\ddot{\phi}}t^2 + \frac{1}{6}\tilde{\dddot{\phi}}t^3, \tag{11}$$

where we have taken $\tilde{t} = 0$. Using this series in (4) yields, for $k_z v_z \ll \omega$,

$$|G| \sim 2\pi\ (1 - k_z\tilde{v}_z/\omega)\left(\frac{2}{\dddot{\phi}}\right)^{1/3}\ Ai(-x), \tag{12}$$

where Ai(-x) is the Airy function and

$$x = 2^{-2/3}\left[\ddot{\phi}^{3/2}\Big/\dddot{\phi}\right]^{4/3}. \tag{13}$$

Note that (12) is still an asymptotic expansion, and applies equally well to midplane and turning point resonance. When $\ddot{\phi} = 0$ (completely merged resonances), (12) simplifies to

$$|G| \sim 2\pi Ai(0) J_{\ell-1}(k_\perp \rho)\left(\frac{2}{\dddot{\phi}}\right)^{1/3}, \tag{14}$$

so that

$$t_e' = 1.33\ J_{\ell-1}(k_\perp \rho)\left(\frac{3\pi}{\dddot{\phi}}\right)^{1/3}, \tag{15}$$

where the prime serves to distinguish t_e' from (9). Although one may retrieve (8) from (12) by letting $x \to \infty$, the reader is cautioned that

(12) is not valid for large finite x. There does not seem to be a uniform expansion that smoothly bridges the gap between (8) and (12). Rigorous asymptotic expansions have been derived by HOWARD and KESNER (1979) for most cases of interest.

B. Interaction Time

Following GUEST (1968) we define an interaction time during which the wave-particle phase is slowly changing via

$$\Delta\phi = |\phi(t_i) - \tilde{\phi}| = \frac{\pi}{2} . \qquad (16)$$

Defined in this way, t_i is a physical quantity, whereas t_e as defined by (6) is a mathematical definition. In solving this equation, we naturally choose the two closest roots $t_i(1)$ and $t_i(2)$ bracketing $\tilde{t} = 0$ and usually find $|t_i(2)| \approx |t_i(1)|$. The interval $2t_i$ is then a rough measure of the full width of the resonance. A similar relation has been studied by CANOBBIO (1969) for the special case of a linear magnetic field. The relationship of t_i to t_e is not obvious, although they are often identified in the literature. We now determine the conditions under which t_e and t_i are interchangeable. When the motion is known, as in a parabolic well, (16) may be solved numerically. However, when the resonance is "narrow" ($t_i \ll t_b$) a useful approximate equation may be derived with the help of the series (11), which gives

$$t_i^3 + \frac{3\ddot{\phi}}{\dddot{\phi}} t_i^2 = \pm \frac{3\pi}{\dddot{\phi}} ; \qquad (17)$$

This cubic is easily solved in the limiting cases corresponding to t_e and t_e'. For off-midplane heating, where $\dddot{\phi} \to 0$,

$$t_i = \sqrt{\frac{\pi}{\ddot{\phi}}} \qquad (18)$$

and for midplane or turning point heating,

$$t_i' = \left(\frac{3\pi}{\dddot{\phi}}\right)^{1/3} . \qquad (19)$$

Comparing t_e with the total interaction time $2t_i$ then gives

$$t_e = \tfrac{1}{2}\sqrt{2} \quad (1 - k_z\tilde{v}_z/\omega)\ J_{\ell-1}(k_\perp\rho)\ (2t_i) \qquad (20)$$

$$t_e' = 0.67\ (1 - k_z\tilde{v}_z/\omega)\ J_{\ell-1}(k_\perp\rho)\ (2t_i') . \qquad (21)$$

For ECRH, $k_z\tilde{v}_z/\omega = \tilde{v}_z/c << 1$, $\ell = 1$, and $k_\perp\rho<<1$ so that $t_e = 0.7(2t_i)$ and $t_e' = 0.7(2t_i)$. Thus, $t_e \simeq 2t_i$ for ECRH in the three limiting cases of (17). For slow wave heating ($\ell = 1$), $k_\perp\rho \simeq 0.5$, so that $J_o(k_\perp\rho) \simeq 1$, but the large Doppler shift can reduce t_e by 50%. Thus, $t_e \simeq 0.35\ (2t_i)$ for slow wave heating. Although hot plasma effects reduce the spatial variation of k_z, the strong falloff of E_+ at the magnetic beach severely limits the value of single particle calculations. For fast wave heating (usually with $\ell = 2$), taking $J_1(k_\perp\rho) \simeq 0.5\ k_\perp\rho$ and ignoring the small Doppler term gives $t_e = 0.35\ k_\perp\rho\ (2t_i)$, so that one can have $t_e<<2t_i$ in this case.

Thus far, in calculating t_i we have examined only resonances near the extrema of an orbit and a (possibly narrow) region about midway where $\ddot{\phi} \to 0$. What happens at the intermediate points where $\dddot{\phi} \simeq \ddot{\phi}^{3/2}$? For a fixed off-midplane resonance position and a given midplane distribution function, which scaling law do most particles obey? In principle, one has only to solve the cubic (17) or at worst the transcendental equation (16). However, solution of the cubic is complicated by coalescing resonance pairs as $\tilde{t}$ approaches the midplane or a turning point. The simplest case to handle seems to be true midplane heating with $k_z v_z/\omega \ll 1$ as in ECRH, for which (19) may safely be used. For fast wave midplane heating, however, $k_z v_z/\omega \simeq 0.025$, which splits the resonance and broadens the interaction time. This case will be examined quantitatively in the following section.

