



**Growth Rates and Transport Coefficients for the  
Trapped Ion/Collisionless Trapped Particle  
Instability**

**W.A. Houlberg**

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***FUSION TECHNOLOGY INSTITUTE  
UNIVERSITY OF WISCONSIN  
MADISON WISCONSIN***

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W.A. Houlberg

Fusion Technology Institute  
University of Wisconsin  
1500 Engineering Drive  
Madison, WI 53706

<http://fti.neep.wisc.edu>

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Wayne A. Houlberg

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Fusion Technology Program  
Nuclear Engineering Department  
University of Wisconsin  
Madison, Wisconsin 53706

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I. Introduction

Numerical simulation of large Tokamak plasmas indicates the possibility that high temperatures (in the keV range), low densities and sharp density gradients can occur near the plasma edge if a poloidal divertor is used.<sup>(1)</sup> The low collisionality can lie in a regime where the usual ordering for the trapped ion mode breaks down, i.e., the "collisionless" regime.<sup>(2)</sup>

In Section II, the dispersion relations for the trapped ion and collisionless trapped particle instabilities are presented as derived from a simplified local model. In Section III, the properties of the collisionless trapped particle instability are discussed. The linear growth rate for the trapped ion mode under the usual ordering in collisionality is discussed in Section IV. An expression for the growth rate at lower collisionality is then derived and an approximate solution to the linear growth rate, valid for all ranges of collisionality, is obtained.

The effects of temperature gradients, ion Landau damping and non local approximations on the linear growth of the trapped ion mode are discussed in Section V. Finally, the difficulties associated with evaluating the diffusion and conduction coefficients in terms of the linear growth rates are discussed in Section VI.

## II. Dispersion Equation

The dispersion equation for the dissipative trapped ion instability as derived by Kadomtsev and Pogutse<sup>(2,3)</sup> is given by:

$$\left( \frac{1}{kT_e} + \frac{1}{kT_i} \right) \frac{1}{\sqrt{\epsilon}} = \frac{\omega + i \nu_{if}^*}{\omega + i \nu_{if} + \omega_{mi}} \frac{1}{kT_i} + \frac{\omega - i \nu_{ef}^*}{\omega + i \nu_{ef} - \omega_{me}} \frac{1}{kT_e} \quad (1)$$

Equation (1) can either be derived from a local kinetic model<sup>(2)</sup> or a fluid model<sup>(3,4)</sup>, neglecting the effects of Landau damping by circulating particles.

The parameters in the above equation are:

$$\epsilon = \frac{r}{R_o} \quad \text{-- local inverse aspect ratio}$$

$$\nu_{jf} = \frac{\nu_j}{\epsilon} \quad \text{-- effective collision frequencies}$$

$$\omega_j^* \equiv \frac{\ell q}{r a_n} \frac{kT_j}{eB_T} \quad \text{-- diamagnetic drift frequency} \quad (2)$$

$$\omega_{mj} \equiv \frac{\ell q}{r} \frac{kT_j}{eB_T R_o} \quad \text{-- magnetic drift frequency} \quad (3)$$

$\ell$  - toroidal mode number

$B_T$  - toroidal field

$$q = \epsilon \frac{B_T}{B_p}$$

$B_p$  - poloidal field

$$a_n = \left( - \frac{1}{n} \frac{dn}{dr} \right)^{-1} \quad \text{-- density gradient scale length}$$

$$\frac{\omega_{mj}}{\omega_j^*} = \frac{a_n}{R_o} \quad (4)$$

The dispersion relation for the collisionless trapped particle mode is the same as eqn. (1) in the limit of  $v_i v_e \rightarrow 0$ .

$$\left[ \frac{1}{kT_e} + \frac{1}{kT_i} \right] \frac{1}{\sqrt{\epsilon}} = \frac{\omega + \omega_i^*}{\omega + \omega_{mi}} \frac{1}{kT_i} + \frac{\omega - \omega_e^*}{\omega - \omega_{me}} \frac{1}{kT_e} \quad (5)$$

The growth rate for the trapped ion mode should transform smoothly to the growth rate for the collisionless trapped particle mode as collisions are decreased.

It is important to note that equations (1) and (5) were derived under the assumption  $\omega_{mj} \ll |\omega + i v_{jf}|$ ; thus, stabilization criteria at low collisionality should be examined with this in mind. Recent work by Tang and Adam<sup>(5)</sup> and Tang, Adam and Ross<sup>(6)</sup> includes higher order terms in  $\omega_m$  using an orbit model developed by Pogutse<sup>(7)</sup>. The ordering used in this paper, however, is consistent with the results of recent divertor studies which indicates density gradients in the outer regions of the plasma which are sharp compared with both magnetic field gradients and temperature gradients. First, some of the properties of the collisionless trapped particle mode will be discussed using eqn. (5), then the dissipative trapped ion mode will be examined.

### III. Collisionless Trapped Particle Instability

Let  $\xi = T_e/T_i$  in eqn. (5) and use  $\omega_{mi}$  and  $\omega_i^*$  as the parameters, then:

$$\omega_e^* = \xi \omega_i^* = \xi \omega^*$$

$$\omega_{me} = \xi \omega_{mi} = \xi \omega_m$$

$$\frac{1+\xi}{\sqrt{\epsilon}} = \frac{\omega + \omega_m^*}{\omega + \omega_m} + \frac{\omega - \xi \omega_i^*}{\omega - \xi \omega_m}$$

$$\omega^2 + (1-\xi)\omega_m + \frac{\xi \omega_m (\sqrt{\epsilon} \omega^* - \omega_m)}{1 - \sqrt{\epsilon}} = 0 \quad (6)$$

Let  $\omega = \omega_r + i\gamma$  and separate the real and imaginary components of the equation.  
 $\omega_r$  and  $\gamma$  should both be real and  $\gamma$  must be positive for growing waves.

$$\text{Real: } \omega_r^2 - \gamma^2 + (1-\xi)\omega_m \omega_r + \frac{\xi \omega_m (\sqrt{\epsilon}\omega^* - \omega_m)}{1 - \sqrt{\epsilon}} = 0 \quad (7)$$

$$\text{Imaginary: } 2\omega_r \gamma + (1-\xi)\omega_m \gamma = 0 \quad (8)$$

For  $\gamma \neq 0$ , eqn. (8) gives the solution for  $\omega_r$  which can then be substituted into eqn. (7) to yield a solution for  $\gamma$ .

$$\begin{aligned} \omega_r &= - \frac{(1-\xi)\omega_m}{2} \\ \omega_r &= \frac{\omega_{me} - \omega_{mi}}{2} \end{aligned} \quad (9)$$

$$\frac{(1-\xi)^2 \omega_m^2}{4} - \gamma^2 - \frac{(1-\xi)^2 \omega_m^2}{2} + \frac{\xi \omega_m (\sqrt{\epsilon}\omega^* - \omega_m)}{1 - \sqrt{\epsilon}} = 0$$

$$\gamma_o^2 \equiv \gamma^2 = - \frac{(1-\xi)^2 \omega_m^2}{4} + \frac{\xi \omega_m (\sqrt{\epsilon}\omega^* - \omega_m)}{1 - \sqrt{\epsilon}}$$

$$\gamma_o^2 = - \frac{(\omega_{mi} - \omega_{me})^2}{4} + \frac{\omega_{me} (\sqrt{\epsilon}\omega_i^* - \omega_{mi})}{1 - \sqrt{\epsilon}} \quad (10)$$

$$\text{or } \gamma_o^2 = - \frac{(\omega_{mi} - \omega_{me})^2}{4} + \frac{1}{2(1-\sqrt{\epsilon})} [ \sqrt{\epsilon}(\omega_{me}\omega_i^* + \omega_{mi}\omega_e^*) - 2\omega_{mi}\omega_{me} ] \quad (10a)$$

The solution given by eqns. (9) and (10) is valid only when  $\gamma_o^2 > 0$ . If  $\gamma_o^2 < 0$  in equation (10) the correct solution is given by:

$$\gamma = 0 \quad (11)$$

$$\omega_r = - \frac{(1-\xi)\omega_m}{2} \pm \left[ \frac{(1-\xi)^2 \omega_m^2}{4} - \frac{\xi \omega_m (\sqrt{\epsilon}\omega^* - \omega_m)}{1 - \sqrt{\epsilon}} \right]^{1/2}$$

$$\omega_r = \frac{(\omega_{me} - \omega_{mi})}{2} \pm (-\gamma_o^2)^{1/2} \quad (12)$$

For equal temperatures and  $\varepsilon \ll 1$  eqns. (9-12) reduce to:

$$\gamma_o^2 > 0: \quad \omega_r = 0 \quad (13)$$

$$\gamma^2 = \gamma_o^2 = \omega_m (\sqrt{\varepsilon} \omega^* - \omega_m) \quad (14)$$

$$\gamma_o^2 < 0: \quad \gamma = 0 \quad (15)$$

$$\omega_r = \pm (-\gamma_o^2)^{1/2} \quad (16)$$

Figure 1 shows plots of  $\gamma$  and  $\omega_r$  as a function of  $\sqrt{\varepsilon} \omega^*$  for the equal temperature case given by eqns. (13-16). For  $\sqrt{\varepsilon} \omega^* > \omega_m$  the solution is aperiodic, i.e., there is no real component to the frequency. For  $\sqrt{\varepsilon} \omega^* < \omega_m$ ,  $\gamma = 0$ , and the wave is neither damped or growing. The conditions for stabilization is

$$\frac{\omega_m}{\sqrt{\varepsilon} \omega^*} \geq 1$$

which is in violation of the assumption made in deriving the dispersion equation in the first place,  $\omega_m \ll \sqrt{\varepsilon} \omega^*$ .

