One-Dimensional, Time-Dependent, Integral Neutron Transport for Inertial Confinement Fusion

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by

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Abstract

Neutron transport is of great importance to inertial confinement fusion (ICF) for several reasons. An accurate neutron energy spectrum is necessary for tritium breeding purposes, and the deposition of energy in the ICF target by energetic neutrons born from fusion may have detrimental effects on the fusion burn. The goal of this research was to develop an accurate neutron transport method that can be incorporated into an existing radiation-hydrodynamics code for modeling ICF implosions.

A novel time-dependent neutron transport method, based on the integral form of the neutron transport equation, was developed. This method utilizes a dimensionless integration space and the Neumann series method to obtain the integral form of the reduced collisions equations.

This neutron transport method was implemented for infinite slab and sphere geometries. Using a pulsed source in space and time, the method was used to reproduce benchmark solutions previously published in the literature, and was found to have excellent agreement with these benchmarks.

The method was expanded to incorporate finite slab and sphere geometries. The method was implemented for a finite slab, and benchmarked against PARTISN, a finite difference, discrete-ordinates code. The method was found to agree with PARTISN at intermediate mean free times, while diverging from PARTISN at late mean free times. The method was used to obtain analytic expression for the first two collided fluxes in a finite sphere geometry. A collision study was performed for both geometries to determine how many collisions were necessary to approximate the total flux at early mean free times. This study showed that only a few collisions were necessary to

approximate the total flux at times of interest to ICF applications.

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Chapter 1

Introduction

1.1 Introduction to Fusion

Fusion energy, which has the potential to create vast amounts of energy, has been under development since the 1950's. The two major advantages of fusion energy over current forms of energy production are: 1) a major component of fusion fuel, deuterium, is plentiful and inexpensive, and 2) the waste produced from fusion is not made up of long-lived, heavy radioactive isotopes, but stable light isotopes such as hydrogen and helium [1]. A disadvantage, however, is that the flux of neutrons produced from the fusion reactions can activate reactor structural materials. In contrast to waste from fission reactors, however, there is no transuranic waste. Though a few radioisotopes created are long-lived, the vast majority of radioisotopes have short half-lives, and the small inventory of radioactive waste will decrease rapidly.

Fusion is the process of combining the nuclei of two light elements together, creating a heavier element. Fusion has been difficult to achieve because the nuclei are both positively charged, and therefore repel each other. The fusion fuel must be

heated to incredibly high temperatures such that the velocities of the nuclei are very large, allowing the nuclei to overcome the repulsive Coulomb force. The nuclei will scatter off each other more often than they will fuse together, therefore the fuel must be confined, allowing the nuclei to collide many millions of times, until they finally fuse [1].

There are two confinement schemes commonly considered for fusion energy. These are the magnetic and inertial confinement fusion concepts. The goal of magnetic confinement fusion is to create a steady-state plasma confined by a magnetic field [2]. Devices currently being considered for magnetic confinement include the tokamak and the stellarator [1]. Inertial confinement fusion involves heating the fusion fuel to thermonuclear temperatures [1] by rapid compression of the fuel pellet so that a large number of fusion reactions occur before the pellet blows apart. Large laser beam generators or light/heavy ion beam accelerators are used as drivers to generate beams, which compress the pellets to high densities and the fuel to thermonuclear temperatures. While the fusion reactions considered for the two concepts are the same, the density and pressure regimes differ by several orders of magnitude [2].

1.2 Introduction to Inertial Confinement Fusion

Unlike magnetic confinement fusion, Inertial confinement fusion, or ICF, does not depend on external means to confine a plasma. Instead, ICF utilizes the mass inertia of the fuel to confine the fuel long enough to achieve thermonuclear burn. The confinement time of an ICF plasma is then very short, usually on the order of 50 ps. Target compression influences the confinement time and the burn yield. Compression to extremely high densities leads to longer confinement times and higher reaction

rates [2].

To protect the walls of the reactor vessel in which the fusion burn takes place, the energy release from the explosion of the fuel must be limited. This in turn limits the mass of fuel in a pellet to only 1 - 10 mg. To burn such a small mass of fuel requires a very high fuel compression [2].

A typical deuterium-tritium fuel pellet consists of four regions, as seen in Fig 1.1. The outer shell of the pellet is an ablator, and is made of plastic, followed by a layer of plastic and deuterium-tritium (DT). The layer of plastic with DT acts as a thermal shield between the ablator layer and the DT ice, increasing the laser absorption efficiency of the pellet. Behind this is a shell of DT ice, and the innermost region is DT vapor. The pellet is uniformly irradiated by a large number of lasers. The energy from the lasers heats up the ablator, which begins to expand. To conserve momentum, the rest of the shell is forced inward. As the fuel pressure increases from the implosion, a hot spot of very high temperature is formed at the center of the pellet. Conduction by electrons from the hot spot to the surrounding cold fuel and radiation cool the hot spot.

