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in Thermal Barriers

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ABSTRACT

Self-consistent electrostatic potential variations are con-
sidered for several model situations which share important features
with proposed tandem mirror thermal barriers (which do not contain
sloshing ions). For some conditions, equivalent to a prescribed
pre-acceleration, the desired potential depressions are found. The
electron to ion temperature ratio is seen to play a critical role in
determining the necessary pre-acceleration. When the parameters are
not properly adjusted, no steady-state negative potential solutions
are found.

I. INTRODUCTION

The mirror approach to controlled nuclear fusion is undergoing a rapid evolution. The latest major stage in this evolution involves the concept of the tandem mirror machine [1-3], which is a straight solenoid with a minimum-B mirror cell at each end. Early studies of this concept showed that there were advantages to having hotter electrons in the end plug than in the solenoid. One way to maintain different electron temperatures in two spatial regions is to impose a potential depression between the two regions, which tends to isolate the two electron populations from each other. This depression is called a thermal barrier.

One method for creating the desired potential depression involves a magnetic field depression. Ions travelling along magnetic field lines will be accelerated in the region of decreasing magnetic field strength and they will be further accelerated by the hoped for accompanying potential depression. Since the ions speed up and the flux tube widens, the ion density is decreased. The assumption of quasineutrality is then invoked to say that the electron density must be decreased. If the electrons are sufficiently collisional so that the electron density is related to the potential by a Boltzmann relation, an electron density decrease is accompanied

by an electrostatic potential decrease. Thus, the scenario appears self-consistent.

A second approach to generating the thermal barrier involves the injection of an energetic, magnetically confined species into a high mirror ratio cell [4-5]. The injection angle or position is so chosen that these ions, characterized as "sloshing ions", exhibit a strong density depression at the cell midplane. Quasineutrality will then require an associated potential depression.

In this paper we study only the former approach, i.e., potential dips (and hills) created by decreasing magnetic fields with no sloshing ions present. We will utilize a model which is: (a) one dimensional and (b) considers only steady-state solutions. One consequence of (a) is that the potential at $x = \pm \infty$ can only vanish if we maintain overall charge neutrality. This follows from the fact that in one dimension a charge separation creates an electric field which does not decrease with distance. This is in contrast to more realistic three-dimensional models where the potentail vanishes at infinity, even though there is a net charge.

A consequence of assumption (b) is that even if we find steady-state solutions, they may turn out to be unstable.

In spite of the primitiveness of our approach, we find the following noteworthy properties: (1) A potential dip can only exist (without trapped ions) if the ratio of electron to ion temperatures is not too large. (2) No solutions to our models exist for ion

distributions which include particles with $v_{\parallel} = 0$, like a half-Maxwellian. If, however, ions with low $v_{\parallel}$ are removed so that we have a "pre-accelerated" distribution, steady-state solutions do exist.

In Section II we will describe the model that we use. Section III contains results assuming a beam approximation for ions, Section IV a water bag model, and Section V a pre-accelerated Maxwellian ion distribution.

II. THE MODEL

Our model has a magnetic mirror field, shown in Fig. 1(a) and 1(b). The field on the z-axis is given by

$$B(z) = B_O(1 - \epsilon(z)) = B_O \, b(z) \quad , \quad \epsilon(z) > 0 \quad , \tag{1}$$

where $b(z) \equiv B(z)/B_O$, and B_O is the field at $z \to \pm \infty$; $B(z)$ is symmetric about the $z = 0$ plane. We ignore the radial dependence on the grounds that the scale length perpendicular to the z-axis is much larger than that parallel to the z-axis, and write Poisson's equation as

$$\frac{d^2}{dz^2} \, \varphi(z) = - 4\pi e(n_i(z) - n_e(z)) \quad ; \tag{2}$$

φ is the electrostatic potential, $e > 0$ the proton charge and $n_i(z)$, $n_e(z)$ are the ion and electron densities.

We consider plasma time scales between an electron and ion self-collision time, which are different by a factor $(m_i/m_e)^{1/2}$. Then the electrons have thermalized and their density satisfies a Boltzmann relation

$$n_e(z) = n_o \exp[e\varphi(z)/T_e] \quad , \tag{3}$$

with T_e the electron temperature. (The assumption of constant elec-
tron temperature is another feature in which our model is over-
simplified compared with the actual tandem mirror. The idea of a
thermal barrier is to allow different electron temperatures on each
side of the barrier.)