We now consider off-midplane heating, which is the way ECRH or ICRH is normally done. While some particles will always penetrate the resonance zone, a number will turn near the resonance point, this fraction increasing as heating increases $v_\perp$. If $t_i \ll t_b$, the transition from off-midplane to turning point resonance is described by the cubic (17). The number of real roots is governed by the sign of the discriminant, which is proportional to

$$D = 3\pi - 4\sigma \frac{\dddot{\phi}^3}{\ddot{\phi}^2} , \qquad (22)$$

where σ is the sign of the right hand side of (17). When $D > 0$ there exists one real root t_i and when $D < 0$ there exist three real roots, two of these merging as $D \to 0$.

In general, $\ddot{\phi}$ is positive for $0 < t < \frac{1}{4}t_b$ while $\dddot{\phi}$ is positive near the midplane, changing sign about halfway to the turning point, where $\dddot{\phi} < 0$. Thus, for $\sigma = 1$ $(\phi(t_i) > \tilde{\phi})$, a transition occurs from two to three roots to one real root as one proceeds from the midway point (where $\dddot{\phi} = 0$ and the cubic becomes a quadratic) toward the turning point.

A single lower root exists for $\sigma = -1$, except in the vicinity of the midway point. Figure 2 illustrates the metamorphosis of $\phi(t)$ for $k_z = 0$ as the resonance point moves from the midplane to a turning point. Clearly, t_i is difficult to define in the regions (b) and (d) where two roots merge. At this point the minimum of a peak roughly coincides with the bottom of its neighboring valley and we take this to be our resonance overlap condition; two resonances overlap when two roots of (17) merge, i.e. D = 0. From (22) we then obtain the general overlap condition,

$$(\ddot{\phi})^{3/2} = \tfrac{1}{2}\sqrt{3\pi}\ \dddot{\phi}. \qquad (23)$$

At this point the total resonance width is about $3t_i$. In order to apply (23) and examine the variation of t_i one must know the particle dynamics; this is done in the next section for the important case of a parabolic well. We shall also calculate the splitting of midplane resonances due to a small Doppler shift and develop an appropriate overlap condition.

A corresponding ambiguity occurs in the meaning of t_e when resonances overlap. For (9) describes the effect of a single resonance, while (15) gives the velocity kick due to two partially merged resonances. It is well known that two neighboring resonances can interfere, so that the net kick is less than the sum of the individual kicks. However, even a small degree of collisionality can destroy phase information (MOMOTA and TAKIZUKA, 1974). If collisions are sufficiently infrequent, the single resonance formula (9) should be modified to include beating among all four resonances occuring in one bounce time. This has been carried out by HOWARD and KESNER (1979) and used to calculate power absorption for

ICRH in a parabolic well. In the limit of very low collisionality and high particle energy, such phase correlations can lead to the ordered motion known as superadiabaticity (LICHTENBERG and LIEBERMAN, 1973). The present work is relevant to superadiabaticity theory, inasmuch as the resonances must be well localized to justify an impulsive heating model.

3. RESONANCE WIDTH IN A PARABOLIC WELL

In the previous section we saw that the concepts of effective time and interaction time become nebulous when resonances partially overlap. The rather complicated behavior of t_e has been studied for cyclotron heating in a magnetic mirror using asymptotic methods, and will be reported elsewhere. In this section we clarify the meaning of t_i by solving the cubic (17) for a parabolic well.

Consider adiabatic particle motion in the parabolic field,

$$B = B_o\,(1 + z^2/L^2)\;. \tag{24}$$

The guiding center solution is then

$$z = Z \sin \theta, \tag{25}$$

with amplitude

$$Z = v_{zo}\,L/v_{\perp o} \tag{26}$$

and orbital angle $\theta = \Omega_b t$, where $\Omega_b = v_{\perp o}/L$ is the total bounce frequency.

The phase slip (5) may then be conveniently written

$$\phi = \lambda f(\theta), \tag{27}$$

where

$$f(\theta) = \theta \cos 2\theta_1 - \tfrac{1}{2}\sin 2\theta - 2\beta \sin \theta. \tag{28}$$

Here θ_1 is the orbital resonance angle, given by $z_1 = Z\sin\theta_1$ and we have introduced the useful dimensionless parameters.

$$\lambda = \frac{\ell\Omega_o}{2\Omega_b}\left(\frac{v_{zo}}{v_{\perp o}}\right)^2 \tag{29}$$

and

$$\beta = \frac{k_z v_{zo}}{\Omega_o}\left(\frac{v_{\perp o}}{v_{zo}}\right)^2. \tag{30}$$