As long as losses due to conduction and radiation are not too high, ignition will occur in the central hot spot. To achieve ignition, the confinement parameter, $\rho \times R$, of the hot spot must be equal to about 0.3 g/cm^2 where ρ is the density and R is the radius [3]. Alpha particles, produced from fusion reactions in the hot spot, propagate the burn by depositing their energy in the surrounding fuel. Meanwhile, the fuel is rapidly expanding, and remains confined for only about 50 ps. Because the fusion products can be used to propagate the burn to the surrounding fuel, only the hot spot needs to be compressed to a very high density at a very high temperature, which in

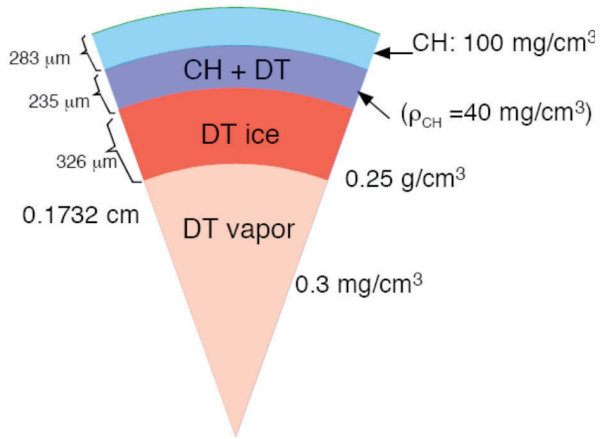

Fig. 1.1.— Inertial Fusion Energy Pellet. Credit: Andy Schmitt, Naval Research Laboratory

turn requires less input energy from the lasers [2].

1.3 Motivating Neutron Transport

Neutron transport is of great importance to the study of Inertial confinement fusion. High-energy neutrons are born from the fusion process. These particles, along with alpha particles, are necessary to propagate the burn from the ignition region to the outlying low-temperature, high-density regions surrounding the ignition region of the target.

The following reactions use the most common fusion fuels, deuterium, tritium, and helium-3, and therefore are of importance to ICF devices:

D + T $\rightarrow$ ^{4}He (3.5 MeV) + n (14.1 MeV)

D + D $\rightarrow$ T (1.0 MeV) + H (3.0 MeV)

$D + D \rightarrow {}^{3}He$ (0.8 MeV) + n (2.45 MeV)

$D + {}^{3}He \rightarrow {}^{4}He$ (3.7 MeV) + H (14.7 MeV).

As can be seen from the above reactions, neutrons figure prominently in two of the fusion reactions. The neutrons carry the bulk of the kinetic energy when present.

The alpha particles and neutrons created during a fusion burn propagate the burn by transferring energy to the low-temperature, high-density regions surrounding the ignition region. However, the alpha particles and neutrons travel at different velocities. In a sense, neutrons can be thought of as pre-heating the areas that are later ignited by the energy from the alpha particles, an effect that may or may not be detrimental to the burn process. A complete understanding of the interplay of these particles is essential to fully characterize a fusion burn.

Accurate neutron modeling is important to ICF for other reasons, as well. The neutrons eventually transport out of the target and collide with the reactor vessel walls. Since the neutrons may suffer collisions before escaping the target, the neutrons emerge with a spectrum of energies. This fact will affect the radioactivity of the reactor vessel walls, tritium breeding, shield designs, and dose rates to reactor personnel.

The deuterium-tritium reaction is of great interest to ICF because the fuel mixture has the lowest ignition temperature of any of the above reactions, in addition to a very high energy yield [2]. However, since tritium has a half-life of only 12.3 years, tritium must be bred. The reactor vessel has specific zones designed for tritium breeding. Two important tritium-breeding reactions are:

${}^{6}Li + n \rightarrow T + \alpha + 4.86$ MeV

^{7}Li + n $\rightarrow$ T + α + n - 2.87 MeV.

Both these reactions require neutrons for breeding. However, the ^{6}Li reaction requires slow neutrons while the ^{7}Li reaction requires fast neutrons. Therefore, it is important to know the energy spectrum of the neutrons escaping the target, to determine if the breeding zone must be enriched with ^{6}Li to ensure that enough tritium is bred.

1.4 Time-Dependent Neutron Transport

The burn time of a fusion target is incredibly fast, taking approximately 50 ps. During this phase, a fusion target fuel region changes rapidly. Given the speed of a 14.1 MeV neutron as roughly 5.2 cm/ns and the approximate radius of the compressed fuel target as 0.012 cm, the fuel traversal time for a neutron is found to be approximately 4.6 ps. Therefore, during the burn phase, a propagating neutron would encounter a rapidly changing medium. For this reason, a steady state approach to neutron transport study would not be appropriate.