For unequal temperatures, eqn. (10a) shows that  $\gamma^2$  is completely symmetric with respect to ions and electrons. Ions and electrons play equal roles in contributing to both the growth and stabilizing terms. For large temperature differences, Pogutse (7) argues the mode can be stabilized:

$$\xi \ll 1, T_i \gg T_e: \quad \gamma^2 = \omega_{mi} [\sqrt{\varepsilon} \omega_e^* - \frac{\omega_{mi}}{4}] \quad (17)$$

$$\text{stable for: } \frac{\omega_{mi}}{4} \geq \sqrt{\varepsilon} \omega_e^*$$

$$\text{or: } \frac{T_i}{T_e} = \frac{1}{\xi} \geq \frac{4}{\sqrt{\varepsilon}} \frac{r}{an}$$



$$\xi \gg 1, T_e \gg T_i: \quad \gamma^2 \approx \omega_{me} \left[ \sqrt{\epsilon} \omega_i^* - \frac{\omega_{me}}{4} \right] \quad (18)$$

which is stable for:  $\frac{\omega_{me}}{4} \geq \sqrt{\epsilon} \omega_i^*$

or:  $\frac{T_e}{T_i} = \xi \geq \frac{4}{\sqrt{\epsilon}} \frac{r}{a_n}$

The mode is therefore unstable in the temperature range, <sup>(7)</sup>

$$\frac{\sqrt{\epsilon} \left( \frac{a_n}{r} \right) < \frac{T_e}{T_i} < \frac{4}{\sqrt{\epsilon}} \left( \frac{r}{a_n} \right) \quad (19)$$

For reasonably sharp density gradients, a large temperature difference is required for stabilization of the mode. The stability conditions are outside the range of validity of the original dispersion relation since they require  $|\omega^2| \ll \omega_m^2$  while the opposite assumption was made in the derivation.

Figure 2 shows plots of  $\omega_r$  and  $\gamma$  from eqns. (9) and (10) for  $\epsilon = .2$  and  $a_n/r = 1, .1$ . To show the symmetry in temperature,  $\gamma$  and  $\omega_r$  are normalized to  $\omega_{me}$  for  $T_e > T_i$  and  $\omega_{mi}$  for  $T_i > T_e$ . For  $a_n/r = 1$ , the mode is stabilized for  $T_e$  and  $T_i$  differing by an order of magnitude while for  $a_n/r = .1$ , more than two orders of magnitude difference in temperature is required for stabilization. For current devices  $\xi \sim 3$  and for larger Tokamaks  $\xi \sim 1$  is expected. Stabilization of the mode with unequal temperatures would be difficult even for modest density gradients.

#### IV. Dissipative Trapped Ion Instability

Using the same notation as in the previous section, eqn. (1) can be expanded to give:

$$\begin{aligned}
 & (1 - \sqrt{\epsilon})\omega^2 + i \left[ v_{ef} \left(1 - \frac{\sqrt{\epsilon}\xi}{1+\xi}\right) + v_{if} \left(1 - \frac{\sqrt{\epsilon}}{1+\xi}\right) \right] \omega + (1 - \sqrt{\epsilon})(1-\xi)\omega_m \omega \\
 & + i (v_{ef} - \xi v_{if})\omega_m - v_{if} v_{ef} - i \frac{\sqrt{\epsilon}\xi (v_{ef} - v_{if})\omega^*}{1 + \xi} \\
 & + \xi \omega_m (\sqrt{\epsilon}\omega^* - \omega_m) = 0
 \end{aligned}$$

Drop the  $v_{if}$  terms where they are added to or subtracted from  $v_{ef}$  since

$$v_{if} \simeq \xi^{3/2} \left(\frac{m_e}{m_i}\right)^{1/2} v_{ef}, \quad \xi \simeq 0(1)$$

Define

$$\omega_o \equiv \frac{\sqrt{\epsilon} \xi \omega^*}{1 + \xi} = \frac{\sqrt{\epsilon} \omega_e^*}{1 + T_e/T_i} \quad (21)$$

Let  $\xi = 1$  in the  $\sqrt{\epsilon}$  coefficient of the second term. Then eqn (20) reduces to:

$$\begin{aligned}
 & (1 - \sqrt{\epsilon})\omega^2 + i (1 - \sqrt{\epsilon}/2) v_{ef} \omega + (1 - \sqrt{\epsilon})(1-\xi)\omega_m \omega - v_{if} v_{ef} \\
 & - i v_{ef} (\omega_o - \omega_m) + \omega_m [(1+\xi)\omega_o - \xi\omega_m] = 0
 \end{aligned} \quad (22)$$

Let  $\omega = \omega_r + i\gamma$  and separate the real and imaginary components of eqn.(22):

$$\begin{aligned}
 \text{Real: } & (1 - \sqrt{\epsilon})(\omega_r^2 - \gamma^2) - (1 - \sqrt{\epsilon}/2)v_{ef} \gamma + (1 - \sqrt{\epsilon})(1-\xi)\omega_m \omega_r \\
 & - v_{if} v_{ef} + \omega_m [(1+\xi)\omega_o - \xi\omega_m] = 0
 \end{aligned} \quad (23)$$

Imaginary:  $2(1 - \sqrt{\epsilon})\omega_r \gamma + (1 - \sqrt{\epsilon}/2)v_{ef}\omega_r + (1 - \sqrt{\epsilon})(1-\xi)\omega_m \gamma$

$$- v_{ef} (\omega_o - \omega_m) = 0 \quad (24)$$

The imaginary component can be solved for  $\omega_r$  in terms of  $\gamma$  and then substituted into eqn. (23) to obtain an equation for  $\gamma$ .

$$\omega_r = \frac{v_{ef} (\omega_o - \omega_m) - (1-\sqrt{\epsilon})(1-\xi)\omega_m \gamma}{2(1-\sqrt{\epsilon})\gamma + (1-\sqrt{\epsilon}/2)v_{ef}} \quad (25)$$

An approximate solution can be obtained from equations (23) and (25) under two different orderings of terms. A "high collisionality" approximation with  $v_{ef} \gg \gamma$  gives the growth rate in the trapped ion mode. A "low collisionality" approximation with  $v_{ef} \ll \gamma$  adds a collisional growth term to the growth rate for the collisionless trapped particle instability.

(a) "High Collisionality"

With the usual ordering for the trapped ion mode<sup>(2)</sup>,  $v_{ef} \gg \gamma$ , eqn.(25) reduces to:

$$\omega_r = \frac{\omega_o - \omega_m}{1 - \sqrt{\epsilon}/2} \quad (26)$$

Substitution into eqn. (23) and using  $(1-\sqrt{\epsilon}/2)^2 \sim (1-\sqrt{\epsilon})$  gives

$$(\omega_o - \omega_m)^2 - (1-\sqrt{\epsilon})\gamma^2 - (1-\sqrt{\epsilon}/2)v_{ef} \gamma + (1-\sqrt{\epsilon}/2)(1-\xi)\omega_m(\omega_o - \omega_m)$$

$$- v_{ef} v_{if} + \omega_m [(1+\xi)\omega_o - \xi \omega_m] = 0$$

$$(1-\sqrt{\epsilon})\gamma^2 + (1-\sqrt{\epsilon}/2)v_{ef} \gamma + v_{ef} v_{if} - \omega_o^2 + \frac{\sqrt{\epsilon}}{2} (1-\xi)\omega_m(\omega_o - \omega_m) = 0$$

The last term is small compared to  $\omega_o^2$  since  $\omega_m \ll \omega_o$  and  $\sqrt{\epsilon} \ll 1$ . The  $\gamma^2$  term is small compared to the second term since  $\gamma \ll v_{ef}$ .

$$\gamma = \frac{1}{1-\sqrt{\epsilon}/2} \left[ \frac{\omega_o^2}{v_{ef}} - v_{if} \right] \quad (27)$$

$$\gamma_H \equiv \gamma = \frac{1}{1-\sqrt{\epsilon}/2} \left[ \left( \frac{\epsilon \omega_e^*}{1+T_e/T_i} \right)^2 \frac{1}{v_e} - \frac{v_i}{\epsilon} \right] \quad (28)$$

The stabilization condition for the mode can be obtained from eqn. (27) or (28) and is given by:

$$\begin{aligned} \frac{\omega_o^2}{v_{ef}} &\leq v_{if} \\ v_{ef} v_{if} &\geq \omega_o^2 \\ v_e v_i &\geq \frac{\epsilon^3 \omega_e^{*2}}{(1+T_e/T_i)^2} \end{aligned} \quad (29)$$

The solution for  $\gamma$  was obtained under the ordering  $v_{ef} \gg \gamma$  and therefore is valid only for:

$$\begin{aligned} \gamma &\approx \frac{\omega_o^2}{v_{ef}} - v_{if} \ll \frac{\omega_o^2}{\gamma} \\ \gamma &\ll \omega_o \end{aligned} \quad (30)$$

When  $v_{ef} < \omega_o$  then  $\gamma > \omega_o$  and the ordering breaks down. For  $v_{ef} < \omega_o$  a different ordering must be used.