Physically, β locates the Doppler-shifted resonance, while $2\pi\lambda$ is the phase slip between E_+ and $v_\perp$ in one complete orbit ($\theta \to \theta + 2\pi$). The required derivatives of (28) are

$$f'' = 2 \quad \sin 2\theta + 2\beta \sin \theta \tag{31}$$

$$f''' = 4 \cos 2\theta + 2\beta \cos \theta,$$

which are to be evaluated at the resonance angle $\tilde{\theta}$ found by setting $f'(\tilde{\theta}) = 0$. This gives

$$\cos\tilde{\theta} = \tfrac{1}{2}\left[-\beta \pm (\beta^2+4\cos^2\theta_1)^{\frac{1}{2}}\right]. \qquad (32)$$

Defining the resonance half-width $\Delta\theta = \theta_i - \tilde{\theta} = \Omega_b t_i$, (17) takes the form

$$(\Delta\theta)^3 + \frac{3f''}{f'''}(\Delta\theta)^2 = \frac{3\pi\sigma}{\lambda f'''}, \qquad (33)$$

where $\sigma = \pm 1$. It is then perfectly straightforward to solve (33) for given v_{zo}, $v_{\perp o}$ and θ_1. Note that $\Delta\theta \sim \lambda^{-1/2}$ for off-midplane heating while $\Delta\theta \sim \lambda^{-1/3}$ for midplane or turning point heating.

A. ECRH

It is instructive to solve (33) for $\beta = 0$, corresponding to the conditions of ECRH. Using (31) yields

$$(\Delta\theta)^3 + \frac{3}{2}\tan 2\theta_1\,(\Delta\theta)^2 = \frac{3\pi\sigma}{4\lambda}\sec 2\theta_1. \qquad (34)$$

The discriminant is proportional to

$$D(\lambda,\theta_1) = 3\pi - 2\sigma\lambda\sin 2\theta_1\tan^2 2\theta_1. \qquad (35)$$

The resonance overlap condition is $D = 0$, or

$$\sin^3 2\theta_1 = \frac{3\pi\sigma}{2\lambda}\cos^2 2\theta_1, \qquad (36)$$

itself a cubic in $\sin 2\theta_1$! The discriminant of (36) vanishes when

$\lambda = \pi/\sqrt{3}$, for which the only real values of θ_1 are 30° and 60°. Thus, (36) merely gives a simple particular solution of (35); the resonances at $\theta_1 = 30°$ and 60° overlap when $\lambda \gtrsim \pi/\sqrt{3}$. When any two resonances just overlap, the total orbital angle over which ϕ is slowly changing increases by 50% to $3\Delta\theta$. For λ greater than the critical value, the total width is $2(\Delta\theta + \theta_1)$ for $\theta_1 \leq 45°$ and $2(\Delta\theta + 90° - \theta_1)$ when $\theta_1 \geq 45°$. From (31) we now see that $f''' = 0$ when $\theta_1 = 45°$, the "midway point" where t_i is given by (18).

To assay the changing resonance width as θ_1 varies from 0 to 90°, we must devise a scheme to choose from among the six roots of (34). Examination of Fig. 2 shows that the relevant root to track is the upper one ($\sigma = +1$) to the right of θ_1 when $\theta_1 \leq 45°$ and to the left of θ_1 for $\theta_1 \geq 45°$. This root was then used to initialize an iterative solution of the exact equation (16), which agrees within 5° of the cubic for $\lambda > 3$, but greatly exceeds it for smaller λ. The resulting half-widths are plotted in Fig. 3 as functions of λ for various θ_1. Note that by symmetry, the curves for $\theta_1 = 15°$ and 75° and for $\theta_1 = 30°$ and 60° coincide. The most striking feature of these curves is the large gap between the $\theta_1 = 90°$ and 75° curves; the particles must turn very close to the resonance layer in order to follow the $\lambda^{-1/3}$ law. It would seem that initially most particles follow the $\lambda^{-1/2}$ law.

Typical values of λ are 1.5 for slow wave heating, 3 for the fast wave and $\lambda = 16$ for ECRH. Reading from Fig. 3 we find half-widths of 59°, 42° and 9° for a $\theta_1 = 45°$ particle. If particles "stick" at a turning point after heating has progressed the widths would be 66° , 53° and 12°. The complications arising from a small $k_{\parallel}$ will be treated in the next section.

B. Fast Wave Midplane Heating

As a second example of enhanced resonance width due to overlapping resonances, we consider fast wave midplane heating in a parabolic well. Here the presence of a small but non-negligible Doppler shift has two related effects on particle heating. In the absence of a Doppler shift there are two midplane resonances, at $\theta = 0°$ and $180°$, each of which may be regarded as a merged pair of slightly off-midplane resonances. If now a small k_z is "turned on" in the negative z-direction, the resonances at $\theta = 0$ will be moved slightly off-midplane to a higher value of B. More importantly, the $\theta = 180°$ pair will be shifted below B_{min}, i.e. they become virtual. We now derive a criterion which simultaneously measures the degree of overlap of the $\theta = 0°$ resonances and the strength (t_e) of the virtual resonances near $180°$.

In practice B(z) is parabolic near the midplane and the analysis of equation (23) *et seq.* applies. Typical Phaedrus parameters are $\lambda = 3$ and $\beta = 0.15$. Even though $k_z v_z/\omega = 0.025$ might seem negligible, the geometric factor $(v_\perp/v_z)^2 = 6.25$ in (30) amplifies its effect, especially as $v_\perp$ increases. When $\beta = 0$, (33) gives a half-width $\theta_i = 52.9°$ ($56.4°$ exactly), already a very broad resonance. If we now let $\beta = 0.15$, the resonances are shifted to $\pm\tilde{\theta} \approx \sqrt{\beta} = 21.9°$, as depicted in Fig. 4 . Solving (16) numerically we find $\theta_i = 66.6°$, a rather modest increase of 18%. What happens as β is increased further?

