Analytic Expressions for Magnetic Forces on Sectored Toroidal Coils

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Summary

The magnetic forces developed in a sectored toroid are slightly different from those found in a uniformly wound toroid of comparable dimensions. A simple expression is derived in closed form for the forces developed in a toroid with a finite number of segments. Although this is an approximation, it is shown to be highly accurate. The use of this result for the precise computation of constant tension D magnets is discussed.

Introduction

As fusion research and development has progressed the need for larger and larger toroidal magnets has increased concurrently. The necessity for even greater size and sophistication in toroidal magnets is a certainty as power producing Tokamaks are developed. The magnetic forces must be known precisely for such machines to keep the structural costs at a minimum and correctly predict the behavior of components such as constant tension D magnets.

The simplest toroidal magnet is a one layer thin shell of uniformly wound conductors. If there are N turns each carrying a current I, then the magnetic field inside the coil is

$$B_\theta = \mu_o NI/2\pi R, \qquad (1)$$

where R is the distance from the major axis to the point where B_θ is measured. Here we are using SI units with $\mu_o = 4\pi \times 10^{-7}$ henrys/meter. The magnetic field pressure is

$$P = \frac{1}{2\mu_o} B^2 = \mu_o N^2 I^2/8\pi^2 R^2. \qquad (2)$$

Multiplying this by $2\pi R/N$ one can get the force per unit length exerted on the conductor,

$$f_\ell = \mu_o NI^2/4\pi R. \qquad (3)$$

Note that the only coordinate required is the major radius of the point on the conductor where the force is to be observed. Naturally the force is outward and normal to the surface.

It must be emphasized that Eq. (3) only applies to one layer toroids where the current is uniformly distributed over the surface; that is, current density is independent of θ. In practice this uniformity is rarely obtainable or desirable. Sectored magnets are required to provide access to internal reactor components. The best design appears to be the constant tension-constant cross-section type of "D" sector introduced by File, Mills and Sheffield.[1] In this paper we derive an analytic correction for Eq. (3) to account for the sectored construction of toroids of general shape.

Others have developed similar corrections for the above force on a conductor. File and Sheffield[2] utilized the analytic expression of B inside a set of straight vertical conductors placed on concentric circles.[3] Purcell[4] calculated the magnetic field numerically to obtain the force. Both authors used the force obtained to calculate the shape of constant tension D magnets.

We believe that our correction for Eq. (3) is simpler than those previously obtained, and we present examples to demonstrate its accuracy. As will be shown later, we consider the solenoidal curvature of a magnet in addition to its toroidal curvature. The corrected expression for f_ℓ will then be used to compute the shapes of a variety of constant tension D magnets.

Correction for Toroidal Curvature

Let us consider a set of vertical conductors placed in concentric circles as shown in Fig. 1. There are N conductors on each circle, with currents going "up" in the inner ring and "down" in the outer one. Each conductor has a radius c.

We have chosen to define a "conductor" as one section of current carrier; hence, this example represents a magnet with N sectors. Each sector could have many current carriers, but since they are closely packed it is acceptable to treat them as one conductor.

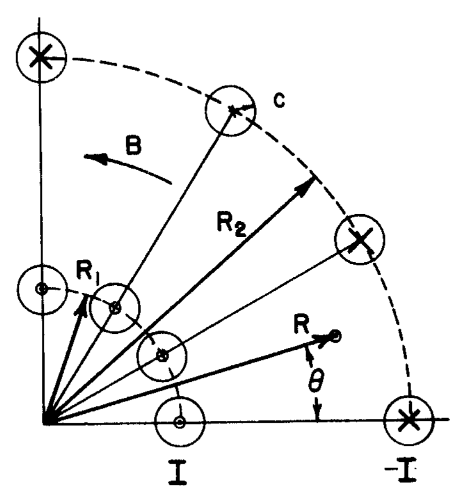

Fig. 1. Toroidal cross section as viewed from above.