#### (b) "Low Collisionality"

For  $v_{ef} \ll \gamma$  a collisional growth term can be obtained for the "collisionless" trapped particle instability. Eqn. (25) reduces to:

$$\omega_r = - \frac{(1-\xi)\omega_m}{2} + \frac{(\omega_o - \omega_m)}{(1-\sqrt{\epsilon})} \frac{v_{ef}}{2\gamma} \quad (31)$$

Substitution into eqn (23) and rearranging terms gives:

$$(1-\sqrt{\epsilon})\gamma^2 + (1-\sqrt{\epsilon}/2) v_{ef}\gamma - \frac{(\omega_o - \omega_m)^2}{1-\sqrt{\epsilon}} \frac{v_{ef}^2}{4\gamma^2} + v_{if} v_{ef} - (1-\sqrt{\epsilon})\gamma_o^2 = 0 \quad (32)$$

where  $\gamma_o^2$  is given in eqn (10) and represents the collisionless growth rate. The ion collision term can be neglected in eqn. (32). The first collision term decreases  $\gamma$  from  $\gamma_o$  as collisions are added while the second collision term increases  $\gamma$  over  $\gamma_o$  as  $v_{ef}$  is increased. The growth term is dominant when  $v_{ef}/\gamma \gg \omega_m/\omega_o$  and  $\omega_m \ll \omega_o$ . Under this ordering, eqn. (32) becomes quadratic in  $\gamma^2$ :

$$\gamma^4 - \gamma_o^2 \gamma^2 - \frac{(\omega_o - \omega_m)^2}{(1-\sqrt{\epsilon})^2} \frac{v_{ef}^2}{4} = 0 \quad (33)$$

$$\gamma_L^2 = \gamma^2 = \frac{\gamma_o^2}{2} + \left[ \frac{\gamma_o^4}{4} + \left( \frac{(\omega_o - \omega_m) v_{ef}}{2(1-\sqrt{\epsilon})} \right)^2 \right]^{1/2} \quad (34)$$

Equation (34) would tend to overestimate the growth of the mode since the terms dropped in eqn. (32) both decrease the growth slightly. Figure 3 shows plots of  $\gamma_H/\omega_o$  and  $\gamma_L/\omega_o$  vs.  $v_{ef}/\omega_o$  (eqns. (27) and (34)) for  $\epsilon = .2$  and  $a_n/r = .001$ . Also shown is a plot of  $\gamma_H/\omega_o$  neglecting the ion collision term. The ion collision term becomes dominant for  $v_{ef}/\omega_o \gtrsim 5$  and completely stabilizes the mode at  $v_{ef}/\omega_o \approx 8$ . A smooth fit to the "high" and "low" collisionality regimes can be made by using

$$\gamma_T = \frac{\gamma_H \gamma_L}{\gamma_H + \gamma_L} \quad (35)$$

$\gamma_T/\omega_o$  is also shown in Figure 3. Note that the approximation  $\gamma \approx \omega_o^2/v_{ef}$ , which is often used for the linear growth rate for the trapped ion mode, is only approximately valid in a very small range of collisionality.

(c) Numerical Solution of Dispersion Relation

Eqns. (23) and (25) can be solved quite easily numerically to determine how good the fit for  $\gamma_T$ , eqn. (35), actually is. The solution can be expressed in terms of a normalized growth rate vs. a normalized collision frequency with the parameters  $\epsilon = r/R$ ,  $v_i/v_e$ ,  $a_n/r$  and  $\xi = T_e/T_i$ . Designate  $\beta$  as the normalizing frequency and define the following dimensionless parameters:

$$\Omega = \frac{(1-\sqrt{\epsilon}/2)\omega_r}{\beta} \quad (36)$$

$$\Gamma = \frac{(1-\sqrt{\epsilon}/2) \gamma}{\beta} \quad (37)$$

$$N_e = \frac{v_{ef}}{\beta}, \quad N_i = \frac{v_{if}}{\beta} \quad (38)$$

For equal temperatures eqns. (23) and (25) reduce to:

$$\Omega^2 - \Gamma^2 - N_e \Gamma - N_e N_i + (1-\sqrt{\epsilon})\gamma_o^2/\beta^2 = 0 \quad (39)$$

$$\Omega = \frac{N_e (\omega_o/\beta - \omega_m/\beta)}{2\Gamma + N_e} \quad (40)$$

where  $\gamma_o$  is the collisionless growth rate. If  $\beta$  is chosen as  $\omega_o$ ,  $\omega_m$  or  $\gamma_o$ , the expressions  $\omega_o/\beta$ ,  $\omega_m/\beta$  and  $\gamma_o/\beta$  either reduce to unity or functions of  $\epsilon$  and  $a_n/r$ . Furthermore,

$$N_i = \left( \frac{m_e}{m_i} \right)^{1/2} N_e \text{ for } T_e = T_i$$

$$\approx .015 N_e$$

Substituting eqn. (40) into eqn. (39) and expanding leads to a fourth order equation in  $\Gamma$ :

$$\begin{aligned}
 & 4\Gamma^4 + 8N_e\Gamma^3 + \left(5 + 4\frac{N_i}{N_e} - \frac{4}{N_e^2} \frac{(1-\sqrt{\epsilon})\gamma_o^2}{\beta^2}\right) N_e^2\Gamma^2 \\
 & + \left(1 + 4\frac{N_i}{N_e} - \frac{4}{N_e^2} \frac{(1-\sqrt{\epsilon})\gamma_o^2}{\beta^2}\right) N_e^3\Gamma + N_e^3N_i \\
 & - \left(\frac{\omega_o}{\beta} - \frac{\omega_m}{\beta}\right)^2 N_e^2 - \frac{(1-\sqrt{\epsilon})\gamma_o^2}{\beta^2} N_e^2 = 0
 \end{aligned} \tag{41}$$

Dividing through by  $N_e^4$  and rearranging yields:

$$\begin{aligned}
 & 4\left(\frac{\Gamma}{N_e}\right)^4 + 8\left(\frac{\Gamma}{N_e}\right)^3 + \left(5 + 4\frac{N_i}{N_e}\right)\left(\frac{\Gamma}{N_e}\right)^2 + \left(1 + 4\frac{N_i}{N_e}\right)\left(\frac{\Gamma}{N_e}\right) + \frac{N_i}{N_e} \\
 & = \frac{1}{N_e^2} \left[ 4\frac{(1-\sqrt{\epsilon})\gamma_o^2}{\beta^2} \frac{\Gamma}{N_e} \left(1 + \frac{\Gamma}{N_e}\right) + \left(\frac{\omega_o}{\beta} - \frac{\omega_m}{\beta}\right)^2 + \frac{(1-\sqrt{\epsilon})\gamma_o^2}{\beta^2} \right]
 \end{aligned} \tag{42}$$

For a given positive value of  $\Gamma/N_e = (1-\sqrt{\epsilon}/2)\gamma/v_{ef}$  the left-hand side of eqn. (42) and the bracketed term on the right-hand side are known positive quantities and a unique positive value of  $N_e$  can then be obtained.  $\Gamma = (\Gamma/N_e) \cdot N_e$  gives the corresponding normalized growth rate.

Figure 4 shows plots of  $\Gamma$  vs  $N_e$  for the fitted solution, eqn. (35), and the numerical solution of eqn. (42) with  $\beta = (1-\sqrt{\epsilon}/2)\gamma_o$ . The fitted solution is as much as 40% high for intermediate ranges of collisionality but is quite accurate for high and low collisionality. Figure 5 is essentially the same as Figure 4 but the normalization parameter is  $\beta = \omega_o$ . Here it can be seen that the magnetic drift frequency,  $\omega_m$ , has little effect on the growth rate for  $v_{ef} > \omega_o$ , i.e., the usual collisionality regime for the trapped ion mode. The cases shown in Figures 3-5 are all for  $\omega_m \ll 2\omega_o = \sqrt{\epsilon}\omega^*$ . The behavior of  $\gamma$  near

$\omega_m = \omega_o$  is questionable since the original dispersion relation was derived with the ordering  $\omega_m \ll 2\omega_o$ .

#### V. Other Factors Affecting the Growth of the Trapped Ion Mode

In deriving the dispersion relation for the trapped ion instability, Kadomtsev and Pogutse<sup>(2)</sup> used a simple Krook collision model and neglected temperature gradients and ion Landau damping. Sagdeev and Galeev<sup>(8)</sup> later used a Landau collision term for the electrons and found the electron collisional growth term in good agreement with Kadomtsev and Pogutse. The ion collision term was neglected in their analysis. Rosenbluth, Ross and Kostomarov<sup>(9)</sup> used a Fokker-Planck collision term for both electron and ion collisions and included temperature gradients in their analysis. A variational technique was used to obtain the electron collisional growth and ion collisional damping terms:

$$\frac{\gamma_e}{\omega_o} = 1.95(1 + 1.41 \eta_e) \frac{\omega_o}{v_{ef}} \quad (43)$$

$$\frac{\gamma_i}{\omega_o} = -2.54 (1 - .57 \eta_i) \left( \frac{v_{if}}{\omega_o} \right)^{1/2} \left[ \ln \left( \frac{4.9 \omega_o}{v_{if}} \right)^{1/2} \right]^{-3/2} \quad (44)$$

where 
$$\omega_o \equiv \frac{\sqrt{\epsilon} \omega_e^*}{1 + T_e/T_i}$$

$$\omega_e^* \equiv \frac{\ell q}{r} \frac{kT_e}{e B_T} \frac{1}{n} \frac{dn}{dr}$$

$$v_{jf} = \frac{v_j}{\epsilon}$$

$$\eta_j \equiv \frac{d \ln T_j}{d \ln n}$$



The electron collisional growth is roughly twice as large as that given by Kadomtsev and Pogutse for the zero temperature gradient case ( $\eta = 0$ ). However, the ion collision term is a stronger function of  $v_{if}$  so the mode is stabilized at a lower effective collision frequency (see Figure 6).

Figure 7 shows plots of  $\gamma = \gamma_e + \gamma_i$  vs.  $v_{ef}$  for equal electron and ion temperatures and various temperature gradients. As the temperature gradients increase, the growth rate of the mode increases because of greater electron collisional growth and decreased ion collisional damping. For  $\eta_i > 1.75$  the ion collisional damping changes to growth and cannot stabilize the mode.