The ions are assumed to be collisionless. In steady state
the energy

$$W = \frac{1}{2} m_i (v_\parallel^2 + v_\perp^2) + e\varphi(z) \tag{4}$$

and the magnetic moment

$$\mu = \frac{1}{2} m_i v_\perp^2 B_o/B(z) \tag{5}$$

are assumed to be constants of the motion. We have added a factor
B_o, the constant field at $z = \pm \infty$ to the definition of μ, so that μ
has the dimensions of energy. The ion distribution function can now
be constructed from the invariants W and μ. We assume a symmetric
distribution with respect to $z = 0$ and assume $\varphi = 0$ and $B(z) = B_o$ at
$z = \pm \infty$. Then $W \geq \mu$. As W and μ are intrinsically positive, the
distribution function $f(W,\mu)$ must only be specified in the first

octant of the W - μ plane. The ion density for $e\varphi < 0$ is then given by

$$n_i(z) = \int_0^\infty dW \int_0^W d\mu \; \frac{f(W,\mu)b(z)}{\sqrt{W - e\varphi(z) - \mu \; b(z)}} \qquad (6)$$

where we have used the Jacobian

$$\frac{\partial(v_z,v_\perp)}{\partial(W,\mu)} = \frac{b(z)}{m_i^2 \; v_z \; v_\perp}$$

to transform the density integral from $v_\parallel$, $v_\perp$ to W, μ.

Equations (6) and (3) together with Eq. (2) determine the electric potential. The resulting equations can be written in dimensionless form as

$$\frac{d^2}{dy^2} \psi = - [n_i(\psi,b(y))/n_o - \exp \psi] \quad , \qquad (7)$$

where $\psi = e\varphi/T_e$, $y = z/\lambda_D$, and $\lambda_D = [T_e/(4\pi e^2 n_o)]^{1/2}$ is the electron Debye length. The resulting potential ψ is created by the variation of the magnetic field b(y) or ε(y) and will, in general, scale with the magnetic field. If b(y) varies slowly over a Debye length, the left-hand side of (7) will be small compared with either

n_i/n_o or exp ψ. In other words, the quasineutral approximation will be a good approximation and we can to that accuracy determine ψ algebraically from

$$n_i(\psi, b(y)) = n_e(\psi) \quad . \tag{8}$$

We can also expand Eq. (7) at $y = \pm \infty$ to obtain

$$\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{y=\pm\infty} = \delta\psi\left(\frac{\partial n_e}{\partial \psi} - \frac{\partial n_i}{\partial \psi}\right)\Bigg|_{\substack{\psi=0 \\ b=1}} - \delta b \left.\frac{\partial n_i}{\partial b}\right|_{\substack{\psi=0 \\ b=1}} \quad .$$

For a magnetic depression ($\delta b < 0$), the second term is positive (since $\partial n_i/\partial b > 0$ by the arguments in the introduction). Since a potential depression ($\delta\psi < 0$) implies $d^2\psi/\partial y^2|_{y=\pm\infty} < 0$, we obtain the following necessary condition for a potential depression:

$$\left.\frac{\partial n_e}{\partial \psi}\right|_{\psi=0} > \left.\frac{\partial n_i}{\partial \psi}\right|_{\substack{\psi=0 \\ b=1}} \quad . \tag{9}$$

This result will be applied to different model ion distribution functions in succeeding sections.

III. ION BEAM APPROXIMATION

We consider two equal beams of monoenergetic ions, one left going, and one right going,

$$f(W,\mu) = C \; \delta(\mu - \mu_o)\delta(W - W_o) \quad .$$

The ion density then becomes

$$n_i(z) = n_o \; b(z) \; \sqrt{\frac{W_o - \mu_o}{W_o - e\varphi(z) - \mu_o \; b(z)}} \quad . \tag{10}$$

We determined the constant of proportionality in such a way as to make $n(z \to \infty) = n_o$. Inserting this result and the electron distribution into Poisson's equation, we have

$$\frac{d^2\varphi(z)}{dz^2} = 4\pi e \; n_o \left[\exp(e\varphi/T_e) - \frac{b(z) \; \sqrt{W_o - \mu_o}}{\sqrt{W_o - e\varphi(z) - \mu_o \; b(z)}} \right] \quad . \tag{11}$$

We assume now $\epsilon(z) \ll 1$ (compare Eq. (1)), and expand Eq. (11) up to linear terms in φ and ϵ. Furthermore, we introduce $\mu_o = \alpha W_o$ $(0 < \alpha < 1)$, $\psi = e\varphi/T_e$, $y = z/\lambda_D$, and the Mach number

$$M^2 = \frac{2W_o(1 - \alpha)}{T_e} \quad .$$

M is the ratio of the speed v_z at $z \rightarrow \pm \infty$ to the ion-sound speed. Then (11) becomes (with a prime denoting a y-derivative)

$$\psi''(y) - a^2 \psi(y) = \widetilde{\epsilon}(y) \quad , \tag{12}$$

where $a^2 = 1 - 1/M^2$ and $\widetilde{\epsilon}(y) = [(1 - \alpha/2)/(1 - 2)]\epsilon(y)$.