If the conductors were infinitely long, the field in the $\theta = 0$ plane would be[2,3]

$$B_\theta(R) = \frac{1}{kR}\left[1 + \frac{1}{(R/R_1)^N - 1} + \frac{1}{(R_2/R)^N - 1}\right] \quad (4)$$

where $k = 2\pi/\mu_o NI$.

We are actually interested in the field at one conductor due to the 2N-1 others. To get this one must subtract the self field of that conductor from the total field given in Eq. (4).

Let us designate the field as $B_{\theta 1}$ when the current at $R = R_1$, $\theta = 0$ is turned off. Likewise $B_{\theta 2}$ indicates that there is no current at $R = R_2$, $\theta = 0$. A simple derivation gives the self fields, then subtracting those from B_θ gives the field at one conductor due to all the others

$$B_{\theta 1}(R) = B_\theta(R) - 1/kN(R-R_1), \quad (5)$$

$$B_{\theta 2}(R) = B_\theta(R) + 1/kN(R-R_2). \quad (6)$$

Here filamentary conductors are acceptable, $c = 0$, since it can be shown that f_ℓ is independent of c in this example.

Both terms on the right hand side of Eq. (5) are singular at R_1; however, the singularities cancel in the subtraction making $B_{\theta 1}$ continuous at R_1. A similar cancellation occurs at R_2 for $B_{\theta 2}$. With some algebraic manipulation one gets the following results

$$B_{\theta 1}(R_1) = \frac{1}{2kR_1}(1 - 1/N + \varepsilon), \quad (7)$$

$$B_{\theta 2}(R_2) = \frac{1}{2kR_2}(1 + 1/N + \varepsilon). \quad (8)$$

where $\varepsilon \equiv 2/[(R_2/R_1)^N - 1]$. In most cases studied ε is very small; hence, we neglect it.

When conductors 1 and 2 carry currents I and -I respectively, the force per unit length on each is given as follows

$$f_{\ell 1} \simeq \frac{I}{2kR_1}(1 - 1/N), \quad (9)$$

$$f_{\ell 2} \simeq \frac{I}{2kR_2}(1 + 1/N). \quad (10)$$

In both cases the force is directed away from the main field region. A derivation using energy principles rather than Eq. (4) gives the same result.

The forces in Eqs. (9) and (10) arise because N conductors have been arranged in a toroidal configuration formed by infinitely long wires. Let us now speculate on what happens when this is replaced by a more reasonable shape such as that shown in Fig. 2. Two significant changes take place: the conductor is no longer vertical at all points studied, and much of it has a substantial curvature.

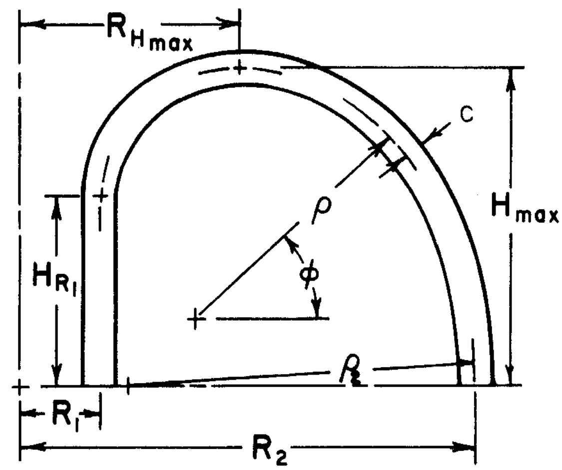

Fig. 2. Side view of a constant tension D magnet.

When the angle ϕ is defined as in Fig. 2 it is natural to rewrite f_ℓ as follows

$$f_\ell \simeq \frac{I}{2kR}\left(1 + \frac{1}{N}\cos\phi\right). \quad (11)$$

Indeed it can be shown that $f_\ell = I/2kR$ for a set of straight wires fanning out from a central core, $\phi = 0$, thus obtaining an additional rational for using $\cos\phi$ above. Further justification is best given in numerical examples presented later in the paper.