There are several limits on the range of applicability of equations (43) and (44). First, the electron collisional growth rate should be less than the effective diamagnetic drift frequency,  $\gamma_e < \omega_o$ , which is violated at low collision frequencies,  $v_{ef} < (1-5) \omega_o$ . For these low collision frequencies the mode starts converting to the "collisionless" trapped particle instability as discussed in Section IV-b. Second, the ion collision term has a singularity at  $v_{if}/\omega_o = 4.9$ . When the  $\ln$  term in eqn. (44) becomes less than about unity the approximations made in the derivation become invalid. The ion collision term is then only valid in the regime.

$$\ln \left( 4.9 \frac{\omega_o}{v_{if}} \right) \lesssim 1$$

$$v_{if} \lesssim \frac{\omega_o}{2} \tag{45}$$

This restriction is only necessary for sharp ion temperature gradients since for  $\eta_i \lesssim 1.5$  the mode is stabilized in the regime where eqn. (45) is violated.

Increasing the toroidal mode number,  $\ell$ , leads to larger  $\omega_o$  and larger growth rates for a given effective collision frequency. An upper limit on the toroidal mode number can be obtained from

$$\omega_o < \omega_{bi} \quad (46)$$

$$\omega_{bi} = \left( \frac{2 kT_i}{m_i} \right)^{1/2} \frac{(2\varepsilon)^{1/2}}{R_o q} \quad \text{ion bounce frequency}$$

which is required for existence of the mode. A more restrictive upper limit on  $\ell$  occurs when Landau damping by circulating ions is included in the analysis.

Sagdeev and Galeev<sup>(8)</sup> found that Landau damping is a strong stabilizing term near a rational  $q$  surface and weakly stabilizing between rational  $q$  surfaces.

$$\frac{\gamma_{LD}}{\omega_o} = -1.92 \left( \frac{2 \omega_o}{\omega_{bi} s^{(0)}} \right)^3 \quad (47)$$

where  $s^{(0)} = |m^{(0)} - \ell q|$

and  $m^{(0)}$  is an integer which makes  $s^{(0)}$  smallest. Rosenbluth, Ross and Kostomarov<sup>(9)</sup> included temperature gradients and obtained a minimum value for Landau damping by evaluating  $s^{(0)}$  midway between rational  $q$  surfaces, i.e.,  $s^{(0)} = .5$ .

$$\frac{\gamma_{LD}}{\omega_o} = -1.73 (1 - 1.5 \eta_i) \left( \frac{4 \omega_o}{\omega_{bi}} \right)^3 \quad (48)$$

The zero temperature gradient case agrees well with the results of Sagdeev and Galeev. Both predict  $\gamma_{LD} \propto \ell^4$  while  $\gamma_e \propto \ell^2$ . For  $\eta_i > 2/3$ , eqn. (48) predicts that Landau damping turns to growth and cannot stabilize the mode. This can only be true if the mode is localized in radius to a region much smaller than the distance between mode rational surfaces.

A radially dependent model should be used to better determine the overall effect of ion Landau damping and whether the modes can be localized between mode rational surfaces. The radial equation can be put into the form: <sup>(10-12)</sup>

$$[K^{-2} \frac{d^2}{dx^2} + Q(x, \omega)] \Phi(x) = 0 \quad (49)$$

$$K = \frac{a}{\Delta r_{bi}} \gg 1 \quad (50)$$

$$\Delta r_{bi} = \frac{q \rho_i}{\sqrt{\epsilon}} \quad \text{trapped ion banana width} \quad (51)$$

$$\rho_i = \frac{(2m_i k T_i)^{1/2}}{e B_T} \quad \text{ion gyroradius} \quad (52)$$

$Q$  can be called the radial potential in analogy with the Schrodinger equation <sup>(11)</sup>. There are contributions to  $Q$  from both trapped and untrapped particles which will be designated  $Q_T$  and  $Q_U$  respectively. For the usual ordering in the trapped ion mode regime,  $v_{ef} > \sqrt{\epsilon \omega} \gg \gamma$ , and equal temperatures, the trapped particle contribution to the radial potential can be reduced to: <sup>(12)</sup>

$$Q_T \simeq 1 - \frac{2 \omega}{\sqrt{\epsilon \omega}^*} - \frac{\omega}{i v_{ef}} \quad (53)$$

where ion collisions, temperature gradients and magnetic drift have been neglected. In a local model which neglects untrapped ion contributions, the spatial derivative term is neglected since  $K \gg 1$  and  $Q_T$  is a smooth function. Then  $Q_T = 0$  gives the local dispersion relation.

Jablon, Laval and Pellat <sup>(12)</sup> used a density profile of

$$n \sim n_0 e^{-x^{3.5}} \quad (54)$$

to simplify the radial dependence of  $Q_T$ . Untrapped ions were neglected in their analysis. The radial equation

$$\frac{d^2\Phi}{dx^2} + K_o^2 \left[ x(1+i\gamma) - \frac{\Omega}{x} \right] \Phi = 0 \quad (55)$$

where  $K_o = K(a)$

$$\gamma = \frac{\omega}{v_{ef}}, \quad \Omega = \frac{\omega}{\omega_o(a)}, \quad v_{ef} = \text{const.}$$

was then solved by a WKBJ analysis. A plot of  $Q_T$  vs  $x$  is shown in Figure 8a. For  $x < x_T$  the wave is expected to be spatially damped since  $\text{Re } Q_T < 0$ . The boundary condition at  $x = 0$  was then chosen to be  $\Phi(0) = 0$ . The most pessimistic assumption of perfect reflection was made at  $x = 1$  giving  $\Phi(1) = 0$ . Ross and Horton<sup>(10)</sup> also used a WKBJ analysis and found the lowest radial eigenmode gives growth rates only slightly less than the electron collisional growth from local models. The reduction in growth is due to finite banana width corrections for the trapped ions. For  $K_o^2 \gg 1$ , the correction term would be expected to be small since  $Q_T$  is a smooth function of radius.

The circulating particle contribution to the radial potential is given by Gladd and Ross<sup>(11)</sup> as:

$$Q_U = \frac{\omega + \omega_{mi} + i v_{if}}{\omega} z \{ Z(z) - Z(\sqrt{\epsilon}z) \} + \frac{\omega_e^*}{\omega_e^* + \omega T_e/T_i} \eta_i [ z(1-\sqrt{\epsilon}) + (z^2 - 1/2)Z(z) - (\epsilon z^2 - 1/2)Z(\sqrt{\epsilon}z) ] \quad (56)$$

where 
$$\eta_i = \frac{d \ln T_i}{d \ln n}$$

$$z = \frac{\omega}{k_{\parallel}} \left( \frac{m_i}{2\epsilon k T_i} \right)^{1/2}$$

$$k_{\parallel} = \frac{|m - \ell q|}{q R}$$

Z - Fried & Conte dispersion function

Let  $x_0$  denote the position of a rational  $q$  surface ( $k_{\parallel}(x_0)=0$ ). Plots of the real and imaginary components of  $Q_U$  near  $x_0$  are shown in Figures 8b and 8c for  $\eta_i = 0$  and 1 respectively. Both the real and imaginary components of  $Q_U$  have resonances at  $x_0$ . For  $\eta_i = 0$ ,  $\text{Im } Q_U$  is negative for all values of  $x$  and gives Landau damping of the mode by circulating ions. For  $\eta_i > 2/3$ ,  $\text{Im } Q_U$  is still negative near  $x_0$  but is positive in the wings of the resonance. Ion Landau growth can occur where  $\text{Im } Q_U > 0$  but is dominated by the damping in the resonance. <sup>(11)</sup> Rosenbluth, Ross and Kostomarov <sup>(9)</sup> evaluated the Landau damping term midway between rational  $q$  surfaces where  $\text{Im } Q_U > 0$  for  $\eta_i > 2/3$  which explains why they found Landau damping turning to growth for  $\eta_i > 2/3$ .

Gladd and Ross <sup>(11)</sup> solved the radial equation numerically using eqn. (56) for  $Q_U$ . The local electron and ion collisional terms of Rosenbluth, Ross and Kostomarov <sup>(9)</sup> were used to evaluate  $Q_T$  with a density profile of  $n \sim n_0 e^{-x^{3.5}}$ . Ormak design parameters were used for the plasma. For  $\eta = 0$  the trapped ion instability was found to be stable for all toroidal mode numbers. For  $\eta = 1$ , ion collisional damping was found to stabilize the instability for  $\ell < 3$  and ion Landau damping stabilized the mode for  $\ell \geq 6$ . The maximum growth rate occurred for  $\ell \sim 3-4$  and was less than half the local value of the growth rate without Landau damping.

Gladd and Ross <sup>(9)</sup> also discuss the possibility of using algebraic models to get a qualitative picture of Landau damping. The spatial derivative is

replaced by the radial wave number and the radial potential is replaced by an averaged quantity:

$$-a^2 k_r^2 + K^2 Q(x, \omega) = 0 \quad (57)$$

$$k_r \cong \pi \ell \frac{dq}{dr}$$

$$Q(x, \omega) = Q_T + L \bar{Q}_U$$

$\bar{Q}_U$  is averaged over a spatial range corresponding to  $m-1/2 < \ell q < m+1/2$  and  $L < 1$  represents the weighting of  $Q_U$  with the wave function.

This model predicts stabilization by ion Landau damping for  $\ell \lesssim 7$  in the Ormak model for  $ak_r/K \gtrsim 1$ , i.e., when the trapped ion banana width becomes greater than the spacing between mode rational surfaces. When this happens the mode cannot be sufficiently localized between mode rational surfaces and ion Landau damping becomes dominant. The maximum growth occurs for  $ak_r/K \sim .5$ . For best simulation of the numerical results, Gladd and Ross found that  $L$  should be an increasing function of the toroidal mode number  $\ell$ . For the Ormak model

$$L \sim 1 - e^{-(D/6)^2}$$

gave the best results where  $D$  is the number of mode rational surfaces between the turning point and the outside of the torus.

