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UWFDM-248

Proc. 3rd ANS Top. Mtg., 843 (1978).

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NUWMAK is a high power density, noncircular conceptual tokamak fusion reactor design. The device is found to be magnetohydrodynamically stable except for vertical instability due to bad curvature of the external B_z field. The MHD equilibrium properties of NUWMAK are obtained by solving the Grad-Shafranov equation numerically. Only 8 equilibrium field coils are required to provide plasma equilibrium and proper shape control. The average plasma power density is about 10.1 MW/m^3 and the neutron wall loading is 4.3 MW/m^2. The plasma current is 7.2 MA, the major radius is 5.13 m, the plasma width is 1.13 m and the plasma height to width ratio is 1.64. The on-axis magnetic field is 6.05 Tesla and toroidal β is 6% while the safety factor q at edge is found to be 2.63.

I. INTRODUCTION

One approach to the design of economic Tokamak fusion reactors is to make the plasma as compact as possible with a high power density. This is most easily accomplished with a high toroidal magnetic field and noncircular plasma cross-section. The NUWMAK reactor design is based on this concept. The MHD equilibrium and stability calculations for this design have been done using the Princeton MHD Code.[1,2] An interesting feature of the design is that it utilizes only four pairs of equilibrium field coils; two pairs are normal coils inside the toroidal field magnets and the other two are superconducting coils outside the toroidal field magnets. In Sec. II we describe the plasma equilibrium and stability conditions; the details for the NUWMAK design are given in Sec. III.

II. EQUILIBRIUM AND STABILITY

1. Equilibrium

In a magnetohydrodynamic (MHD) model, the equilibrium configuration is determined by the pressure balance equation

$$\nabla p = \vec{J} \times \vec{B} , \qquad (1)$$

and the Maxwell's equations

$$\nabla \times \vec{B} = \mu_o \vec{J} , \qquad (2)$$

$$\nabla \cdot \vec{B} = 0 . \qquad (3)$$

From Eq. (1), $\vec{B} \cdot \nabla p = 0$, $\vec{J} \cdot \nabla p = 0$, we then see that both the magnetic field $\vec{B}$ and plasma current $\vec{J}$ lie on the surfaces of constant pressure, or magnetic surfaces.

In the axisymmetric toroidal system we choose a magnetic flux coordinate system ψ, θ, ϕ, where ϕ denotes the toroidal angle, θ is the poloidal angle and ψ is poloidal magnetic flux label,

$$\psi(\vec{x}) = \frac{1}{2\pi} \int d^3\vec{x}\, \vec{B} \cdot \nabla\theta , \qquad (4)$$

which has the property that $\psi(\vec{x})$ = constant specifies the magnetic surfaces. In these coordinates, the magnetic field can be decomposed into poloidal and toroidal components using[3,4,5]

$$\vec{B} = \frac{1}{2\pi} \nabla\phi \times \nabla\psi + R_o B_o g(\psi) \nabla\phi , \qquad (5)$$

where Ro is the major radius and B_o is the toroidal magnetic field strength at R_o. g is function to be defined later. Combining Eqs. (1) and (2) we obtain the Grad-Shafranov equation for the

This work was supported by DOE.

equilibrium[6]

$$X^2\nabla \cdot \left(\frac{1}{X^2}\nabla\psi\right) = 2\,\pi\mu_o X J_\phi$$
$$= -4\,\pi^2\left(X^2\frac{dP}{d\psi} + R_o^2 B_o^2\, g\,\frac{dg}{d\psi}\right), \quad (6)$$

where X is the distance from the axis of symmetry, and J_ϕ is the toroidal current density. In cylindrical coordinates (x,z,ϕ), Eq. (1) is usually written as

$$\frac{\partial^2\phi}{\partial X^2} - \frac{1}{X}\frac{\partial\psi}{\partial X} + \frac{\partial^2\phi}{\partial z^2} = -\,4\pi^2\left(X^2\frac{dp}{d\psi} + R_o^2B_o^2 g\frac{dg}{d\psi}\right) \quad (7)$$