The general solution of (12) can be written as

$$\psi(y) = \int_{-\infty}^{y} \frac{1}{a} \sinh a(y - y')\widetilde{\epsilon}(y')dy' + A\,e^{-ay} + B\,e^{+ay} \quad . \tag{13}$$

We first discuss the solution for the supersonic case $M > 1$ or $a^2 > 0$. We assume that $\widetilde{\epsilon}(y)$ is an even function of y and require that $\psi(\pm \infty) = 0$, which determines the so far undetermined constants A and B in (13). Thus, Eq. (13) can be written

$$\psi(y) = -\frac{1}{2a}\left[\exp(ay)\int_y^\infty \exp(-ay')\widetilde{\epsilon}(y')dy'\right.$$

$$\left. + \exp(-ay)\int_{-\infty}^y \exp(ay')\widetilde{\epsilon}(y')dy'\right] \quad , \tag{14}$$

where we assume that $\epsilon(y)$ vanishes fast enough at $y \to \pm\infty$ to ensure the convergence of all integrations. The solution (14) is symmetric about $y = 0$, is negative definite, and has a minimum at the origin. It is sketched in Fig. 2. This solution clearly exhibits the desired potential shape.

In the subsonic case, the ion speed v_z at $z \to -\infty$ is less than the ion-sound speed. Defining $\bar{a} \equiv (1/M^2 - 1)^{1/2}$, we have to solve

$$\psi'' + \bar{a}^2 \psi = \widetilde{\epsilon}(y) \tag{15}$$

which has the solution

$$\psi(y) = (1/\bar{a})\int_{-\infty}^y dy' \sin\bar{a}(y - y')\widetilde{\epsilon}(y') \quad .$$

This solution satisfies $\psi(y \to -\infty) = 0$ whenever $\widetilde{\epsilon}(z \to -\infty) = 0$. However, the symmetric condition $\psi(y \to +\infty) = 0$ is only satisfied if

$$F \equiv \int_{-\infty}^{+\infty} dy' \cos \bar{a} \, y' \, \tilde{\epsilon}(y') = 0 \quad , \tag{16}$$

which is, in general, for arbitrary $\tilde{\epsilon}(y)$, not the case. The potential for $y \rightarrow +\infty$ will therefore not vanish exactly, but will exhibit oscillations like (see Fig. 3)

$$\psi(y \rightarrow +\infty) = \frac{F}{a} \sin \bar{a} \, y \quad . \tag{17}$$

If we insist on a symmetric solution, we have to write

$$\psi = \frac{1}{2\bar{a}} \left\{ \sin \bar{a} \, y \int_{0}^{y} \cos a \, y' \, \tilde{\epsilon}(y') dy' + \cos \bar{a} \, y \right.$$

$$\left. \times \int_{y}^{\infty} \sin \bar{a} \, y \, \tilde{\epsilon}(y') dy' \right\} \quad . \tag{18}$$

It is easily seen that in (18) $\psi(y) = \psi(-y)$; and $\psi(y)$ also satisfies Eq. (15). This solution is, in general, oscillatory for both $z \rightarrow -\infty$ and $z \rightarrow +\infty$. If $\bar{a}$ is of order one and $\tilde{\epsilon}$ varies slowly over a Debye length, F will be quite small. Also, the fact that the amplitude of the oscillations in (18) does not decrease, is clearly a consequence of the one dimensionality of our model.

The quasineutral solutions of (12) and (15) reveal a striking difference between the supersonic and subsonic solutions. In the

supersonic case $\psi(y) = - a^{-2}\tilde{\epsilon}(y)$, i.e., a decrease of the magnetic field $(\tilde{\epsilon} > 0)$ causes a potential hole $(\psi < 0)$. The ions are accelerated in a decreasing magnetic field, and the resulting electric field tends to enhance this effect. In the subsonic case, the action of the electric field is reversed, i.e., we have $\psi(y) = a^{-2}\tilde{\epsilon}(y)$. Ions tend to be accelerated into the magnetic well by the magnetic field, but are slowed down by the resulting electric field.

In order to find the full solution in the quasineutral approximation, we have to solve

$$e^{\psi} = \frac{1 - \epsilon}{\sqrt{1 - \dfrac{\psi}{M^2} + \dfrac{\alpha\epsilon}{1 - \alpha}}} \quad . \tag{19}$$

For fixed $\epsilon > 0$, the graphical solution of (19) with $\alpha = 1/2$ is sketched in Fig. 4. We find two points A and B representing possible solutions. As we are interested only in the solution for which $\psi \to 0$ as $z \to -\infty$ and $\epsilon \to 0$, the physically acceptable solution is the one which approaches $\psi \to 0$ as $\epsilon \to 0$, i.e., as the point C moves upward toward 1. Now the slope of the left side of (19) at $\psi = 0$ is 1, while the slope of the right side of (19), evaluated at $\epsilon = 0$ and $\psi = 0$, is $1/M^2$. Thus, in the subsonic case $(M < 1)$, it is point B which approaches 1 as point C moves up to 1, while in the supersonic case $(M > 1)$, it is point A which approaches 1 as point C

moves up to 1. Therefore, we can recover Eq. (9) for this particular case.