The algebraic model discussed above breaks down when  $ak_r \gtrsim K$  and thus should not be used to predict stabilization of the trapped ion mode. It does however give a reasonably good qualitative picture of the effects of ion Landau

damping for  $ak_r/K < 1$ . Gladd and Ross use another model valid for  $ak_r/K > 1$  which will not be discussed here.

In general the diffusion will be dominated by intermediate values of  $\ell$  where

$$\frac{ak_r}{K} \approx .5$$

or 
$$\ell_{\max} \approx \frac{.5}{\pi \frac{dq}{dr} \Delta r_{bi}} \quad (58)$$

gives the maximum growth rate. The maximum growth rate from the radial model is smaller than the growth rate from linear theory for the same mode number.

#### VI. Turbulent Diffusion and Conduction Coefficients

From a strong turbulence analysis<sup>(13-15)</sup>, the diffusion coefficient can be expressed as:

$$D_{\perp} \approx \sqrt{\epsilon} \frac{\bar{\gamma}}{\bar{k}_r} 2 \quad (59)$$

where  $\bar{\gamma}$  and  $\bar{k}_r$  are the growth rate and radial wave number averaged over the nonlinear spectrum. Ignorance of the spectral distribution makes  $D_{\perp}$  impossible to evaluate in Eqn. (59). Dupree<sup>(14)</sup> suggests that at saturation, a single mode may dominate the diffusion and stabilize all other modes. For a given mode, the radial wave number is determined by the distance between rational surfaces:

$$k_r \approx \pi \ell \frac{dq}{dr} \quad (60)$$

In Figure 9 the normalized diffusion coefficient for several toroidal mode numbers is plotted vs. the effective electron collision frequency for equal temperatures and a typical set of plasma parameters. The results of Rosenbluth, Ross and Kostomarov were used for the electron and ion collisional growth terms,

eqns. (43-44), in the trapped ion mode regime. The results of Section IV-b of this paper were used for the growth rate at lower collision frequencies.

At low collision frequencies, the lowest toroidal mode numbers give the lowest linear growth rates but the highest diffusion coefficients. At higher collision frequencies,  $v_{ef}/\omega_{o1} > 5$ , the largest growth rate is determined by the largest allowable toroidal mode number (limited by Landau damping). However, the largest diffusion coefficient is determined by some intermediate value of the toroidal mode number, i.e., a toroidal mode number large enough such that ion collisional damping is negligible ( $v_{ef}/\omega_o \ll 10$ ) but small enough such that the mode is still in the trapped ion mode regime ( $v_{ef}/\omega_o \gg 1$ ). The maximum growth rate then occurs for  $v_{ef}/\omega_o \simeq 5$ . Since  $\omega_o = \ell \omega_{o1}$ , the mode number, which gives the largest diffusion coefficient, is then given by:

$$\ell \simeq \frac{1}{5} \frac{v_{ef}}{\omega_{o1}}$$

The diffusion coefficient in the trapped ion mode regime is bounded by

$$D_{\max} \simeq \frac{\sqrt{\epsilon} \gamma_e}{k_r^2} \quad (61)$$

which is independent of the toroidal mode number since the electron collisional growth rate,  $\gamma_e$ , and  $k_r^2$  are both proportional to  $\ell^2$ .

A nonlinear analysis of the evolution of the modes is required to determine which mode will dominate the diffusion if indeed only one mode is responsible for the diffusion at saturation. For turbulent drift waves, Dupree<sup>(14)</sup> argues



that the modes with the largest values of  $\gamma/k_{\perp}^2$  will dominate the diffusion. In a subsequent paper on low frequency instabilities, Dupree <sup>(15)</sup> cites experimental support of a single dominant mode. He argues that the mode which requires the largest amplitude at saturation will stabilize the other modes. It is not necessarily the mode with the largest linear growth rate. However, the mode which requires the largest amplitude at saturation also yields the largest diffusion coefficient. Waddell <sup>(16)</sup> argues that the dominant mode for the collisionless trapped particle instability will be the one with the largest linear growth rate. Although his analysis seems to apply for a fixed radial wave number, the diffusion coefficient he finds is still given by eqn. (59).

Even if there is a single dominant mode at each localized position in the plasma, the question still arises as to whether there will be a single mode responsible for the diffusion over the entire region where the trapped ion/collisionless mode exists. Rather than using a single mode, a safe upper limit on the diffusion in the trapped ion mode regime would still be given by eqn. (61) and in the "collisionless" regime by the  $\ell = 1$  mode.

Dobrowolny and Nocentini <sup>(17)</sup> evaluated the particle and heat fluxes for the trapped ion mode assuming small amplitude electrostatic fluctuations. Their analysis admittedly is not valid when the mode becomes strongly turbulent but does indicate general relationships between particle and heat fluxes during the early stages of evolution of the mode. The ion and electron heat fluxes were found to be comparable to the particle flux. This seems to rule out an enhanced electron or ion thermal conductivity similar to the enhanced ion thermal conductivity for classical transport.

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REFERENCES

- (1) A. T. Mense, Ph.D. thesis, Nuclear Engineering Dept., Univ. of Wis.-Madison, 1976.
- (2) B. B. Kadomtsev and O. P. Pogutse, Sov. Phys. JETP 24, 1172 (1967).
- (3) B. B. Kadomtsev, Sov. Phys. JETP Lett. 4, 10 (1966).
- (4) A. T. Mense, Univ. of Wis. Fusion Design Memo 133, (1975).
- (5)** W. M. Tang and J. C. Adam, Bull. Am. Phys. Soc. 20, 1288 (1975).
- (6) W. M. Tang, J. C. Adam and D. W. Ross, Proc. of Annual Meeting on Theoretical Aspects of Controlled Thermonuclear Research (Sherwood Theory Meeting - Madison, Wisconsin) April 1976, paper 1B-18.
- (7) O. P. Pogutse, Nucl. Fusion 9, 157 (1969).
- (8) R. Z. Sagdeev and A. A. Galeev, Sov. Phys. Dokl. 13, 562 (1968).
- (9) M. N. Rosenbluth, D. W. Ross and D. P. Kostomarov, Nucl. Fusion 12, 3(1972).
- (10) D. W. Ross and C. W. Horton, Jr., Phys. Rev. Lett. 28, 484 (1972).
- (11) N. T. Gladd and D. W. Ross, Phys. Fluids, 16, 1706 (1973).
- (12) C. Jablon, G. Laval and R. Pellat, Phys. Rev. Lett. 27, 83 (1971).
- (13) T. H. Dupree, Phys. Fluids 9, 1773 (1966).
- (14) T. H. Dupree, Phys. Fluids 10, 1049 (1967).
- (15)** T. H. Dupree, Phys. Fluids 11, 2680 (1968).
- (16) B. V. Waddell, Nucl. Fusion 15, 803 (1975).
- (17) M. Dobrowolny and A. Nocentini, Nucl. Fusion Let. 13, 629 (1973).