On the other hand, since the fusion target is so small, a neutron would only experience a few collisions before escaping the target. The number of collisions a neutron would experience can be calculated from an escape probability estimate as a function of the target density, ρ, multiplied by the target radius, R. Such a study indicates that, for a $\rho \times R$ value of 2.0 g/cm^2, less than 30% of the neutrons experience a collision [3]. The number of collisions a neutron experiences in a fusion target is small. Therefore, a Neumann series approach is considered appropriate for calculating the total neutron flux. As discussed in Section 2.1, a Neumann series decomposes

the total flux into the uncollided flux and the collided fluxes. The first collided flux is calculated using the uncollided flux as a source term, the second collided flux is calculated using the first collided flux as a source term, and so on. Since the neutrons only make a few collisions before escaping the target, only the first several collided fluxes need to be calculated to obtain an accurate result for the total flux.

Given the importance of neutron transport to ICF, the goal of this work was to develop a novel neutron transport method that could be incorporated into an existing radiation hydrodynamics code. The ensuing chapters discuss common methods for simplifying the neutron transport equation, and present a novel method for solving the transport equation utilizing dimensionless variables. A literature review is provided in Chapter 2, wherein the differential form of the reduced collision equations is introduced and reviewed. Further developments referenced in Section 2.5 show how to convert the equation to its integral form.

The integral formulation of the transport equation is the basis for collision probability methods, which are known to produce highly accurate results for unit cell calculations in reactor physics. The integral equation will be examined for its applicability to time-dependent neutron transport within inertial fusion targets. Using a dimensionless variable transformation for the time and space domain, the collision-by-collision, time-dependent integral equations will be converted into dimensionless form known as the reduced collision equations. The differential form of the reduced collision equations has been used to produce highly accurate benchmark solutions to time-dependent, one-dimensional infinite medium problems in Cartesian and spherical coordinate systems. Chapter 3 highlights the one-dimensional infinite medium work using the integral form of the reduced collision equations. Benchmark solutions for the

infinite slab and sphere case were generated and compared to existing solutions in the literature. The results for one-dimensional finite media are contained in Chapter 4. For finite media, the solution for the first few collisions is compared to solutions from the PARTISN code. Finally, Chapter 5 provides a conclusion on the feasibility of this integral transport method for ICF applications, final comments and future work.

Chapter 2

Literature Review

The neutral particle transport equation is shown below:

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma(\boldsymbol{r}, E, t)\right]\psi\left(\boldsymbol{r}, \hat{\Omega}, E, t\right) = \\ \int dE' \int d\hat{\Omega}'\Sigma_s\left(\boldsymbol{r}; \hat{\Omega}' \to \hat{\Omega}, E' \to E\right)\psi\left(\boldsymbol{r}, \hat{\Omega}', E', t\right) + S\left(\boldsymbol{r}, \hat{\Omega}, E, t\right). \tag{2.1}$$

where ψ is the angular neutron flux, Σ_s is the macroscopic scattering cross-section, Σ is the total cross-section, E is the energy, S is the external source, and $\hat{\Omega}$ is the scattering angle. This equation is a linearized form of the Boltzmann equation describing the statistical distribution of particles in a fluid [4]. The equation describes the balance of neutrons lost through exiting the volume or suffering collisions to those gained as secondaries from collisions or produced by sources.

To obtain solutions to equation 2.1 requires several simplifying assumptions. These assumptions include time-independence, isotropic scattering, one- or two- dimensional transport, monoenergetic particles, and several others. By making several such assumptions, it is possible to find analytic solutions or numerical approximations

to the neutral particle transport equation.

2.1 Multiple Collision Method

The multiple collision method is alternatively known as the Neumann series solution, the Order of Scattering Theory, or the Method of Successive Generations. The multiple collision method expands the total flux into an infinite sum of the collided fluxes:

$$\psi_{tot}\left(\boldsymbol{r},\hat{\Omega},t\right)=\sum_{n=0}^{\infty}\psi_n\left(\boldsymbol{r},\hat{\Omega},t\right) \tag{2.2}$$

and

$$\Phi_{tot}\left(\boldsymbol{r},t\right)=\sum_{n=0}^{\infty}\phi_n\left(\boldsymbol{r},t\right) \tag{2.3}$$

where ψ is the angular flux and Φ is the scalar flux. The scalar flux is calculated from the angular flux as:

$$\Phi(\boldsymbol{r},t)=\int d\hat{\Omega}'\psi\left(\boldsymbol{r},\hat{\Omega}',t\right). \tag{2.4}$$

The source for the uncollided flux is simply the external source. The source for the first collided flux is the uncollided flux, and so on. Using the multiple collision method, the neutron transport equation becomes an infinite series of coupled equations:

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma(\boldsymbol{r}, E, t)\right]\psi_0\left(\boldsymbol{r}, \hat{\Omega}, E, t\right) = S\left(\boldsymbol{r}, \hat{\Omega}, t\right) \tag{2.5}$$

$$\begin{aligned}&\left[\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma(\boldsymbol{r}, E, t)\right]\psi_1\left(\boldsymbol{r}, \hat{\Omega}, E, t\right) = \\ &\quad\int dE'\int d\hat{\Omega}'\ \Sigma_s\left(\boldsymbol{r}, \hat{\Omega}' \to \hat{\Omega}, E' \to E\right)\psi_0\left(\boldsymbol{r}, \hat{\Omega}', E', t\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\vdots\end{aligned} \tag{2.6}$$

$$\begin{aligned}&\left[\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma(\boldsymbol{r}, E, t)\right]\psi_n\left(\boldsymbol{r}, \hat{\Omega}, E, t\right) = \\ &\quad\int dE'\int d\hat{\Omega}'\ \Sigma_s\left(\boldsymbol{r}, \hat{\Omega}' \to \hat{\Omega}, E' \to E\right)\psi_{n-1}\left(\boldsymbol{r}, \hat{\Omega}', E', t\right).\end{aligned} \tag{2.7}$$

The multiple collision method has been a popular tool for analytically solving the neutron transport equation. Syros has used the multiple collision method to obtain closed form solutions in infinite slab geometries with anisotropic scattering [5, 6]. Windhofer and Pucker applied the multiple collision method to finite slab geometries [7]. Ganapol, in particular, has had considerable success with the multiple collision method, applying it in conjunction with Legendre Polynomial Expansion (see Section 2.2) to find solutions to a number of infinite medium geometries, including infinite and semi-infinite slab geometries [8], and spherical and cylindrical geometries [9, 10]. Ganapol has also generated a number of benchmark solutions for various infinite geometries [11].

2.2 Expansion in Legendre Polynomials

The mono-energetic neutron transport equation in one spatial dimension, x, is shown below:

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + \Sigma(x,t)\right]\psi\left(x,\mu,t\right) = \int_{-1}^{1} d\mu' g\left(\mu' \to \mu\right)\psi\left(x,\mu',t\right) + S\left(x,\mu,t\right) \tag{2.8}$$

where $g\left(\mu' \to \mu\right)$ is the scattering kernel and $\mu = \cos(\theta)$, and has the domain

$$\mu \in \left[-1 \leq \mu \leq 1\right]. \tag{2.9}$$

The scattering kernel can be expressed as an infinite sum of Legendre polynomials [12]:

$$g\left(\mu' \to \mu\right) = \sum_{l=0}^{\infty} \frac{2l+1}{2}\omega_l P_l(\mu) P_l(\mu') \tag{2.10}$$

where ω_l are the quadrature weights and $\omega_0 \equiv 1$.

The solution to equation 2.8, with the scattering kernel expressed as in equation 2.10 can be written as an infinite sum of collision components [13]:

$$\psi(x,t) = \frac{e^{-\Sigma vt}}{vt}\sum_{n=0}^{\infty}\frac{\left(\Sigma_s vt\right)^n}{n!}F_n(\eta,\mu) \tag{2.11}$$

where $\eta = x/vt$. Substituting the expression for ψ into equation 2.8, the "reduced collision equations," an infinite set of integro-differential equations in F_n is obtained:

$$\left[n-1+(\mu-\eta)\frac{\partial}{\partial\eta}\right]F_n(\eta,\mu)=n\sum_{l=0}^{L}\frac{2l+1}{2}w_lP_l(\mu)\int_{-1}^{1}d\mu' P_l(\mu')F_{n-1}(\eta,\mu') \quad (2.12)$$

where the infinite sum of Legendre polynomials has been truncated to order L [12].

If the sum of Legendre polynomials is truncated to $L = 1$, then the P_1 approximation is obtained. This corresponds to allowing only two discrete directions of motion [8]. The two directions of motion are $\pm\frac{1}{\sqrt{3}}$. Then the reduced collision equations in an infinite medium become:

$$\left[(\pm 1-\eta)\frac{\partial}{\partial\eta}+n-1\right]F_{n\pm}(\eta)=\frac{n}{2}F_{n-1\mp}(\eta) \quad (2.13)$$

where here $\eta=\frac{\sqrt{3}x}{vt}$, and F_+ and F_- correspond to the angular flux in the $+\frac{1}{\sqrt{3}}$ and $-\frac{1}{\sqrt{3}}$ directions, respectively. Ganapol has successfully used this method to obtain solutions to the time-dependent monoenergetic transport equation in infinite and semi-infinite geometry [8]. It should be noted, however, that the P_1 solution gives an incorrect wave front velocity of $\frac{v}{\sqrt{3}}$ [14].