The code we use to solve Eq. (7) numerically was developed at Princeton Plasma Physics Laboratory. The nonlinear equation (7) is solved by prescribing the functions $P(\psi)$ and $g(\psi)$. The form employed for the functions $P(\psi)$ and $g(\psi)$ in equation (7) can have a significant effect on the equilibrium and stability of the system. $P(\psi)$ and $g(\psi)$ are chosen have the following forms[1]

$$P(\psi) = Po\left(\frac{\psi_* - \psi}{\psi_* - \psi_o}\right)^{\alpha_1}, \quad (8)$$

and

$$g(\psi) = 1 - g_p\left(\frac{\psi_* - \psi}{\psi_* - \psi_o}\right)^{\alpha_2}, \quad (9)$$

where Po is the given peak plasma pressure at magnetic axis, ψ_* and ψ_o are the fluxes at the plasma boundary and magnetic axis respectively. The values of parameters α_1 and α_2 are of the order of unity. The constant g_p is varied until ψ converges. The plasma boundary is determined consistently with the location of the external equilibrium (EF) coils. A rectangular mesh of arbitrary size is used as an outer boundary for the problem and the boundary conditions on this border are fixed by specifying the currents in EF coils. As such, the procedure does not require the use of a conducting external boundary. The plasma boundary can be fixed either by a limiter or by the separatrix.

2. Stability

In designing a tokamak fusion reactor, not only should it reach the certain fusion power level, it is also necessary to satisfy the equilibrium and stability criteria associated with the device.

Before we discuss the stability problem, several important parameters such as the safety factor, toroidal β and poloidal β will be defined first. The volume enclosed by a flux surface ψ is given by

$$V = \oint d^3\vec{x}\ ; \quad (10)$$

its derivative with respect to ψ has the following form

$$V' = \frac{dv}{d\psi}$$
$$= 2\pi\oint\frac{x\,d\ell}{|\nabla\psi|}, \quad (11)$$

where the line integral is obtained by integrating along the poloidal field line at constant ψ. Defining the toroidal flux ψ_T as

$$\psi_T \equiv \frac{1}{2\pi}\oint d^3\vec{x}\ \underline{B}\cdot\nabla\phi\ , \quad (12)$$

then the safety factor $q(\psi)$ is given by

$$q(\psi) = \frac{d\psi_T}{d\psi}$$
$$= R_oB_o g(\psi)\oint\frac{d\ell}{X|\nabla\psi|} \quad (13)$$

The volume average plasma pressure is

$$\overline{P} = \frac{\int P\frac{dv}{d\psi}d\psi}{\int\frac{dv}{d\psi}d\psi}. \quad (14)$$

The toroidal-β of the system is defined as

$$\beta_T = \frac{\overline{P}}{\frac{1}{2\mu_o}\overline{B}_T^2}, \quad (15)$$

Where $\overline{B}_T^2 = R_o^2B_o^2\int g(\psi)\left\langle\frac{1}{x^2}\right\rangle dV$. For any quantity A, its flux surface average value is given by

$$\langle A\rangle = \frac{\oint\frac{XA(\psi)d\ell}{|\nabla\psi|}}{\oint\frac{Xd\ell}{|\nabla\psi|}} \quad (16)$$

The poloidal-β is written as

$$\beta_p = \frac{\overline{P}}{\frac{1}{2\mu_o}\overline{B}_p^2}, \quad (17)$$

where $\overline{B}_p^2 = \int\left\langle\left(\frac{|\nabla\psi|}{2\pi X}\right)^2\right\rangle dV$.

Following Zakharov and Shafranov[7], another definition of poloidal-β (for non-circular cross section tokamaks) is

$$\bar{\beta}_p = \frac{\bar{P}}{\bar{P} - \frac{R_o^2 B_o^2}{2\mu_o} \int V(\psi) g(\psi) \frac{dg}{d\psi} \langle \frac{1}{x^2} \rangle \, d\psi} \quad . \tag{18}$$

The stability of the plasma with respect to the rigid motions of the plasma column, ideal and resistive MHD modes will be considered in next section. The MHD stability criteria are obtained by using the energy principle under with or without plasma restivity[3,4] included.