We conclude that in the subsonic case $(M < 1)$, the physical branch is the one with $\psi > 0$, in qualitative agreement with our solution of the linearized Poisson equation. Likewise, in the supersonic case $(M > 1)$, the physical branch is the one with $\psi < 0$, in qualitative agreement with our solution of the linearized Poisson equation. The necessary condition, $M > 1$, for a negative potential depression could have been obtained directly by applying criterion (9) to Eq. (10).

In the next section, we consider the case of a distribution of ion energies and magnetic moments, treating the simple water bag model.

IV.   THE WATER BAG MODEL

We consider now a distribution of ions of the form

$$f(W,\mu) = \text{const} \tag{20}$$

in the triangle defined by the straight lines $\mu = 0$, $W = W_o$ and $\mu = W - D$ as sketched in Fig. 5(a). Figure 5(b) shows the same distribution in the conventional $v_\perp$, $v_{\parallel}$ coordinate system. By choosing $D > 0$, we have eliminated particles with very small $v_{\parallel}$, i.e., we are dealing with a distribution drifting along B. Using (6) and assuming that φ becomes negative (as in Fig. 2) we find

$$n_i(z) = n_o \frac{\left(1 - \frac{e\varphi}{W_o}\right)^{\frac{3}{2}} - \delta^{\frac{3}{2}}\left(1 - \frac{e\varphi}{\delta W_o}\right)^{\frac{3}{2}} - \delta^{\frac{3}{2}}\epsilon^{-1}\left[\left(1 + \epsilon\frac{1 - \delta}{\delta} - \frac{e\varphi}{\delta W_o}\right)^{\frac{3}{2}} - \left(1 - \frac{e\varphi}{\delta W_o}\right)^{\frac{3}{2}}\right]}{1 - \frac{3}{2}\delta^{\frac{1}{2}} + \frac{1}{2}\delta^{\frac{3}{2}}} \tag{21}$$

where we have introduced $\delta = D/W_o$, $0 \le \delta < 1$. As in the last section, we expand this expression to first order in ϵ and $e\varphi/W_o$, where we assume δ to be a finite quantity, and find

$$n_i(z) = n_o\left(1 + \frac{e\varphi}{W_o} A - \epsilon B\right) \quad , \tag{22}$$

where

$$A = \frac{3}{2} \frac{\frac{1}{2}(\delta^{1/2} + \delta^{-1/2}) - 1}{1 - \frac{3}{2}\delta^{1/2} + \frac{1}{2}\delta^{3/2}} > 0 \quad ;$$

$$B = \frac{3}{8} \frac{(1 - \delta)^2}{\delta^{1/2}(1 - \frac{3}{2}\delta^{1/2} + \frac{1}{2}\delta^{3/2})} > 0 \quad .$$

The Poisson equation can be written in the form

$$\frac{d^2}{dy^2} \psi - \left(1 - \frac{A(\delta)}{M^2}\right) \psi = B \epsilon(y) \quad , \tag{23}$$

where $M^2 = W_o/T_e$. This equation is exactly of the form (11) in Section III.

We conclude that a potential dip as in Fig. 2 will be formed if $M^2 > A(\delta)$. This conclusion also follows directly from criterion (9). If $M^2 < A(\delta)$, a potential hill will be formed, however, only as long as $e\varphi < D$. As soon as $e\varphi > D$, the slow particles will be turned around and we can no longer integrate over the whole triangle of the distribution in Fig. 5 (see Fig. 6). It follows from (4) and (5) that the separatrix, which separates the particles which

contribute to the ion density at a location with potential $\varphi(z)$ is given by the line

$$W - e\varphi - b\,\mu = 0 \quad . \tag{24}$$

In order to obtain the contributing particles for the symmetric case, we have to subtract from expression (21) the integral over the triangle abc, which is given by

$$n_o\,\frac{1 - \epsilon}{\epsilon}\,(D - e\varphi)^{3/2}\,(1 - \frac{3}{2}\,\delta^{1/2} + \frac{1}{2}\,\delta^{3/2})^{-1}$$

and obtain

$$n_i(z) = n_o\,\frac{(1 - e\varphi/W_o)^{3/2} - \epsilon^{-1}(\delta + (1 - \delta)\epsilon - e\varphi/W_o)^{3/2}}{1 - \frac{3}{2}\,\delta^{1/2} + \frac{1}{2}\,\delta^{3/2}} \quad .$$