The scalar flux may also be expanded in Legendre polynomials [10]:

$$\phi(x,t)=\sum_{k=0}^{\infty}\frac{2k+1}{2}f_k(t)P_k(\eta) \quad (2.14)$$

where the expansion coefficients are given by

$$f_k(t) = \int_{-1}^{1} d\eta' P_k(\eta')\phi(x,t). \tag{2.15}$$

This expansion is valid because ϕ is only defined on $-1 \leq \eta \leq 1$. Ganapol has used this method to determine the scalar flux in infinite slab, cylindrical, and spherical geometries, and in conjunction with the multiple collision method, to obtain results for linearly anisotropic scattering in these same geometries [14].

2.3 Diffusion Equation

The one-speed neutron diffusion equation is shown in equation 2.16 below:

$$\frac{1}{v}\frac{\partial \phi}{\partial t} - \nabla \cdot D(\boldsymbol{r})\nabla\phi + \Sigma_a(\boldsymbol{r})\phi(\boldsymbol{r},t) = S(\boldsymbol{r},t) \tag{2.16}$$

where $\phi(\boldsymbol{r},t)$ is the scalar flux in three dimensions.

The diffusion equation is found by first finding the P_1 equations, two equations that describe the neutron scalar flux, ϕ and the neutron current, $\mathbf{J}$. To obtain the diffusion equation, we must next relate the neutron current to the neutron scalar flux. This is done by using Fick's law:

$$\mathbf{J}(\boldsymbol{r},t) = -D(\boldsymbol{r})\nabla\phi(\boldsymbol{r},t) \tag{2.17}$$

where D is the neutron diffusion coefficient. By using Fick's law, we are assuming that the neutron current is proportional to the spatial gradient of the flux. By using the

P_1 equations, we are assuming that the angular flux exhibits only linearly anisotropic angular dependence.

The time-dependent neutron diffusion equation is a parabolic differential equation. The response of a system governed by a parabolic equation to a source is such that the source is immediately felt over the entire system. This is not physically how neutrons emitted from a short burst transport, however, as they will be confined by wavefronts. The neutron transport equation is a hyperbolic differential equation. The neutrons move at a finite velocity, and therefore need a finite amount of time to move to a new location, a phenomenon known as causality. Therefore, the diffusion equation is a poor approximation to a system that exhibits well-defined wavefronts. The diffusion equation is also known to be a poor approximation near boundaries of a medium, near sources, and in strongly absorbing media [15]. This is a consequence of assuming that the angular flux is only weakly dependent on angle.

2.4 Discrete Ordinates

The discrete ordinates method solves for the angular flux in discrete directions. The discrete ordinates method is often called the S_N method, where N is related to the number of directions. As the number of directions increases, the accuracy of the results also increases, as does the difficulty of the computations [16].

To obtain the discrete ordinates approximation to the neutron transport equation, the scattering kernel must first be expressed in terms of Legendre polynomials, as discussed in Section 2.2. The equation obtained is then assumed to hold for each distinct angle, μ_n, where $n = 1, ..., N$. The one-dimensional, time-dependent, one-speed discrete ordinates approximation is [17]

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \mu_n \frac{\partial}{\partial x} + \Sigma(x)\right] \psi_n(x) = \sum_{l=0}^{L} \frac{2l+1}{2} P_l(\mu_n)\Sigma_l(x)\phi_l(x) + S(x,\mu_n) \tag{2.18}$$

where $\psi(x,\mu_n)$ is written as $\psi_n(x)$ and the Legendre moments of the scalar flux are [17]

$$\phi_l = \sum_{n=l}^{N} w_n P_l(\mu_n)\psi_n(x). \tag{2.19}$$

The TIMEX code, developed at Los Alamos National Laboratory, is a code that utilizes the discrete ordinates method. TIMEX solves the time-dependent, one-dimensional multigroup transport equation in a variety of geometries, including plane, cylindrical, spherical, and two- angle plane geometries. TIMEX can incorporate a number of boundary conditions and allows for anisotropic scattering [18].

The PARTISN code, also developed at Los Alamos National Laboratory, is the successor of the TIMEX and DANTSYS discrete ordinates code packages. PARTISN solves the multigroup form of the neutral particle Boltzmann transport equation. PARTISN can solve either the time-dependent or the static form of the transport equation. PARTISN can solve the transport equation for a number of multi- dimensional geometries, and for a variety of boundary conditions. PARTISN allows for anisotropic scattering and inhomogeneous sources [19].