A. Rigid Motions

First we consider rigid displacements of the plasma. The stabilizing force tending to bring back the plasma column to its original equilibrium position requires[8]

$$\frac{\partial}{\partial R} [B_R R]_{R=R_o} > 0$$

for stability against horizontal displacements and

$$\frac{\partial B_R}{\partial Z} < 0$$

for vertical motions. Using $\nabla X \vec{B} = 0$, the stability condition is obtained and related to the sign and magnitude of decay index N by

$$0 \leq N \lesssim 3/2 \ , \tag{19}$$

where $N = - \frac{R}{B_Z} \frac{dB_Z}{dR} \Big|_{R=R_o}$. The positive value of decay index assures stability against the vertical displacement. On the other hand, the decay index must not exceed 3/2 for horizontal stability. If the plasma cross section is noncircular, vertical stability is more difficult to obtain. Zakharov[9] shows that for a small elongated plasma with flat current profile the relation between decay index and elongation ration (b/a) is

$$\frac{b}{a} = 1 + (\frac{a}{R_o})^2 \left[\frac{3}{4} \ln(\frac{8R_o}{a}) - \frac{17}{16}\right] - N \left[\ln(\frac{8R_o}{a}) \pm \beta_p - \frac{5}{4}\right] . \tag{20}$$

Thus, for vertical stability, the elongation must satisfy

$$\frac{b}{a} - 1 < (\frac{a}{R_o})^2 \left[\frac{3}{4} \ln (\frac{8R_o}{a}) - \frac{17}{16}\right] .$$

For moderate aspect ratio tokamaks, such as Ro = 5m and minor radius a = 1.25m, a stable configuration can be obtained only if

$$\frac{b}{a} \lesssim 1.1 \ .$$

A recent paper by Haas[10] et al. indicates that for arbitrary cross section, aspect ratio toroidal plasma, the system will be unstable if the decay index is negative and the toroidal current decreases monotonically towards the boundary. Therefore, additional feedback stabilization systems are required for such devices.

B. Ideal MHD Modes

(i) Kink Modes

The most important MHD instabilities are non-local, helical-type kink modes. Thus are characterized by a displacement of the form $\xi(r) \exp[i(m\theta - n\phi)]$, with integer numbers for m and n (these are the poloidal and toroidal mode number respectively). In circular cross section tokamaks, if the safety factor $q > 1$, all "m" modes are stabilized, this is the well-known Kruskal-Shafranov limit.[11] This places an upper limit on the plasma current. The critical q value of stability is found to be larger for noncircular cross section tokamaks.

(ii) Ideal Interchange Modes

The localized internal or interchange modes are driven by plasma pressure gradients. In the singular layer near to the rational surfaces ($\vec{K} \cdot \vec{B} = 0$ or $q(r_s) = m/n$), the pressure driving displacements can "interchange" with the magnetic field lines without bending them. In toroidal systems, the relevant stability condition is called the Mercier criterion[5]

$$(- \frac{dp}{dr}) \ (q^2 - 1) + \frac{rB_\phi^2}{8} \left(\frac{dq/dr}{q}\right)^2 > 0 \ . \tag{21}$$

The negative pressure gradient is destabilizing and the second term is stabilizing by the influence of the magnetic shear effects. However, the stability condition has a more complicated form for general noncircular tokamaks[4]

$$DI = D + H - \frac{1}{4} < 0 \,, \tag{22}$$

where

$$D = \langle \frac{B^2}{|\nabla\psi|^2} \rangle \frac{P'V'}{(q')^2} \left[\frac{2\pi R_o g q'}{\langle B^2 \rangle} - V'' + 4\pi^2 p'v' \langle \frac{X^2}{|\nabla\psi|^2} \rangle \right]$$

$$- \left(\frac{P'V'}{q'}\right)^2 (2\pi R_o g)^2 \langle \frac{1}{|\nabla\psi|^2} \rangle \,, \tag{23}$$

and

$$H = - \frac{2\pi R_o g p'v'}{q'} \left[\frac{1}{\langle B^2 \rangle} \langle \frac{B^2}{|\nabla\psi|^2} \rangle - \langle \frac{1}{|\nabla\psi|^2} \rangle \right] \,. \tag{24}$$

Where the prime of a quantity denotes its derivative with respect to the flux function ψ. H is small for highly symmetric devices, such as circular tokamaks, and can be small for high shear systems also. The shear term $-\frac{1}{4}$ in Eq. (22) has an important contribution to the stabilization of interchange modes.