From Fig. 4 we see that the resonances will be resolved when

$$|f(\tilde{\theta}) - f(-\tilde{\theta})| \gtrsim \frac{\pi}{2\lambda}\ , \qquad (37)$$

which by symmetry becomes

$$|f(\tilde{\theta})| \gtrsim \frac{\pi}{4\lambda} . \tag{38}$$

For $\beta<<1$ we may use $\tilde{\theta} \approx \sqrt{\beta}$ in (29) with $\theta_1 = 0$ to obtain $f(\tilde{\theta}) \approx -4\beta^{3/2}/3$, so that (38) simplifies to

$$\lambda\beta^{3/2} \gtrsim \frac{3\pi}{16} . \tag{39}$$

Thus, with $\beta = 0.15$ we would need at least $\lambda = 10$ to resolve the resonances. It is more meaningful to fix $\lambda = 3$ and increase β (by increasing $k_{\parallel}$) until (39) is satisfied. This gives $\beta = 0.338$, for which $\tilde{\theta} = 32.31^{\circ}$ and $\theta_i = 71^{\circ}$, an increase of 26% over the $\beta = 0$ value. It is interesting that in a real heating situation with $k_{\parallel}$ fixed, the resolution parameter $\lambda\beta^{3/2}$ is independent of $v_{\perp}$; the resonances do not split as $v_{\perp}$ increases. However, the phase slip λ decreases rapidly with increasing $v_{\perp}$, so that θ_i rapidly approaches 180°. The consequences of this broadening will be discussed in the next section.

An interesting connection exists between the parameter $\delta \equiv \lambda\beta^{3/2}$ and the strength of the virtual resonances. If one calculates the G-integral through the pair of virtual resonances in the complex t-plane and assumes that δ is large enough for them to be resolved, one finds $t_e \sim \exp(-4\delta/3)$ for $\beta<<1$. (The analogous result for the real resonances is $t_e \sim \exp(4i\delta/3)$). Since the real pair was well merged in the above case, we conclude that the virtual pair may not be ignored. The magnitude of their contribution will be discussed in detail elsewhere.

4. DISCUSSION

We have compared the concepts of effective and interaction time under very general conditions and found them to be interchangable for nonrelativistic ECRH but significantly different for ICRH. A simple resonance overlap condition was derived by requiring that the discriminant of a cubic equation vanish. The cubic was then solved numerically for a parabolic well, showing that most particles initially follow the off-midplane scaling law. Resonance widths were found to be much larger in typical ICRH experiments than for ECRH, and to broaden as heating takes place. This extreme broadening has profound consequences for ICRH calculations. For example, the customary impulse model of ECRH does not apply to ion heating when $\lambda \tilde{<} 1$, since energy is then absorbed throughout the orbit. Mathematically this is reflected in the breakdown of all asymptotic formulas for t_e as $\lambda \to 0$. Since $\lambda \sim v_\perp^{-3}$, an initial value of $\lambda = 3$ can easily be reduced to $\lambda < 1$ by doubling $v_\perp$. Physically we have the important corollary that heating becomes insensitive to the spatial location of the resonance, which may lie outside the orbit! This mode of off-resonance heating differs from that discussed by GRAWE (1969), which depended on the relativistic mass increase. Impulse models of superadiabaticity also fail for fast wave heating for the same reasons. However, in this case the resonance overlap method of SMITH et al. (1980) may still be applicable.

Finally, we examined the effect of a small Doppler shift on a typical fast wave midplane heating experiment, and derived a relevant overlap condition. We found that while a small $k_\parallel$ did not greatly increase the interaction time, the negative Doppler shift of one resonance pair to imaginary values of $\tilde{\theta}$ could significantly decrease particle heating, and that this decrease was directly related to the separation of the real resonances.

ACKNOWLEDGEMENT

I would like to thank Professor J.C. Sprott for useful discussions of ECRH experiments. This work was supported by U.S. Department of Energy Contract ET-79-S-02-5125.

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FIGURE CAPTIONS

Fig. 1 - Schematic picture of cyclotron resonances in a magnetic trap. The resonance locations (shown as crosses) are Doppler shifted from $\pm z_1$, where the zero order resonance condition $\omega = \ell\Omega_1$ is met. When $z_1 = 0$ we speak of midplane resonance while $z_1 = Z$ defines turning point resonance.

Fig. 2 - Phase function $f(\theta)$ for $k_{\parallel} = 0$ resonances in a parabolic well with resonances at (a) midplane, (b) $\omega_b\tilde{t} = 30^\circ$ (where resonances merge), (c) $\omega_b\tilde{t} = 45^\circ$ (the midway point where $\dddot{\phi} = 0$), (d) $\omega_b\tilde{t} = 60^\circ$ (merging resonances), (e) turning point. $\Delta\theta = \theta_i - \tilde{\theta}$ is the resonance halfwidth, corresponding to $\lambda(f - \tilde{f}) = \frac{\pi}{2}$.