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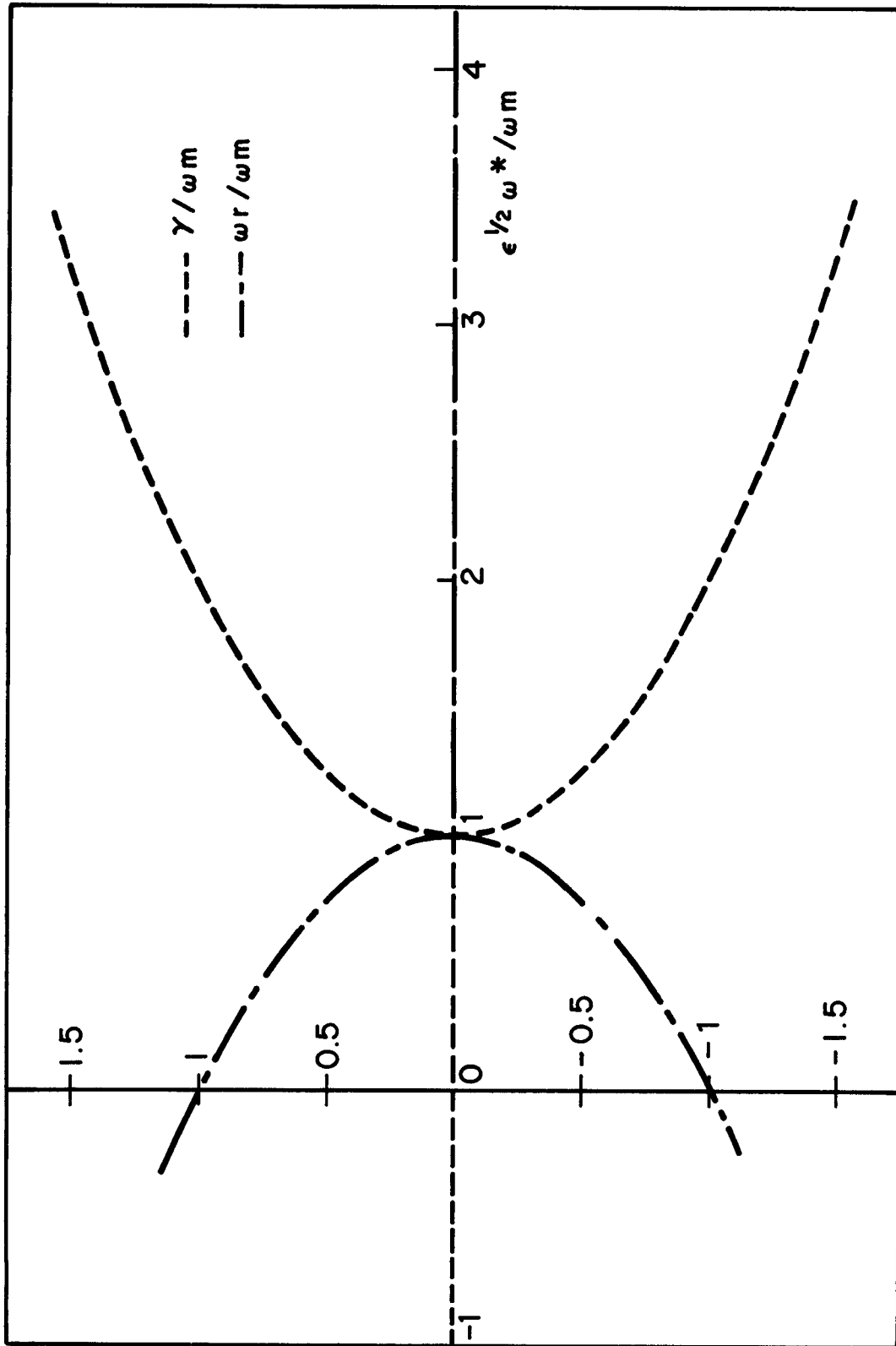
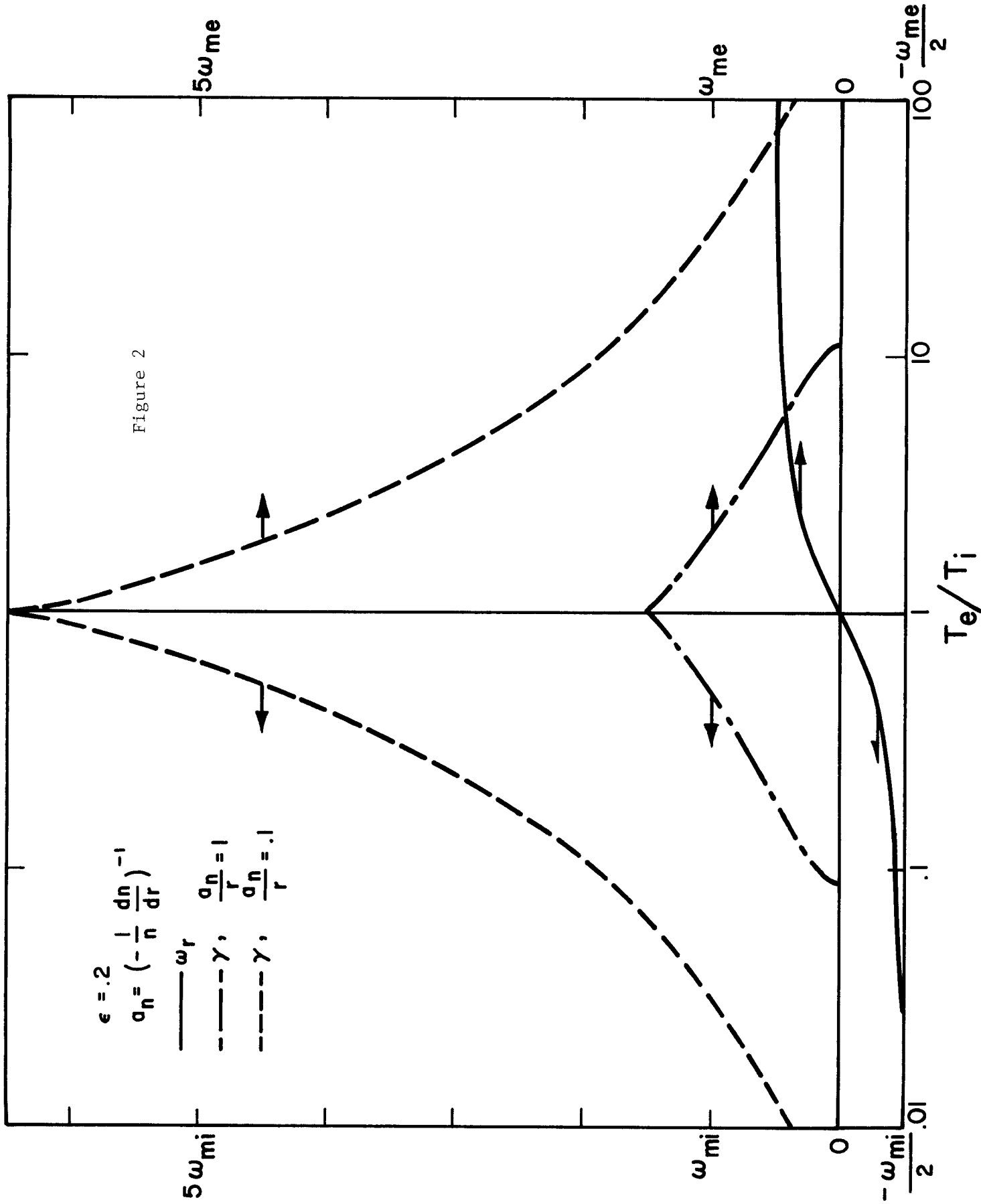
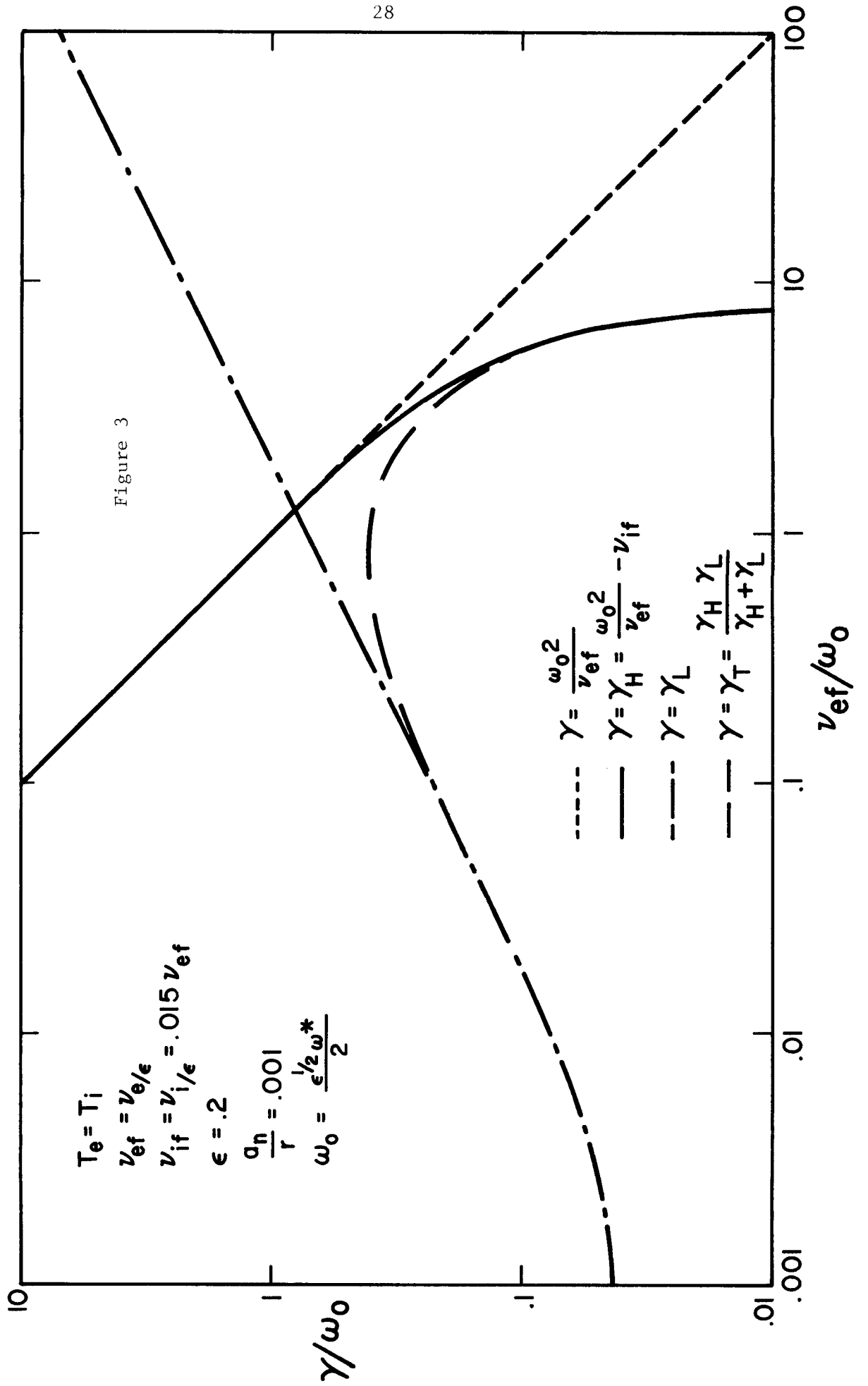
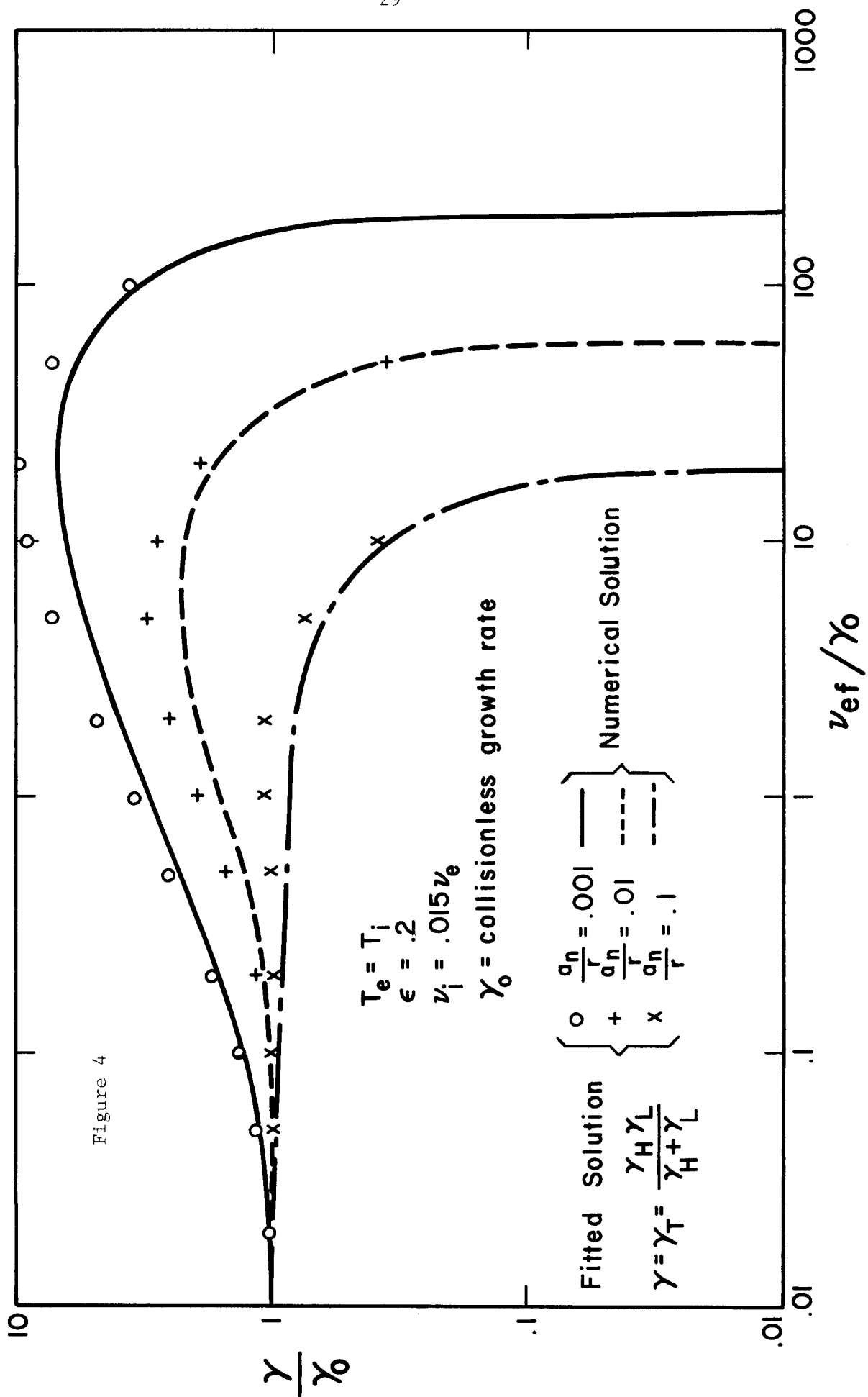