C. Resistive MHD Modes

(i) Resistive Interchange Modes

If the plasma has finite resistivity, the stabilizing effect of magnetic shear in ideal modes vanishes. The stability condition is modified as[4]

$$DR = D + H^2 < 0 \,. \tag{25}$$

Since $DR = DI + (H - \frac{1}{2})^2$, the resistive interchange modes are more stringent than the ideal interchange modes. The effect of non-vanishing H is always destabilizing. When DR > 0, the growth rate scales as $\eta^{1/3}$, where η is the resistivity of the plasma; this is the gravitational mode in the study of the resistive sheet-pinch by Furth et al.[12]

(ii) Tearing Modes

The tearing modes are unstable if DR=0 and $\Delta>0$, where Δ is computed from matching the solutions of MHD equation inside the resistive layer to the solutions in the ideal region on both sides of it.[12] The growth rate can be scaled as a power of the resistivity, $\eta^{(3-2H)/(5+2H)}$. If H = 1/2, the growth rate scales as $\eta^{1/3}$, which is similar to the interchange mode. When H=0, the growth rate scales as $\eta^{3/5}$; this recovers the usual tearing mode scaling.[12] Numerical calculations show that stable zones should occur if the safety factor at plasma edge is greater than 2.5.[13]

(iii) Modified-Tearing Modes

When DR < 0, the unstable modified tearing modes appear with complex frequencies if

$$\Delta > \Delta c \,, \tag{26}$$

where Δc is defined in Eq. (111) of Reference 4. As $H \simeq 1/2$, Δc approaches infinity; therefore the mode is stable. Generally $\Delta_c \sim \eta^{(2H-1)/3}$; in a typical tokamak scaling, we have $1/2 \gtrsim H \gtrsim 0$. This indicates that high temperature will stabilize the modified-tearing modes.

D. β-Limits and Ballooning Modes

Tokamak devices always have some pressure limits either in the form of β_T or β_p. One widely used upper limit of the permissible β_p at equilibrium is that it can not exceed the aspect ratio. At this Shafranov limit[14], the separatrix will coincide with the plasma column boundary. If the plasma β_p is larger than Shafranov limit, then the localized "ballooning" modes will appear. In the study of ballooning modes in the neighborhood of the magnetic axis, it is found that stability condition for these modes is just the same as that for the localized interchange modes [15]. Consequently there is no limit on β as long as the localized modes are stable. However, using the collisionless energy principle Rutherford et al. obtained a more general form of β-limit due to the ballooning instabilities as[16]

$$\beta < \beta_c = \frac{0.18\, a}{R_o\, q_a^{0.75}\, q_o^{1.25}} \left[1 + 0.2 \left(\frac{a}{R_o}\right)^{3/2}\right] \,. \tag{27}$$

The value of this β_c is only a few percent. Numerical studies show that the value of β_c is about 5%[17] for D-shape plasma, and it can reach 12%[18] by proper choices of the pressure and current profiles. Generally, the critical-β value of tokamaks increases with vertical elongation but decreases with larger triangularity [19].

III. RESULTS FOR NUWMAK

The cross-sectional view of NUWMAK is shown in Fig. 1.