Equation (11) does not account for force changes caused by the curvature of the conductor, ρ. These will be derived in the next section for a solenoid and then added to Eq. (1).

Correction for Solenoidal Curvature

Now we consider a set of long straight conductors at a distance x_1 from a uniform sheet of current as shown in Fig. 3. The conductors have radius c and separations s. The sheet current forms the return path.

By careful consideration of flux linkages one can derive the energy attributed to one meter of one conductor which is stored in space to the right of $x = 0$

$$E_\ell = \frac{\mu_o I^2}{2}\left[\frac{x}{s} + \frac{1}{2\pi}\ln(0.2044\ s/c)\right] \quad (12)$$

where $x/s >> 1$. In the limit of very large values of x/s the logarithmic term becomes insignificant and Eq. (12) reduces to the energy stored between two uniform sheets of current. Here, however, we are primarily interested in the correction.

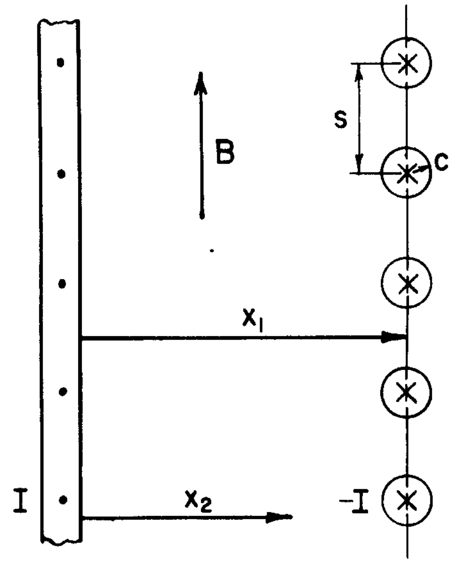

Fig. 3. Cross section of a set of straight conductors parallel to a uniform conducting sheet.

The logarithmic term in Eq. (12) represents field effects near the circular conductors. Even at $x = x_1 - s$ the field ripple is 0.25% of the average field between the sheet and the "wires." This means it is possible to deform the straight conductors above into a solenoid and get accurate expressions for the energy. In Fig. 3 imagine that the current sheet is replaced with the central axis of a solenoid and x_1 becomes the radius ρ. Now the stored energy per conductor becomes

$$E_c = \frac{\mu_o I^2}{2} \left[\frac{\pi\rho^2}{s} + \rho\ell n(0.2044\ s/c)\right]. \qquad (13)$$

The force per unit length of conductor is then

$$f_\ell = \frac{1}{2\pi\rho}\frac{\partial E_c}{\partial\rho} \approx \frac{\mu_o I^2}{2s}\left[1 + \frac{s}{2\pi\rho}\ \ell n(0.2044\ s/c)\right] \qquad (14)$$

The central axis of the conductor is the reference length for this expression of force per unit length.

Since the field distribution that produces the correction is local to the conductor, Eq. (14) should be good for noncircular solenoids. Here ρ would be the radius of curvature at one location rather than for all of the magnet.

General Force Expression

Separate corrections to the force have now been derived to account for both toroidal and solenoidal curvatures. It is very interesting and important to see that both corrections depend on local parameters: radii of curvature, conductor cross section, and separation. The overall shape of a magnet appears to have little effect on the force when compared to the basic local features noted above.