This expression is correct as long as the line (24), coming from below intersects the line $W = \mu + D$ before $W = W_o$. The range of the potential is thus given by

$$D < e\varphi < \epsilon\,W_o + (1 - \epsilon)D \quad .$$

Introducing $e\varphi = D + W_o\chi$, we find

$$0 < \chi < \epsilon(1 - \delta) \quad .$$

If we want to find the quasineutral approximation (8), we have to find a solution for χ from

$$(1 - \delta - \chi)^{3/2} - \epsilon^{-1}((1 - \delta)\epsilon - \chi)^{3/2}$$
$$= (1 - \frac{3}{2}\delta^{1/2} + \frac{1}{2}\delta^{3/2})\exp(M^2 \delta + M^2 \chi) \quad , \tag{25}$$

where δ, ϵ and M are given quantities.

We now consider a nondrifting distribution by putting $\delta = 0$ in (21) and obtain

$$n_i(z) = n_0\{(1 - e\varphi/W_0)^{3/2} + (- e\varphi/W_0)^{3/2}$$
$$- \epsilon^{-1}[(\epsilon - e\varphi/W_0)^{3/2} - (- e\varphi/W_0)^{3/2}]\} \quad . \tag{26}$$

Inserting this into (7) we find

$$\psi'' = \exp\psi + \epsilon^{-1}\left[\left(\epsilon - \frac{\psi}{M^2}\right)^{3/2} - \left(- \frac{\psi}{M^2}\right)^{3/2}\right]$$
$$- \left(- \frac{\psi}{M^2}\right)^{3/2} - \left(1 - \frac{\psi}{M^2}\right)^{3/2} \quad , \tag{27}$$

where the Mach number M is defined as $M^2 = W_o/T_e$. We have assumed $\psi < 0$; and we must have $\psi'' < 0$ as we come in from $y = -\infty$ where we assumed $\psi = 0$. If we regard ϵ and ψ as small quantities of the same order, then the leading terms on the right of (27) are of the order of one-half; keeping only these terms we have

$$\psi''(y) \simeq \epsilon^{-1}[(\epsilon - \psi/M^2)^{3/2} - (- \psi/M^2)^{3/2}] \quad .$$

However, the right side is always positive, and we have obtained a contradiction. We conclude that the water bag model (20) with $D = 0$ does not have acceptable solutions in which ψ becomes negative as it comes in from $y \to -\infty$.

V. MAXWELLIAN ION MODEL

For the case of a Maxwellian ion distribution, the integrals in n_i can no longer be given in closed form. The physical results, however, are the same as in the two previously treated simple cases. As in Section IV, we exclude the particles with very small $v_\parallel$ by integrating to the line $\mu = W - D$, $D > 0$ rather than to the line $\mu = W$ in Fig. 5. The ion density is then given by

$$n_i(z) = \int_D^\infty dW \int_0^{W-D} d\mu \; f \; \frac{b(z)\exp(-W/T_i)}{\sqrt{W - e\varphi - b(z)\mu}} \quad . \tag{28}$$

The result of this integration can be expressed in terms of the error function. However, we believe it is more instructive to leave the result in terms of integrals and then obtain closed results for certain asymptotic cases. This integral can, after several substitutions, be written as

$$n_i(z) = \frac{n_o}{N} \left[e^{-\psi} \int_{\delta-\psi}^\infty d\xi \; e^{-\xi} \; \xi^{1/2} - \epsilon^{1/2} \; e^{\left(-\frac{\psi}{\epsilon} + \frac{b\delta}{\epsilon}\right)} \right.$$

$$\left. \times \int_{(\delta-\psi)/\epsilon}^\infty d\xi \; e^{-\xi} \; \xi^{1/2} \right] \quad , \tag{29}$$

where $\psi = e\varphi/T_i$, $\delta = D/T_i$, and

$$N = \int_\delta^\infty dw \ e^{-w}(w^{1/2} - \delta^{1/2}) \quad .$$

If we assume that $(\delta - \psi)$ is a small quantity, the first integral in (29) can be written approximately as

$$\int_{\delta-\psi}^\infty d\xi \ e^{-\xi} \ \xi^{1/2} = \frac{1}{2}\sqrt{\pi} - \frac{2}{3}(\delta - \psi)^{3/2} \quad .$$

In the second integral we assume $(\delta - \psi)/\epsilon$ to be large. We then find asymptotically

$$\int_{(\delta-\psi)/\epsilon}^\infty d\xi \ e^{-\xi} \ \xi^{1/2} = e^{-\frac{\delta-\psi}{\epsilon}}\left[\left(\frac{\delta - \psi}{\epsilon}\right)^{1/2} + \frac{1}{2}\left(\frac{\epsilon}{\delta - \psi}\right)^{1/2}\right] \quad .$$