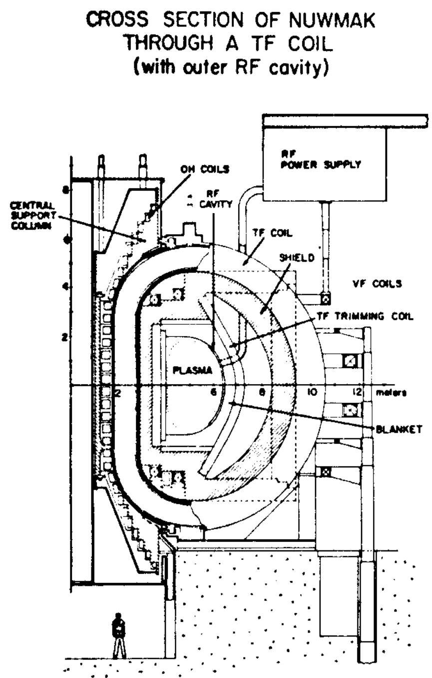

FIGURE 1: Cross Section of NUWMAK Through a TF Coil.

There are only 4 pairs of EF coils required to provide plasma equilibrium and shaping effects. The location of these coils and their currents are listed in Table 1.

TABLE 1. Equilibrium Field Coils of NUWMAK

	Z(m)	R(m)	I(MA)
1.	3.50	3.50	2.30
2.	3.75	4.50	2.00
3.	3.50	10.50	-2.00
4.	1.00	11.50	-3.50

The 4 demountable cyrogenic Al coils are located inside the toroidal field (TF) coils. The other larger 4 superconducting coils are situated outside the TF coil set; they can be moved freely when necessary. There are several advantages for some coils are placed inside the TF coils. Since they are closer to the plasma column, the required currents are found to be much smaller than that of outside coils and still can provide the same effect of equilibrium and shaping. These cryogenic coils are demountable, consequently they will not interlock with TF coil set. Furthermore, they can save space for the Ohmic (OH) coils and support structure. The available space problem can be an important issue especially when the aspect ratio is small and when the device is compact.

Table 2 lists the plasma parameters and power density of NUWMAK.

TABLE 2. Plasma Parameters of NUWMAK

Plasma current	I_p = 7.2 MA
Toroidal field	B_o = 6.05 T
Major radius	R_o = 5.125 m
Minor radius	a = 1.125 m
Elongation	b/a = 1.64
Aspect ratio	Ro/a = 4.55
Toroidal β	β_T = 6%
Poloidal β	β_p = 3.67
Average poloidal β	$\overline{\beta}$ = 2.56
Safety factor (center)	q(o) = 1.09
Safety factor (edge)	q(a) = 2.63
Decay index	N = -1.13
Average power density	$\overline{P}$ = 10.1 MW/m^3
Neutron wall loading	Wd = 4.34 MW/m^2

The plasma current is 7.2 MA, the major radius is about 5.13 m with minor radius a = 1.13 m and elongation ratio about 1.64. The on-axis toroidal magnetic field is 6.05 Tesla, the toroidal β about 6%, the poloidal β is 3.67 and the safety factor at the magnetic axis and plasma edge are 1.09 and 2.63 respectively. The neutron wall loading in this system is about 4.3 MW/m^2 while the plasma power density is 10.1 MW/m^3. It should also be noted that the decay index is -1.13; therefore a beedback stabilization system is required for vertical stability. Figure 2 shows the flux surface plot. As can be seen, the plasma has a vertically elongated cross section and these are two stagnation points located outside the plasma boundary. The pressure and current profiles as a function of radius on the midplane are shown in Fig. 3. It is inter-

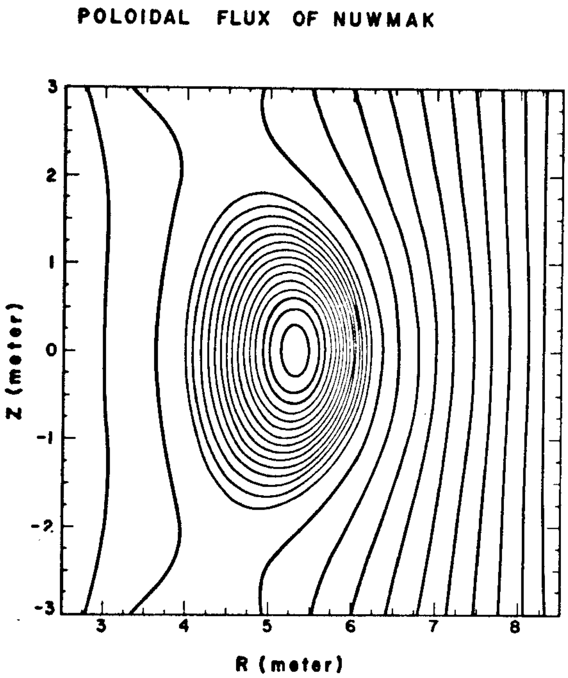

FIGURE 2: Flux Surfaces ψ in NUWMAK.