Fig. 3 - Resonance halfwidth $\Delta\theta$ for $k_{\parallel} = 0$ wave in a parabolic well, plotted versus phase slip λ for various resonance positions. Note that the midplane ($\theta_1 = 0$) and turning point ($\theta_1 = 90^\circ$) resonance curves are identical. Data for $\theta_1 = 15^\circ$ (75°) are plotted as points.

Fig. 4 - Phase function with and without Doppler shift for fast wave midplane heating in a parabolic well. In this case $k_{\parallel}$ is too small to resolve the resonance pair. The nonzero $k_{\parallel}$ slightly increases the already substantial resonance width.

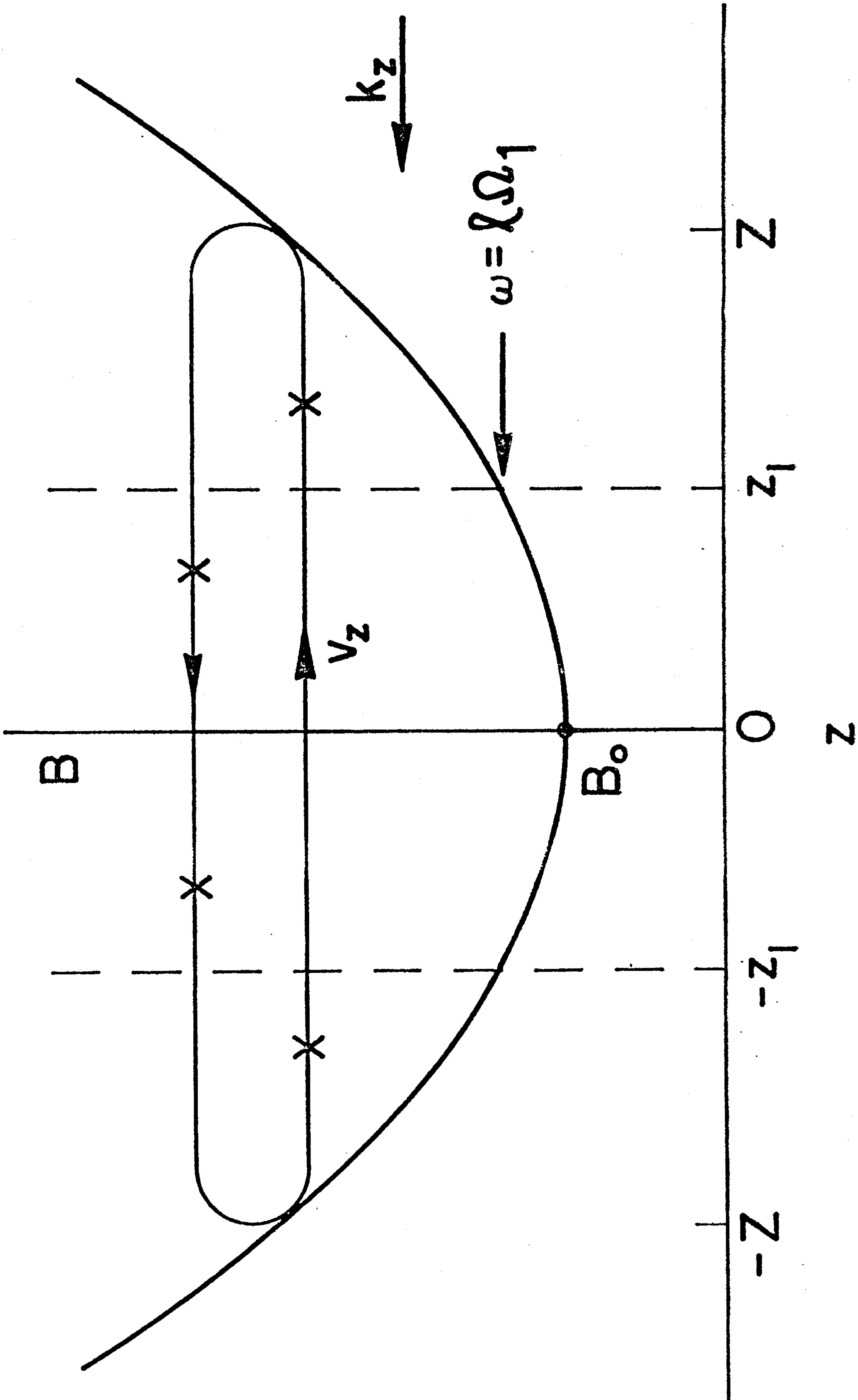
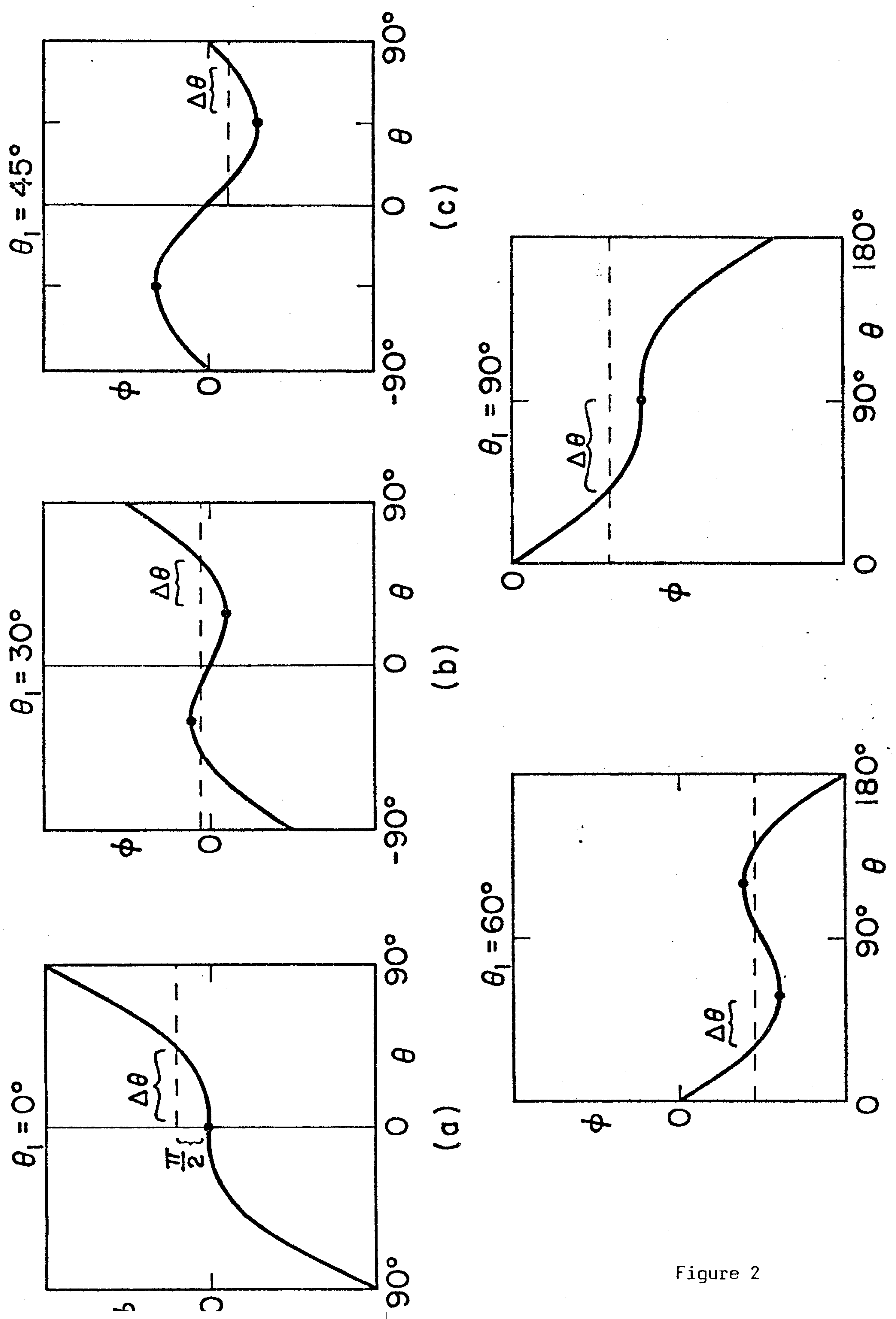
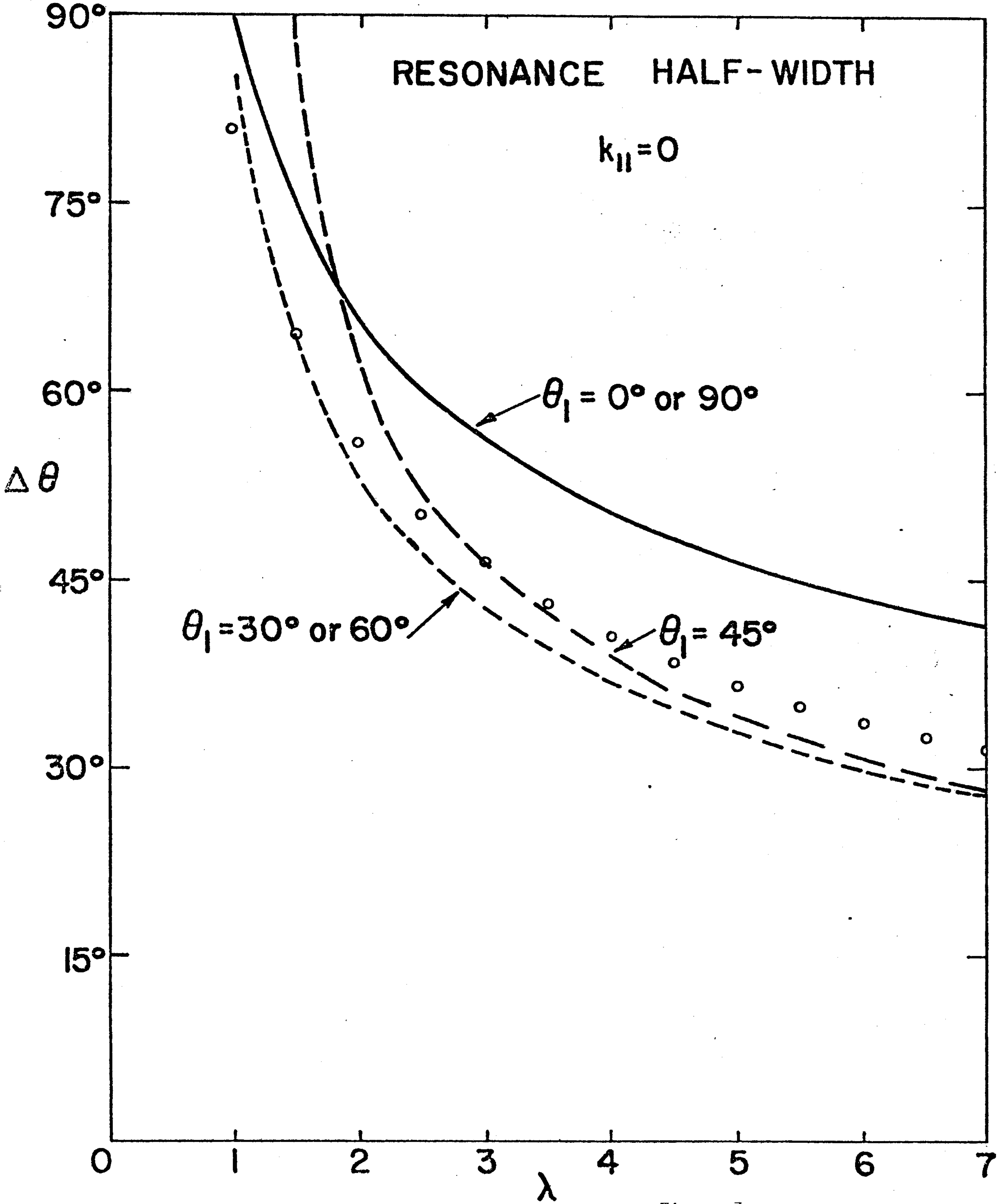