We assume that the force corrections in Eqs. (11) and (14) can be added to Eq. (3) to give a general expression of the force on a toroid. In this case, one has $s = 2\pi R/N$ which leads to the result:

$$\begin{aligned} f_\ell &\simeq \frac{\mu_o N I^2}{4\pi R}\left[1 + \frac{1}{N}\left(\cos\phi + \frac{R}{\rho}\ \ell n\ \frac{1.284R}{cN}\right)\right] \\ &= \frac{\mu_o N I^2}{4\pi R}\ F(R,\rho,c,\phi,N) \end{aligned} \qquad (15)$$

Equation (15) is applicable to a wide variety of toroid shapes. Those most commonly used are the toroid of circular cross section and the constant tension D magnet. It is relatively easy to calculate precise values of fields and forces for the former, so the "circular" toroid will be used as a test of Eq. (15). As an application of Eq. (15) we will calculate the shapes of a variety of constant tension magnets.

So far our discussion has only utilized conductors with a circular cross section of radius c; however, square conductors are often desirable. If a round conductor is replaced by a square one of width w, the self inductance of a new magnet is given with the following substitution:

$$c \Rightarrow 0.573\ w \qquad (16)$$

This is obtained with a simple numerical computation. For square conductors Eq. (16) may be entered directly into Eq. (15).

Accuracy Tests

The fields and forces were calculated numerically for a variety of toroids with circular coils. The subroutine HRHZ, developed by Ballou,[5] was used to obtain the field at one coil due to all the others. The self force was obtained from an inductance formula presented by Smythe[6]

$$f_{\ell s} = \frac{\mu_o I^2}{4\pi\rho}\ (\ell n\ 8\rho/c - 0.75). \qquad (17)$$

Simultaneously the force was calculated using Eq. (15). These are represented by numerical and analytic correction factors, F_{nu} and F_{an} that correspond to F in Eq. (15). The results for two toroids are given in Table 1.

Table I. Numerical and analytic forces for circular toroids with N = 8 and 18

			N=8, c=8.7cm		N=18, c=19.6cm	
ϕ	R (m)	Z (m)	F_{nu}	F_{an}	F_{nu}	F_{an}
0	2.50	0	1.337	1.348	1.154	1.155
$\pi/4$	2.21	0.71	1.247	1.251	1.111	1.112
$\pi/2$	1.50	1.00	1.044	1.038	1.018	1.017
$3\pi/4$	0.79	0.71	0.874	0.869	0.942	0.942
π	0.50	0	0.830	0.819	0.920	0.920

The largest error for 18 sectors is of the order of 0.1%. In the case of 8 sectors the largest error is 1.3%. It would be very unusual to consider a fusion reactor with only 8 coils; this is shown just as an example of the capability of the theory.

The proportion of $R_2/R_1 = 5$ is the same as that used in the toroidal field magnet in the UWMAK-II[7] report; however, the circular shape is quite different from the UWMAK designs. The use of a circular toroid for testing is justifiable because Eq. (15) is not dependent on the coils being circular. Circles were chosen only to have the simplicity and accuracy available in the subroutine HRHZ. In time Eq. (15) will be checked against other more general computer codes.

Constant Tension D Magnets

When a normal force/unit length, f_ℓ is applied to a flexible conductor under tension T, the conductor will take on a radius of curvature

$$\rho = T/f_\ell . \qquad (18)$$

Equations (3) and (18) have been used to calculate the shapes of constant tension D magnets.[8] Equation (15) can now be used to compute these shapes for a sectored toroid. The radius of curvature is expressed as follows:

$$\rho = \frac{R}{N}\left(\frac{4\pi T}{\mu_o I^2} - \ell n \frac{1.284R}{cN}\right)/\left(1 + \frac{1}{N}\cos\phi\right) \qquad (19)$$

If ρ is specified as ρ_2 at R_2, Eq. (19) can be rewritten as follows

$$\frac{\rho}{R} = \left[\frac{\rho_2}{R_2}\left(1 + \frac{1}{N}\right) - \frac{1}{N}\ell n \frac{R}{R_2}\right]/\left(1 + \frac{1}{N}\cos\phi\right). \qquad (20)$$

T, I and c have been absorbed in ρ_2/R_2. Once ρ_2/R_2 is selected, ρ/R is independent of c. This indicates that the shape of a constant tension magnet is independent of the conductor cross section as long as that cross section is not changed within a sector coil.