2.5 Integral Neutron Transport

2.5.1 Time-Independent Neutron Transport

The time-independent neutron transport equation for one-speed neutrons can be written as:

$$\left[\hat{\Omega}\cdot\nabla+\Sigma\left(\boldsymbol{r},\hat{\Omega}\right)\right]\psi\left(\boldsymbol{r},\hat{\Omega}\right)=\Sigma(\boldsymbol{r})c(\boldsymbol{r})\int d\hat{\Omega}' f\left(\boldsymbol{r},\hat{\Omega}'\rightarrow\hat{\Omega}\right)\psi\left(\boldsymbol{r},\hat{\Omega}'\right)+S\left(\boldsymbol{r},\hat{\Omega}\right) \tag{2.20}$$

where $c(\boldsymbol{r})$ is the number of secondary neutrons produced in a collision, $\Sigma(\boldsymbol{r})$ is the total macroscopic cross section, including fission, and $f\left(\boldsymbol{r},\hat{\Omega}'\rightarrow\hat{\Omega}\right)$ is the scattering kernel. This can be transformed into an integral equation for the flux for Cartesian or spherical coordinates [20]:

$$\begin{aligned}\phi(x)=\frac{ca}{2}\Bigg\{&\int_{-1}^{1}\phi(y)E_1\left(a|x-y|\right)dy-3\bar{\mu}(c-1)\Bigg[\int_{-1}^{1}\phi(y)E_3\left(a|x-y|\right)dy\\&-\frac{E_3\left(a|1-x|\right)+(-1)^q E_3\left(a|1+x|\right)}{2}\int_{-1}^{1}\phi(y)y^q dy\Bigg]\Bigg\}\end{aligned} \tag{2.21}$$

where

ϕ = the flux in Cartesian coordinates or the flux times the radial coordinate in spherical coordinates

a = half-thickness or radius, measured in mean free paths

$\bar{\mu}$ = average cosine of the scattering angle

$q = 0$ in the slab case or 1 in the spherical case

E_n = exponential integral of order n.

This form of the integral equation is used to solve time-independent neutron transport problems with both isotropic and linearly anisotropic scattering. Higher orders of anisotropy can be accounted for by keeping more terms in the scattering kernel expansion, equation (2.10).

To solve equation 2.21, the flux is expanded in Legendre polynomials:

$$\phi(x) = \sum_{n=0}^{\infty} F_n(2n+1)P_n(x). \tag{2.22}$$

Inserting this series into equation 2.21, multiplying by $P_m(x)$, and integrating from -1 to 1 results in an eigenvalue equation for F_n [20]:

$$[\mathbf{A} - 3\bar{\mu}(c-1)\mathbf{B}]\,\boldsymbol{F} = \frac{1}{2ca}\boldsymbol{F}. \tag{2.23}$$

The eigenvectors and eigenvalues of equation 2.23 have been found for both infinite and finite medium, for slab and spherical geometries, with both isotropic and anisotropic scattering [20].

2.5.2 Time-Dependent Neutron Transport

The time-dependent integral neutron transport kernels can be found in two different manners, the method of characteristics or the Laplace Transform method [21]. To solve for the time-dependent kernels, we begin with the time-dependent monenergetic neutron transport equation for isotropic scattering:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma\right)\psi\left(\boldsymbol{r},\hat{\Omega},t\right) = \frac{Q(\boldsymbol{r},t)}{4\pi} \tag{2.24}$$

where Q is the scattered and external source,

$$Q(\boldsymbol{r},t) = \Sigma_s\Phi(\boldsymbol{r},t) + S(\boldsymbol{r},t). \tag{2.25}$$

Using the method of characteristics, the resulting, time-dependent integral equation to be solved is

$$\Phi(\boldsymbol{r},t) = \int_{V'}\int_0^t K\left(\boldsymbol{r},\boldsymbol{r}';t-t'\right)Q(\boldsymbol{r}',t')dt'd\boldsymbol{r}' \tag{2.26}$$

where $K\left(\boldsymbol{r},\boldsymbol{r}';t-t'\right)$ is the time-dependent kernel. The time-dependent kernels for homogeneous media have been derived using the Laplace Transform method from the steady-state kernels [21], and are shown in Table 2.1.

The multiple collision method has been used successfully with the integral transport equation [21, 22] to solve for the neutron scalar flux in various geometries. The individual collided fluxes are calculated as:

$$\phi_n(\boldsymbol{r},t) = \int_{V'}\int_0^t K\left(\boldsymbol{r},\boldsymbol{r}';t-t'\right)Q_n(\boldsymbol{r}',t')dt'd\boldsymbol{r}' \qquad n = 0,1,2,... \tag{2.27}$$

where $Q_0(\boldsymbol{r},t) = S(\boldsymbol{r},t)$ is the external source, and the nth collided source is given by