Figure 1



$\theta_1 = 0°$
ϕ
0
-90°
0
θ
90°
$\frac{\pi}{2}$
$\Delta\theta$
(a)
$\theta_1 = 30°$
ϕ
0
-90°
0
θ
90°
$\Delta\theta$
(b)
$\theta_1 = 45°$
ϕ
0
-90°
0
θ
90°
$\Delta\theta$
(c)
$\theta_1 = 60°$
ϕ
0
0
90°
θ
180°
$\Delta\theta$
(d)
$\theta_1 = 90°$
ϕ
0
0
90°
θ
180°
$\Delta\theta$
(e)

Figure 2



RESONANCE HALF-WIDTH
$k_{\parallel}=0$
$\theta_1 = 0° \text{ or } 90°$
$\theta_1 = 30° \text{ or } 60°$
$\theta_1 = 45°$
90°
75°
60°
45°
30°
15°
$\Delta\theta$
0
1
2
3
4
5
6
7
λ

Figure 3

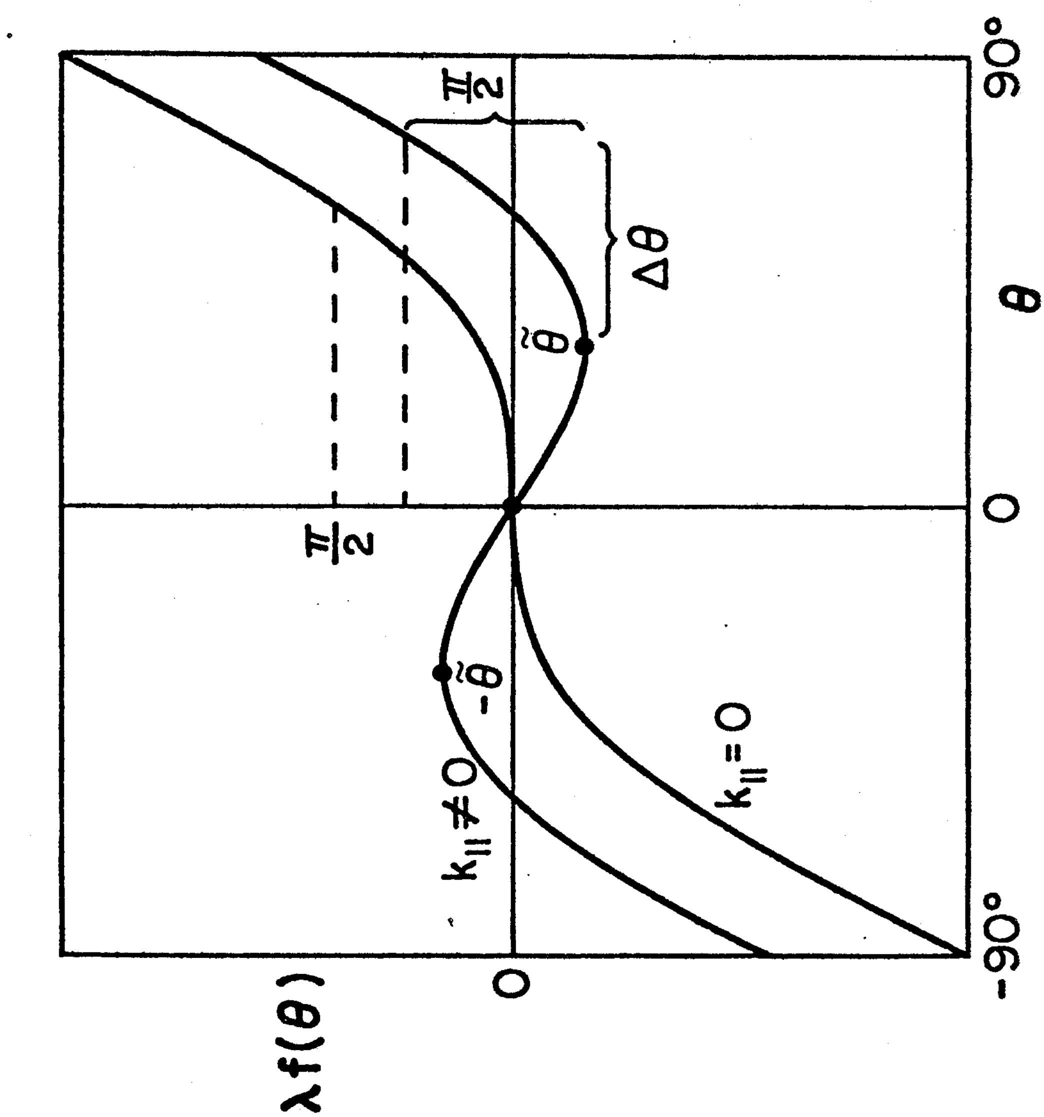

λf(θ)
0
π/2
$k_{\parallel} \neq 0$
$k_{\parallel} = 0$
$\tilde{\theta}$
$-\tilde{\theta}$
Δθ
θ
-90°
0
90°

Figure 4

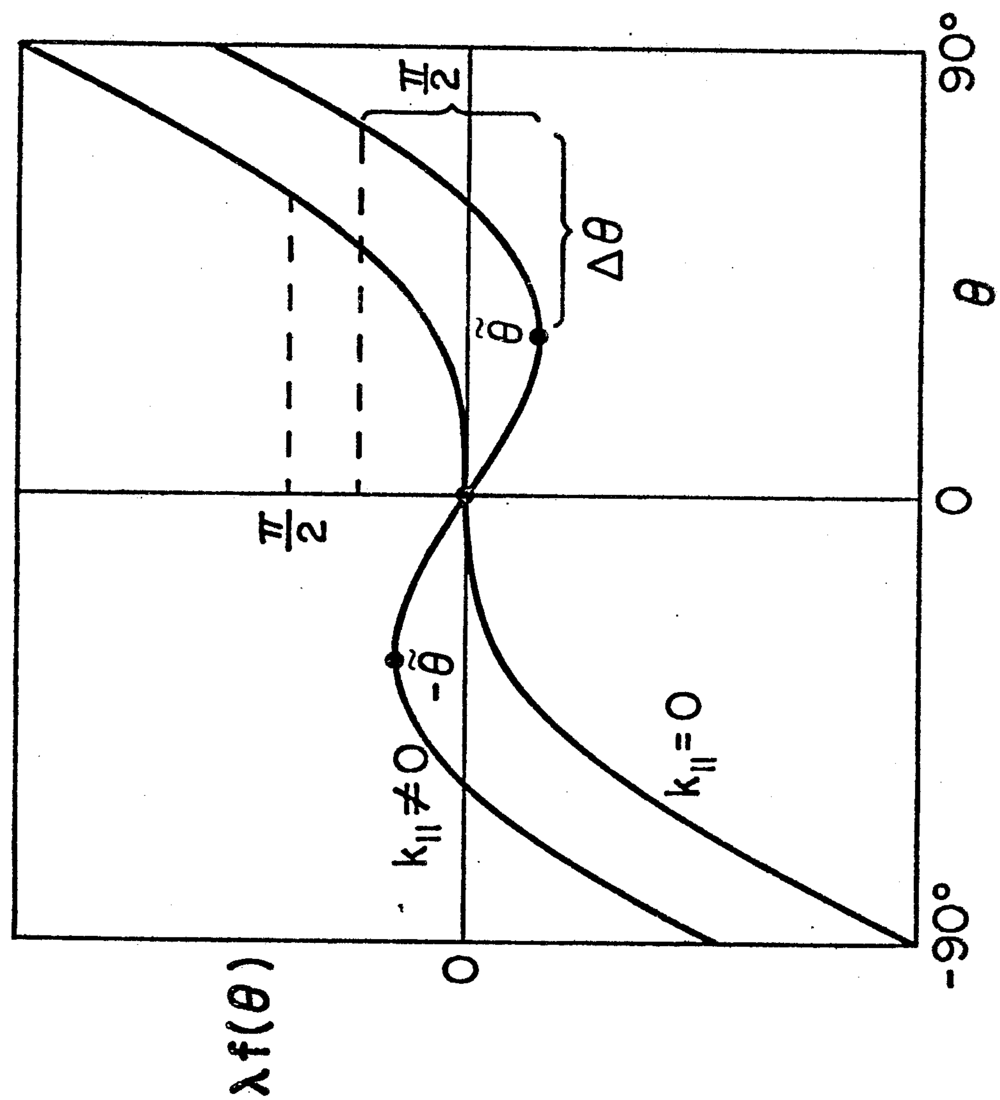


Figure 4