Design Parameters for D Magnets

While a designer needs the exact shape of the D magnet which is finally chosen, the parameters needed to assist one in making the proper choice of coil can be narrowed to a small finite number. The following list gives these parameters and the reasons for their importance. The main parameters as seen in Fig. 2 and reasons for choice are:

1. The number of coils, determined by the permissible field ripple at the outer edge of the plasma and the space required between coils for blanket and shield removal, injection ports for neutral beam heating, vacuum ports for diverted particle removal, etc.
2. The radial distance to the outer leg of the magnet, R_2, determined by the same reasons as in item 1.
3. The maximum height, H_{max} and the radial position of the point of maximum height, $R_{H_{max}}$ determined by the blanket, shield, and diverter components enclosed.
4. The radial position of the vertical inner leg of the magnet, R_1, and the height at which this inner leg becomes tangent to the outer curved portion of the D coil, H_{R_1}, determined by the space needed for ohmic heating coils, the maximum field for which the coils must be designed, and the size and cost of the inner structural cylinder supporting the unbalanced radial loading on the toroidal field magnet.

Equation (20) was substituted into the differential equation for curvature and numerical solutions carried out for selected values of N and ρ_2/R_2. From the numerical values of the coordinates of the D shapes the parameters just described were extracted. A family of curves is plotted in Fig. 4 which depict the variations in these parameters in the form of the dimensionless ratios R_1/R_2, H_{R_1}/R_2, H_{max}/R_2, and $R_{H_{max}}/R_2$, for variations in the independent variable ρ_2/R_2. Each of the curves shown is for a given value of N, the number of discrete coils in the toroidal field magnet. The curves labeled $N = \infty$ are for the case where a continuous winding is assumed over the surface of the D cross section toroid.

The choice of ρ_2/R_2, the ratio of the radius of curvature of the perimeter at the outer end of the horizontal axis of symmetry divided by the radial position of this point, as an independent variable was made for several reasons, which are: to gain an independence of the curves presented from the finite size of a cross section, and computer codes used to design D shaped toroidal field magnets of finite size, where thick-walled theory with distributed magnetic field loading is used,[8] were developed to use initial values of ρ_2/R_2 and can easily be modified to accommodate the field perturbations caused by the finite number of coils.

To illustrate the use of these curves in a typical design let us assume that the following requirements have been imposed on the designer: (1) the field at the plasma center at a radius of 8.12 m must be 3.6 T, (2) the maximum field of the magnet is limited to 8.5 T, (3) the circumferential distance between coil centerlines must be 6 m at the outer end of the horizontal plane of symmetry, (4) the inside surface of the toroidal field coil must clear the diverter channel which extends 12 m above the midplane, and (5) a choice must be made between 18 and 24 coils in the toroidal system. The designer is called upon to make a comparison of the two design choices.

Solution: First note that $R_1(8.5) = 8.12(3.6)$ or $R_1 = 3.44$ m. To obtain the 6 meter spacing, for N = 18, $2\pi R_2/18 = 6$ or $R_2 = 17.19$ m. Therefore, $R_1/R_2 = 3.44/17.19 = 0.20$. From Fig. 4 following the dotted lines originating at $R_1/R_2 = 0.2$ and using the curves labeled N = 18 one finds the necessary parameters. These parameters and the resulting dimensions are listed in Table II.

Similarly for N = 24, R_{max} = 22.92, R_1/R_2 = 0.15. For the dashed line on Fig. 4 the values are also given in Table II.