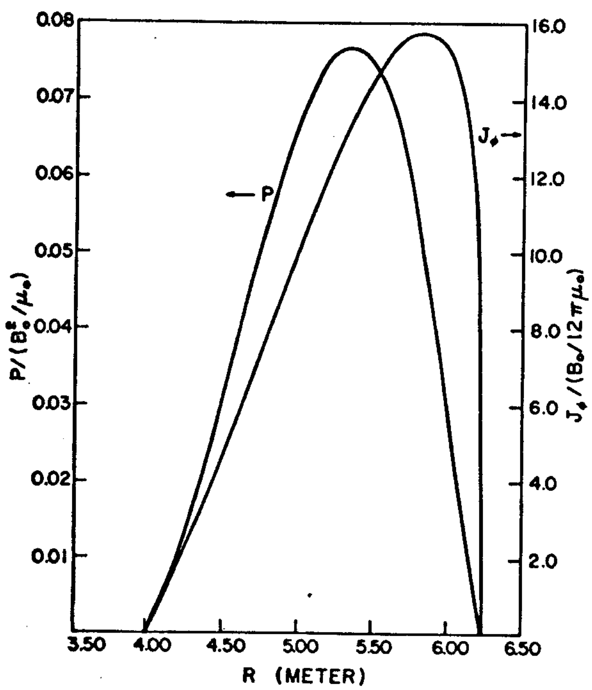

FIGURE 3: Pressure and Toroidal Current Profiles in the Mid-Plane.

esting to note that the magnetic axis is shifted about 0.18 m from the plasma geometric center, while the position of peaked current is shifted about 0.47 m further away from the magnetic axis.

Figure 4 indicates the $|B|$ surfaces in NUWMAK.

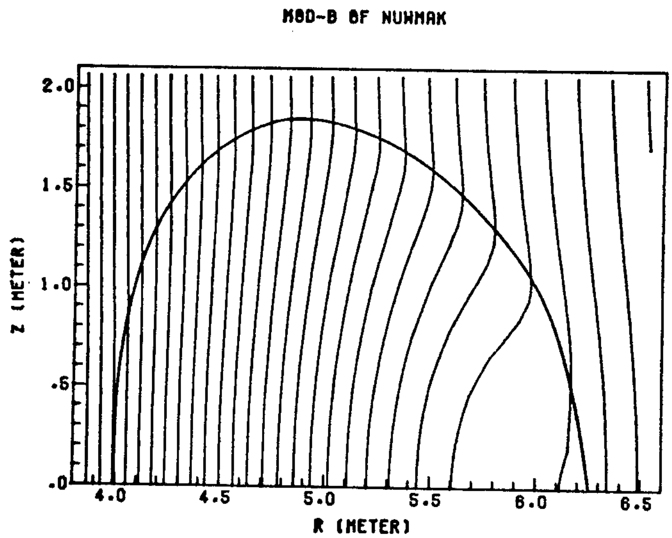

FIGURE 4: Plot of Constant $|B|$ Surfaces. The Heavy Closed Line Indicates the Plasma Boundary.

The diamagnetic effects dominate inside the plasma while the poloidal effect due to plasma current is important at outside edge as seen from the direction of curvature of the $|B|$ contours.

Two important parameters H and D for the MHD stability analysis are shown in Fig. 5 as functions of ψ.