Figure 1









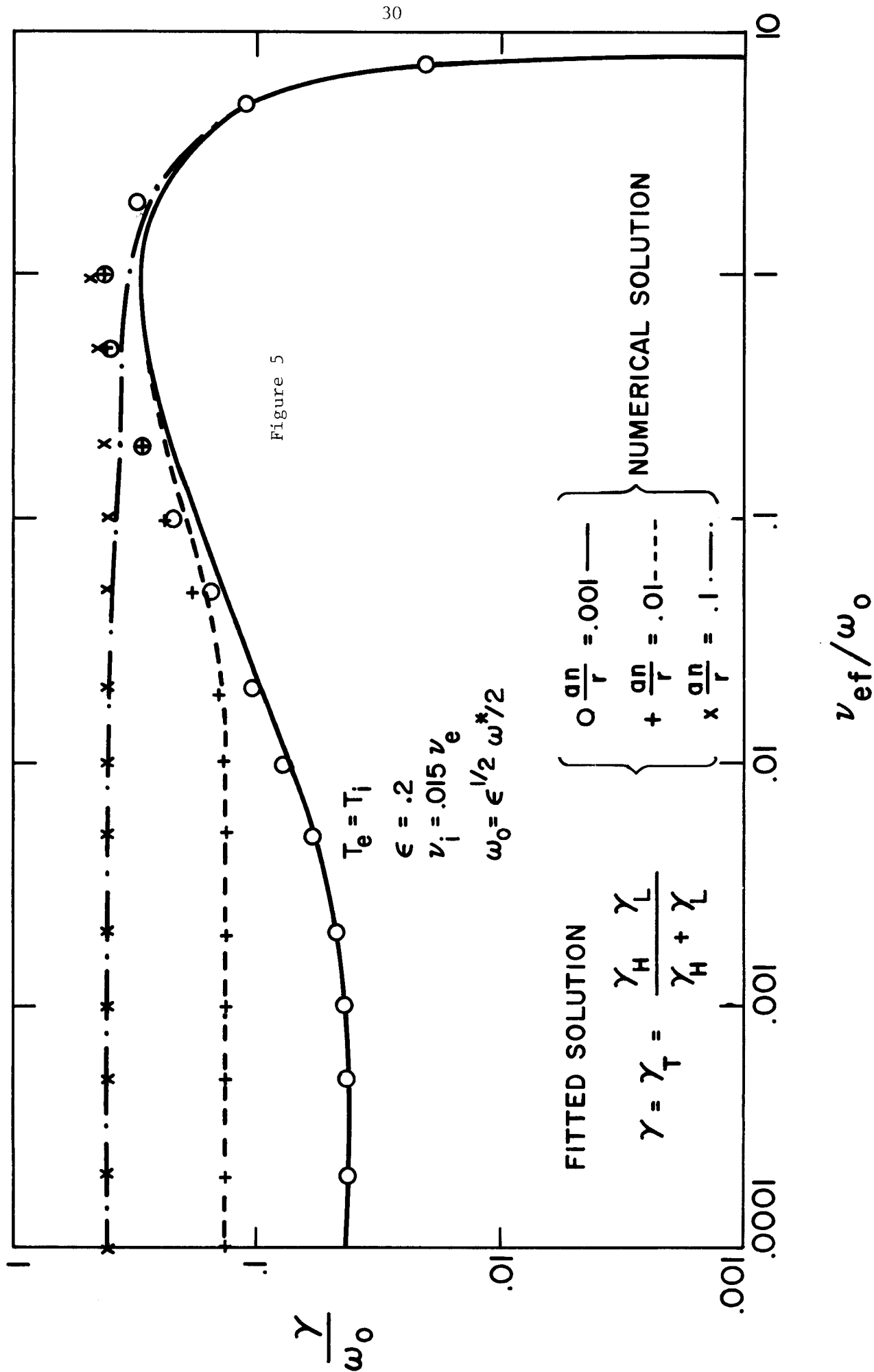


Figure 6

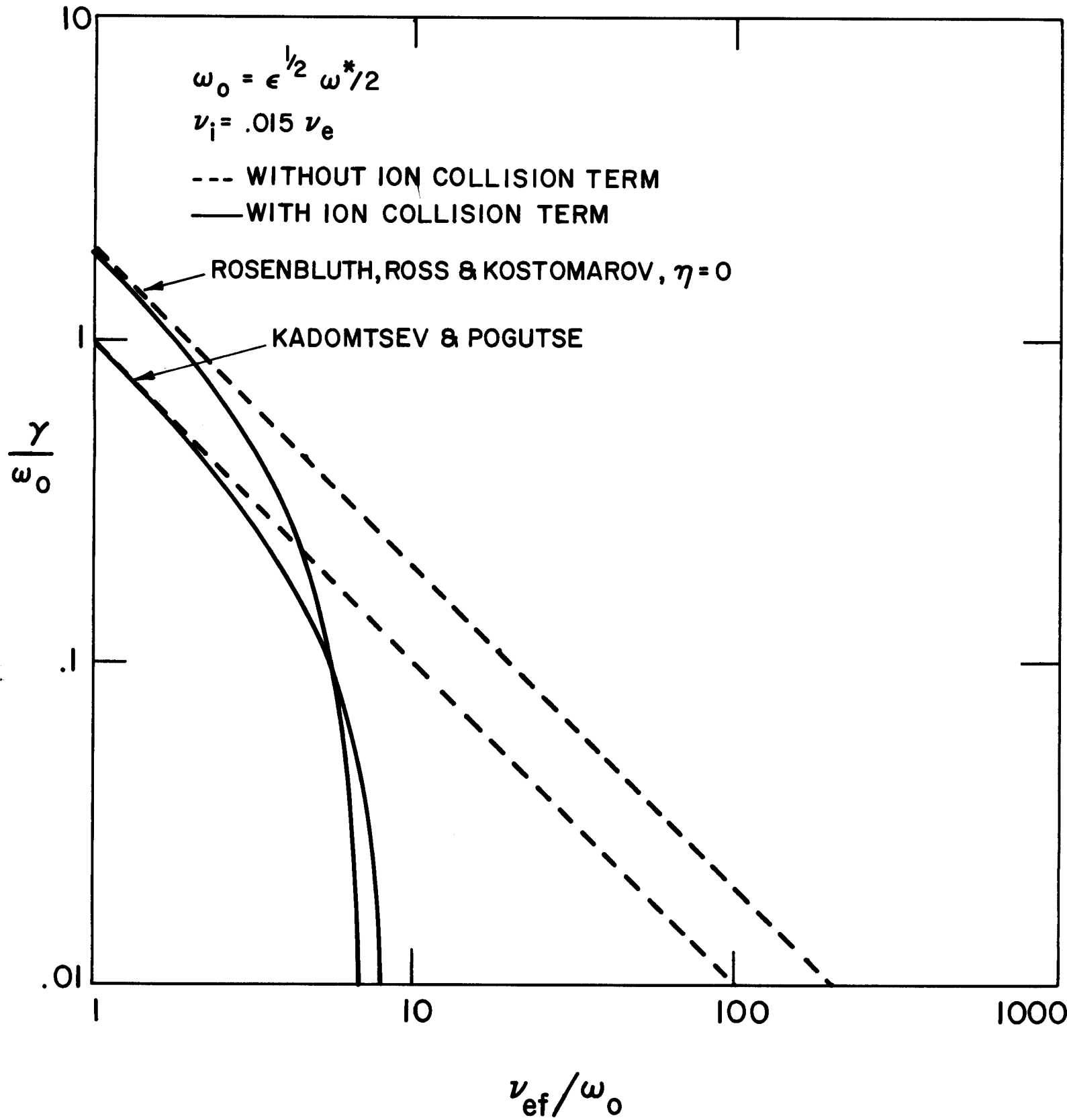


Figure 7

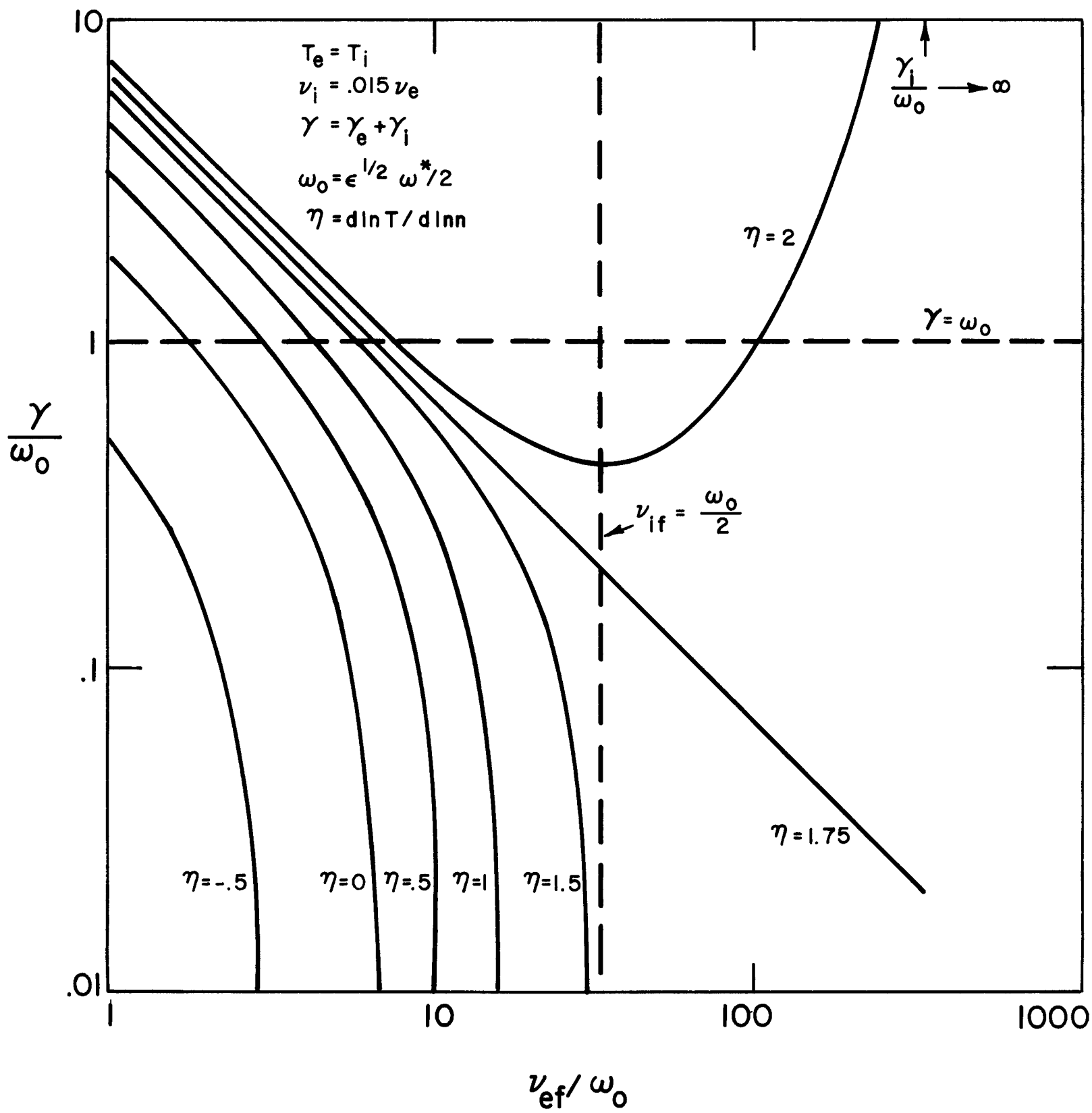


FIGURE 8a\*

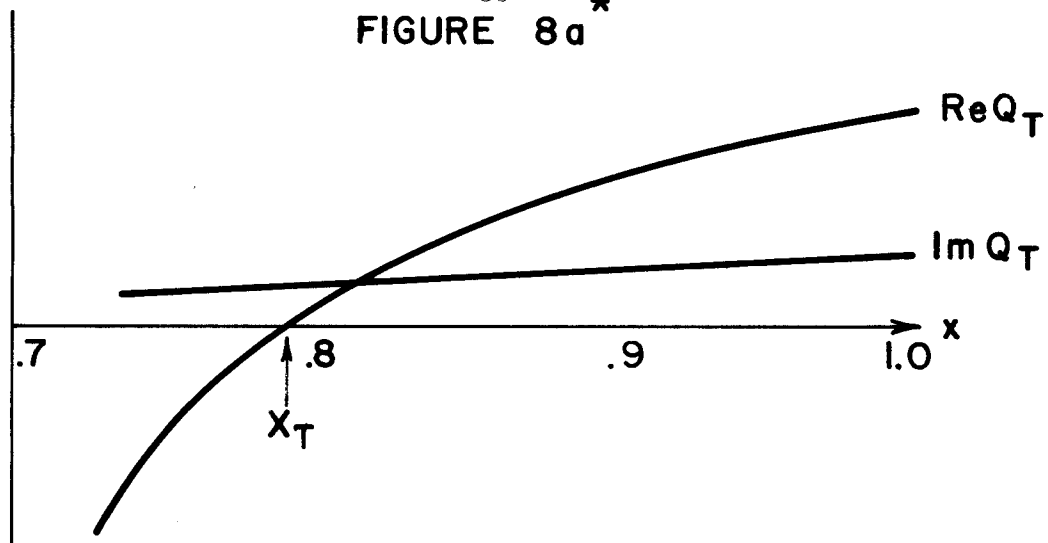


FIGURE 8b\*\*