Using all these results, we finally obtain

$$n_i(y) = n_o\left(1 + A \frac{\psi}{M^2} - B \epsilon\right) \quad , \tag{30}$$

where

$$A = \frac{\frac{2}{3} \delta^{3/2} + \frac{1}{2} e^{-\delta} \delta^{-1/2} - \delta^{1/2} - \frac{1}{2} \sqrt{\pi}}{\frac{1}{2} \sqrt{\pi} - \frac{2}{3} \delta^{3/2} - e^{-\delta} \delta^{1/2}}$$

and

$$B = \frac{\frac{1}{2} e^{-\delta} \delta^{-1/2}}{\frac{1}{2} \sqrt{\pi} - \frac{2}{3} \delta^{3/2} - e^{-\delta} \delta^{1/2}} \quad .$$

Poisson's equation can now be written in the familiar form

$$\psi''(y) - \left(1 - \frac{A}{M^2}\right) \psi = B \, \epsilon(y) \tag{31}$$

with $M^2 = T_i/T_e$. The supersonic solution is obtained for small δ if $\delta > 1/(\pi M^4)$ and exhibits a potential dip $\psi = - [B/(1 - A/M^2)]\epsilon$.

As in Section IV we do not find an acceptable solution if the particles with very small $v_\parallel$ are retained by putting $D = 0$.

The quasineutrality approximation can be invoked by equating (29) and (3). It is useful to express this equality in terms of the complementary error function (erfc) as follows:

$$n_e - n_i = 0 = e^{M^2 \psi} - \frac{1}{\text{erfc } \sqrt{\delta}} \left\{ e^{-\psi} \text{ erfc } \sqrt{(\delta - \psi)} \right.$$

$$\left. - \sqrt{\epsilon} \ e^{-\delta} \ e^{(\delta - \psi)/\epsilon} \text{ erfc } \sqrt{(\delta - \psi)/\epsilon} \right\} \quad .$$

In the large ψ limit, this equation reduces to the form given in Ref. [3].

The criterion for minimum acceleration results directly from the inequality Eq. (9). Substituting (28) and (3) into (9) yields the result

$$M^2 > \frac{e^{-\delta}}{\sqrt{\pi \delta} \text{ erfc } \sqrt{\delta}} - 1 \quad . \tag{32}$$

(An equal sign would yield the minimum pre-acceleration δ_{min}.) In the small δ limit (32) gives

$$M^2 > \frac{1}{\sqrt{\pi \delta}} + \left(\frac{2}{\pi} - 1 \right) \tag{33}$$

or

$$\delta_{min} \approx \frac{1}{\pi (T_i/T_e + 0.36)^2} \quad .$$

The minimum required pre-acceleration (the solution of (32)) is plotted in Fig. 7 and a comparison is shown with the small δ approximation (33). For $T_e/T_i \sim 1$ the required minimum pre-acceleration is $\sim 0.2\ T_i$. Notice that this pre-acceleration increases rapidly at large T_e/T_i.

VI. CONCLUSIONS

Three simplified ion distribution functions have been studied in a physical context which shares some of the proposed features of electron thermal barriers in tandem mirror fusion devices. A mono-energetic fluid model yields potential dips in the supersonic case (ion-beam speed greater than ion-sound speed), but yields potential hills in the subsonic case. Two distributions with a spread in ion energy, the water bag model and the Maxwellian model, have no solutions at all for reasonable boundary conditions. However, when particles with small v_z are removed from the distribution, which amounts to an ion pre-acceleration, satisfactory potential dips are found. The minimum required pre-acceleration is seen to increase with T_e/T_i.

These results indicate a difficulty in creating a potential dip for a plasma streaming into a magnetic field depression. It should be noted that in a Q-machine experiment of this sort a drop in density (more precisely a drop in Langmuir probe ion saturation current) was observed [6]. However, in this experiment the sheath of the hot tungsten plate Q-machine source was seen to form an ion acceleration of about 5 T_e ($T_e \approx 0.20$ eV) and so this experiment

could be interpreted as an example of the supersonic fluid case $(M \simeq 10)$.

We remind the reader that the present results are not directly applicable to the case of a barrier with sloshing ions. This topic will be addressed in a future paper.

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FIGURE CAPTIONS

Fig. 1(a). Magnetic field lines which can be rotated out of the
plane of the paper about the z-axis. (b) Magnetic field
intensity along the z-axis; $B(z) = B_O(1 - \epsilon(z)) = B_O\, b(z)$.
The typical scale length for variation of $\epsilon(z)$ is L_B.

Fig. 2. Self-consistent electrostatic potential vs distance for
$M > 1$.

Fig. 3. Self-consistent electrostatic potential vs distance for
$M < 1$.