Table 2.1: Time-Dependent Integral Transport Kernels in Homogeneous Materials

Geometry	**Time-Dependent Integral Transport Kernels**
Point	$K_{pt}(r,r';t,t') = \dfrac{e^{-\Sigma\, v(t-t')}}{4\pi\,\lvert r-r'\rvert\, v(t-t')}\,\delta\!\left(t-t'-\dfrac{\lvert r-r'\rvert}{v}\right)$
Plane	$K_{pl}(x,x';t,t') = \dfrac{e^{-\Sigma\, v(t-t')}}{2\,(t-t')}\,H\left(t-t'-\dfrac{\lvert x-x'\rvert}{v}\right)$
Spherical Shell	$K_{ss}(r,r';t,t') = \dfrac{e^{-\Sigma\, v(t-t')}}{8\pi\, rr'\,(t-t')}\left[H\left(t-t'-\dfrac{\lvert r-r'\rvert}{v}\right) - H\left(t-t'-\dfrac{\lvert r+r'\rvert}{v}\right)\right]$
Line	$K_l(r,r';\phi,\phi';t,t') = \dfrac{e^{-\Sigma\, v(t-t')}}{2\pi\,(t-t')}\dfrac{1}{\left\{[v(t-t')]^2-\rho^2\right\}^{\frac{1}{2}}}H\left(t-t'-\dfrac{\rho}{v}\right)$
Cylindrical Shell	$K_{cy}(r,r';t,t') = \dfrac{e^{-\Sigma v(t-t')}}{\pi^2(t-t')} \displaystyle\int_{\lvert r-r'\rvert}^{\lvert r+r'\rvert} \dfrac{yH\left(t-t'-\frac{y}{v}\right)dy}{\left(\lvert r+r'\rvert^2-y^2\right)^{\frac{1}{2}}\left(y^2-\lvert r-r'\rvert^2\right)^{\frac{1}{2}}\left\{[v(t-t')]^2-y^2\right\}^{\frac{1}{2}}}$

$$Q_n(\boldsymbol{r},t) = \Sigma_s\phi_{n-1}(\boldsymbol{r},t) \qquad n = 1,2,3,\ldots\,. \tag{2.28}$$

The total flux then is simply the sum of the individual collided fluxes.

Using the multiple-collision formalism discussed in Section 2.1, it has been possible to derive analytic solutions for the uncollided and first collided fluxes for a pulsed source in space and time, in a variety of geometries, including semi-infinite and finite slabs and infinite and finite spheres. Additionally, analytic expressions for the uncol-

lided and n^{th} collided fluxes were derived for the case of a uniform source in space, pulsed source in time, for an infinite one-dimensional slab [21].

Further work with integral neutron transport has been in the area of numerical evaluation. Olson and Henderson [22] solved equation 2.26 for the total scalar flux where the source term includes both the external and scattered source. The resulting integral equation when equation 2.25 is substituted into equation 2.26 is a Volterra type in the temporal dimension and a Fredholm type in the spatial dimension. The integral equation was solved using an iterative process. It is important to note that the time domain grows with each iteration. The authors duplicated the Ganapol benchmark solutions [11] of a pulsed source in space and time in infinite slab and spherical coordinates. Additionally, for the case of a pulsed source in infinite spherical coordinates, the authors derived an analytic solution to the second collided flux. Finally, the authors produced new benchmark solutions for a number of finite medium geometries, including uniform and localized sources in a finite slab geometry, and spherical shell and uniform source in a finite sphere geometry.

A particularly important advantage of integral neutron transport is the inclusion of Heaviside functions in the kernels and analytic solutions, which explicitly incorporate causality. Causality is the notion that a particle traveling at a finite speed will need a finite amount of time to travel a given distance.

An important disadvantage of previous work in numerical integral neutron transport is that the time domain continually grows. The limits of integration for the time domain run from $t = 0$ to the current time, t. Therefore, as time increases, the upper limit of integration increases, leading to long computation times. This disadvantage is addressed through the current research, and is discussed in Section 3.1.

Integral neutron transport is preferred to other methods for solving neutron transport problems in radiation hydrodynamics codes. The source of neutrons from the fusion target burn is such that the neutrons are emitted in short bursts and the flux of neutrons will be confined by wavefronts. As discussed above, the diffusion equation is a poor approximation to systems that have well-defined wavefronts. The discrete ordinates method can result in ray effects, in the time domain as well as the spatial domain in optically low dense media. Ray effects are a phenomenon in which neutrons transport only along the angles for which the neutron transport equation is solved. For this reason, the discrete ordinates method can be a poor choice for solving the time-dependent neutron transport equation in hydrodynamics codes.

Chapter 3

Infinite Media

This chapter presents the work performed for infinite media. This work includes developing the integral form of the "reduced collisions" equations for both Cartesian and spherical coordinate systems, and reproducing benchmark solutions from the literature for these coordinate systems.

3.1 Infinite Slab Geometry

3.1.1 Mathematical Development

A Green's function is derived for the integral form of the "reduced collision equations" for an infinite slab geometry with an arbitrary isotropic source, and will be used to determine the time-dependent neutron flux.