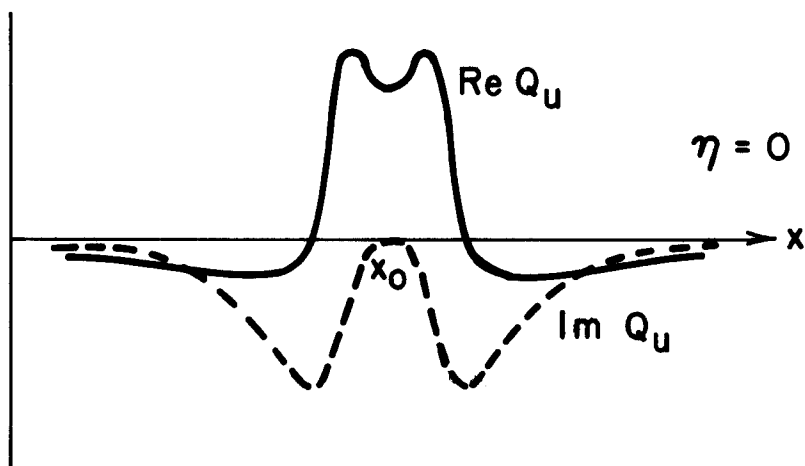
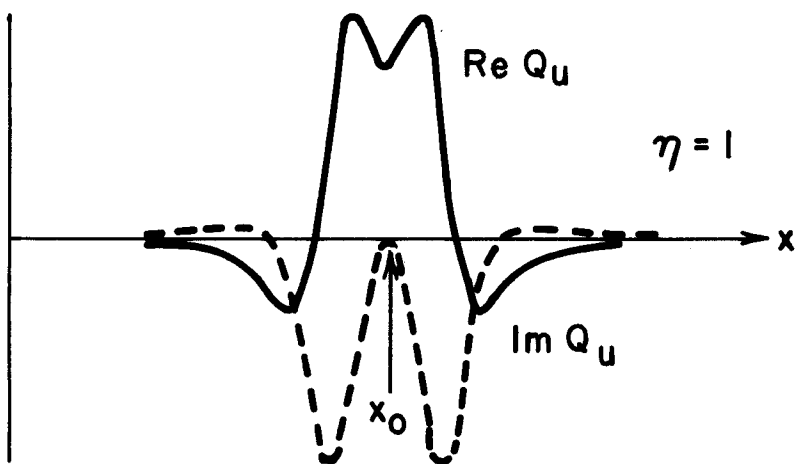


FIGURE 8c\*\*



\* N. T. Gladd, Ph.D. Thesis, Univ. of Texas-Austin, CPPT 44 (1972).

\*\* N. T. Gladd and D. W. Ross, Phys. Fluids 16, 1706 (1973).