Fig. 4. Graphical solution of the quasineutral equation (19).
Solution A is the physically acceptable solution for the
supersonic case $(M > 1)$, while solution B is the physically
acceptable solution for the subsonic case $(M < 1)$.

Fig. 5. Water bag model. (a) In μ - W space, (b) in conventional
$v_\perp$ - $v_\parallel$ space.

Fig. 6. Symmetric potential hill.

Fig. 7. Minimum required pre-acceleration for obtaining a potential
depression as a function of T_e/T_i. The solid curve is the
quasineutrality result and the dashed curve plots the
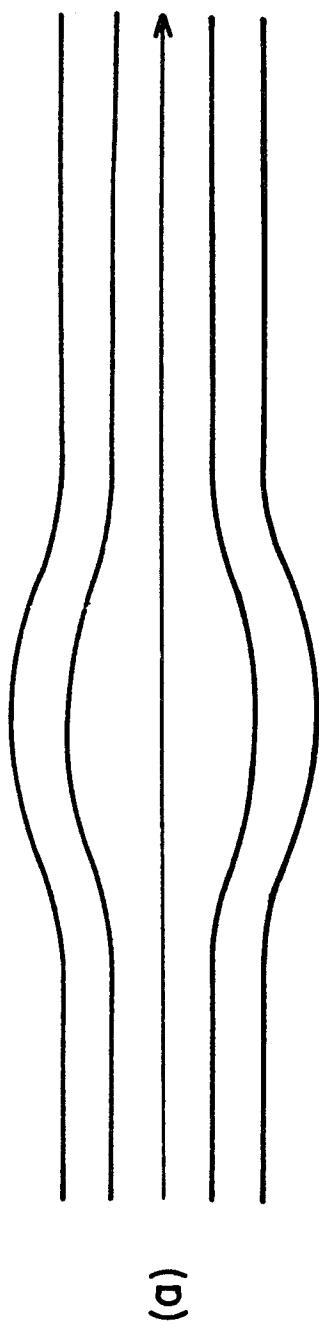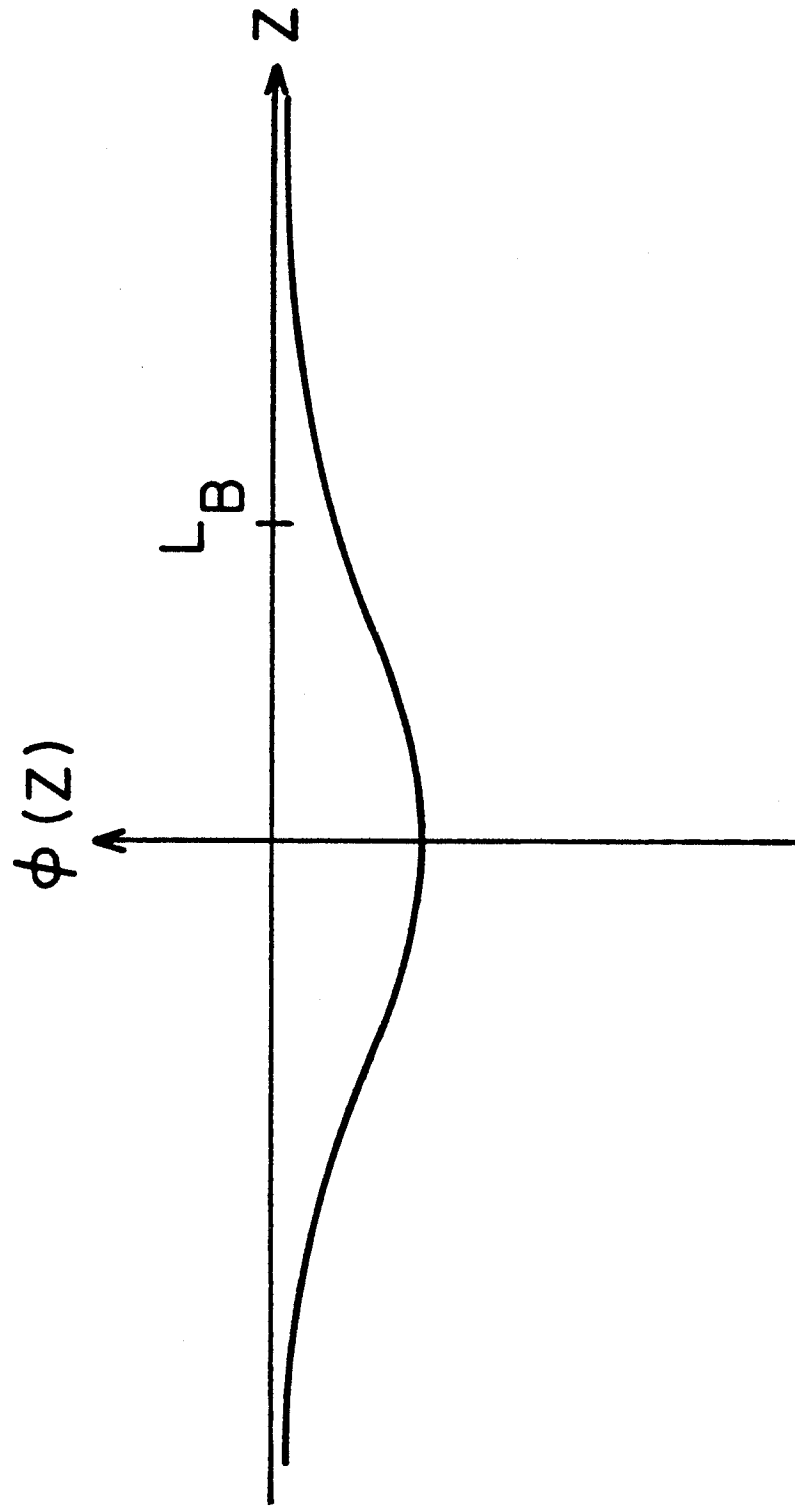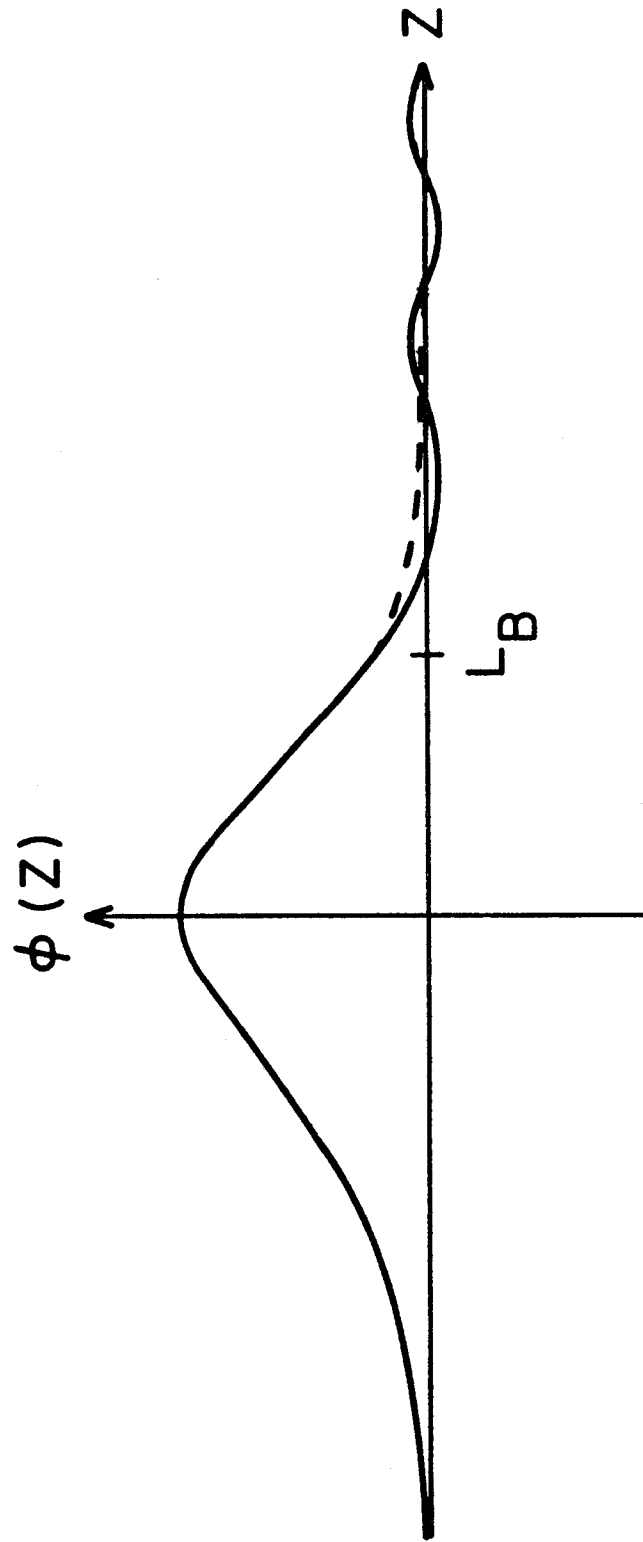approximate solution (Eq. (33)).

(a)

(b)

B(Z)

B_O

O

L_B

Z

Fig. 1

Z

L_B

$\phi(Z)$

Fig. 2

Fig. 3

35

Fig. 4

(a)

(b)

$W - e\phi - b\mu = 0$

$W_0 - D$

W_0

D

$\sqrt{\dfrac{2D}{m}}$

$\sqrt{\dfrac{2W_0}{m}}$

Fig. 5

Fig. 6

Fig. 7