Table II
D Shaped Parameters and Dimensions

N	ρ_2/R_2	H_{R_1}/R_2	H_{R_1} (m)	H_{max}/R_2	H_{max} (m)	$R_{H_{max}}/R_2$	$R_{H_{max}}$ (m)
18	0.719	0.419	7.21	0.634	10.90	0.461	7.92
24	0.870	0.550	12.61	0.744	17.06	0.404	9.26

It is obvious that a magnet with 18 coils will not clear the diverter. The designer can now proceed with the next iteration on his requirements.

To further improve the utility of this work to the designer of sectored toroids, polynomials have been fitted to the data used to plot the family of curves shown in Fig. 4. All of the parameters are taken to vary in the form of a 4th order polynomial in ρ_2/R_2. That is:

$$P_i/R_2 = \sum_{j=0}^{4} A_j(\rho_2/R_2)^j \qquad (21)$$

where P_i is one of the four parameters discussed. Table III lists the numerical values of A_j for the several values of N. A linear interpolation in the factor 1/N can be used to determine the values of A_j for other values of N. A comparison to numerical values obtained by Purcell[4] shows agreement to within 0.8% for both the maximum height and its radial position for the given ratio R_1/R_2 and for 16 coils.

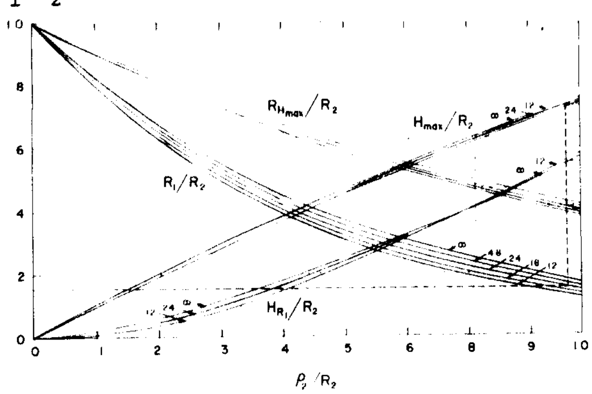

Fig. 4. Variation in design parameters with ρ_2/R_2 for values of N from 12 to ∞.

Conclusions

The use of the analytical expression for the radius of curvature given in Eq. (20) allows one to numerically integrate the differential equation for curvature for thin coils or to adapt the procedure to existing numerical solutions for thick coils. Costly iterative procedures can be avoided and preliminary design parameters chosen directly from the graphs or the empirical equations presented. The numerical example presented shows that a few minutes work will allow intelligent choices in the number of coils, choice of maximum radius and field ripple.

Table III
Design parameters for D shaped toroidal field coils

Parameter P_i/R_2	N	A_0	A_1	A_2	A_3	A_4
	∞	1.000	-1.992	1.923	-1.076	0.281
	48	1.000	-2.075	2.082	-1.202	0.321
R_1/R_2	24	1.000	-2.162	2.252	-1.339	0.364
	18	1.000	-2.222	2.374	-1.441	0.398
	12	1.000	-2.346	2.636	-1.664	0.472
	∞	0.000	0.013	1.450	-1.161	0.353
	48	0.000	-0.052	1.630	-1.330	0.412
H_{R_1}/R_2	24	0.000	-0.123	1.831	-1.524	0.480
	18	0.000	-0.173	1.976	-1.667	0.531
	12	0.000	-0.281	2.298	-1.991	0.648
	∞	0.000	1.000	-0.214	0.046	-0.006
	48	0.000	1.009	-0.220	0.047	-0.006
H_{max}/R_2	24	0.000	1.018	-0.226	0.051	-0.007
	18	0.000	1.023	-0.230	0.052	-0.008
	12	0.000	1.035	-0.238	0.055	-0.008
	∞	1.000	-1.000	0.497	-0.156	0.027
	48	1.000	-1.020	0.517	-0.164	0.028
$R_{H_{max}}/R_2$	24	1.000	-1.041	0.538	-0.175	0.031
	18	1.000	-1.055	0.553	-0.182	0.033
	12	1.000	-1.083	0.582	-0.196	0.035

Acknowledgements

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