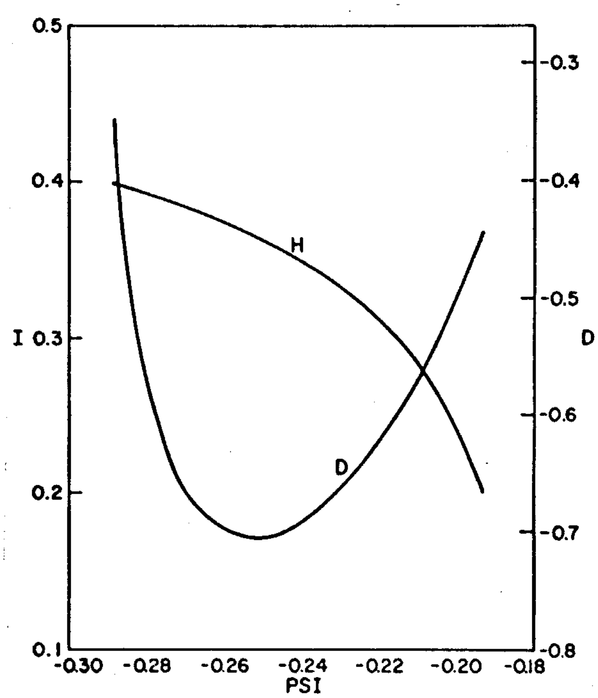

FIGURE 5: Parameters of H and D as Functions of ψ.

The minimum ψ is about -0.30; this corresponds to the location of magnetic axis. The ψ value at the plasma edge is -0.18. The profiles of DI (ideal interchange), DR (resistive interchange) and q are shown in Fig. 6.

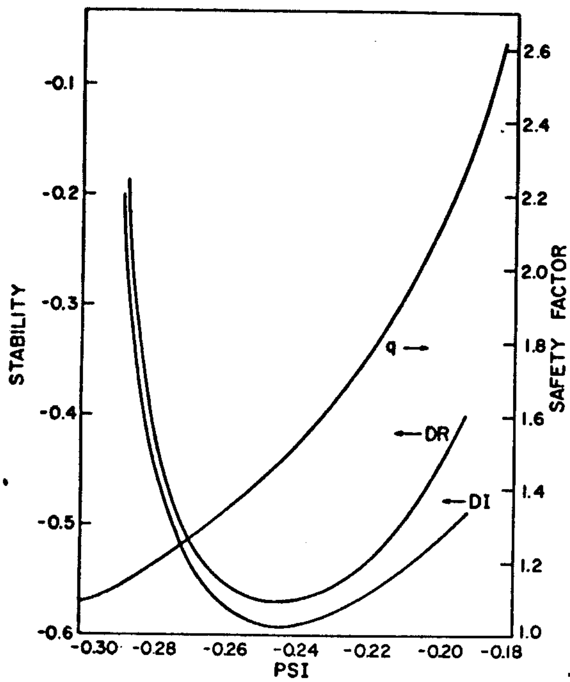

FIGURE 6: Stability Criteria DI,DR and Safety Factor q as Functions of ψ.

We see that both ideal and resistive interchanges are stable in NUWMAK. Since DR < 0 and H > 0 everywhere, the tearing mode is stabilized. On the other hand, the modified tearing modes are stabilized due to small positive value of H, $0.2 \lesssim H \lesssim 0.4$ and high temperature of the plasmas. The safety factor at plasma edge is about 2.63, this can ensure the stability of general kink modes in this system.

Although Eq. (27) gives a value for β_c of about 2% for NUWMAK parameters, this analytical criterion is good for circular tokamaks only. As indicated earlier, D-shape plasmas can have a higher β-limit; for the moderate value of β_T(~ 6%) and since β_p(~ 3.67) is still below the Shafranov limit (~ 4.54), it appears that ballooning modes are stable in NUWMAK.

IV. CONCLUSION

For a reactor design without a divertor such as the NUWMAK system the overall design tends to be simple and serviceable. There are only 4 pairs of EF coils required for plasma equilibrium and shaping. The MHD modes are found to be stable but a feedback stabilization system is required for vertical stability.

ACKNOWLEDGMENTS

We are happy to acknowledge the useful discussions and suggestions with Dr. T. F. Yang of Westinghouse Corporation. Thanks are also due to Miss Gail Herrington and Mrs. Bonnie Smart.

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