To derive the Green's function, we begin with the time-dependent differential transport equation in planar coordinates for a one-dimensional infinite medium with an arbitrary source, $Q(x,t)$:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + \Sigma\right)\Psi(x,\mu,t) = \frac{Q(x,t)}{2}. \tag{3.1}$$

The differential form of the transport equation can be converted to an integral equation for the scalar flux through either the method of characteristics or Laplace transforms [21]. Integrating over the angular variable gives the time-dependent integral equation:

$$\Phi(x,t) = \int_0^t \int_{-\infty}^{\infty} K(x,x';t,t')Q(x',t')dx'dt' \tag{3.2}$$

where

$$\Phi(x,t) = \int \Psi(x,\mu,t)d\mu \tag{3.3}$$

is the scalar flux, $K(x,x';t,t')$ is the time-dependent kernel and $Q(x',t')$ is the time-dependent source. The source $Q(x',t')$ consists of both the external source, $S(x',t')$, and the isotropically scattered source, $\Sigma_s(x')\phi(x',t')$. Inserting the explicit expressions for the planar geometry scalar flux kernel [21] and the time-dependent arbitrary source into equation 3.2, and expanding the integral, one obtains

$$\Phi(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\Sigma v(t-t')}}{2(t-t')} H\left(t - t' - \frac{|x-x'|}{v}\right)\Phi(x',t')dx'dt' + \Phi_0(x,t) \tag{3.4}$$

where $\Phi_0(x,t)$ is the uncollided flux. The uncollided flux is calculated as

$$\Phi_0(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\Sigma v(t-t')}}{2(t-t')} H\left(t - t' - \frac{|x-x'|}{v}\right) S(x',t') dx' dt'. \tag{3.5}$$

The above equation is applied to the case of a unit planar source of pulsed neutrons located at the origin of an infinite medium, $S(x,t) = S_0\delta(x)\delta(t)$. Using this source in equation 3.5, the uncollided flux is found to be:

$$\Phi_0(x,t) = \frac{S_0}{2}\left(\frac{e^{-\Sigma vt}}{t}\right) H\left(t + \frac{x}{v}\right) H\left(t - \frac{x}{v}\right). \tag{3.6}$$

The above solution for the uncollided flux describes an outgoing planar wave of particles moving to the left and right. The neutrons are confined between the wavefronts at $x = vt$ and $x = -vt$ [21].

The Neumann series method is used to decompose the time-dependent integral equation into a series of equations for the individual collided fluxes. The integral equation for the nth collided flux is

$$\Phi_n(x,t) = \Sigma_s \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\Sigma v(t-t')}}{2(t-t')} H\left(t - t' - \frac{|x-x'|}{v}\right) \Phi_{n-1}(x',t') dx' dt' \tag{3.7}$$

for $n \geq 1$. The reduced collision equation ansatz for the nth collided flux has the form [13]:

$$\Phi_n(x,t) = \frac{S_0}{2}\left(\frac{e^{-\Sigma vt}}{t}\right)\left(\frac{(\Sigma_s vt)^n}{n!}\right) H\left(t + \frac{x}{v}\right) H\left(t - \frac{x}{v}\right) F_n(x,t) \tag{3.8}$$

where $F_0(x,t) = 1$. Inserting the ansatz into equation 3.7 and simplifying, the following expression is found:

$$F_n(x,t)H(t+\frac{x}{v})H(t-\frac{x}{v}) = \\ \frac{n}{2}\int_0^t\int_{-\infty}^{\infty}\frac{dx'dt'}{(t-t')vt'}\left(\frac{(t')^{n-1}}{t^{n-1}}\right)F_{n-1}(x',t')H\left(t-t'-\frac{|x-x'|}{v}\right)H(t'+\frac{x'}{v})H(t'-\frac{x'}{v}). \tag{3.9}$$

F_n is called the shape factor for the nth collided flux.

Next the integration variables x' and t' are transformed to a dimensionless domain. The transformed variables are defined as $\tau' = \frac{t'}{t}$ and $\eta' = \frac{x'}{vt'}$. The Jacobian $\left|\frac{\partial(x',t')}{\partial(\eta',\tau')}\right|$ evaluates to $vt^2\tau'$. Substituting the transformed variables into equation 3.9 and extracting the step functions, the following is obtained:

$$F_n(\eta) = \frac{n}{2}\left[\int_{-1}^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}}\frac{(\tau')^{n-1}}{1-\tau'}F_{n-1}(\eta')d\tau'd\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}}\frac{(\tau')^{n-1}}{1-\tau'}F_{n-1}(\eta')d\tau'd\eta'\right]. \tag{3.10}$$

The shape factors, F_n, depend only on the variable η. This can be seen from the fact that $F_0 = 1$, and that the limits of the outer integration only contain the variable η. As a result, the inner τ' integration can always be performed analytically. The numerical integration over the η' variable can be performed using simple integration methods, such as quadrature rules.

Equation 3.10 can be written more compactly in terms of kernels as:

$$F_n(\eta) = \frac{n}{2}\left[\int_{-1}^{\eta} K_{n,A}(\eta,\eta')F_{n-1}(\eta')d\eta' + \int_{\eta}^{1} K_{n,B}(\eta,\eta')F_{n-1}(\eta')d\eta'\right] \tag{3.11}$$

where

$$K_{n,A}(\eta,\eta') = \int_{0}^{\frac{1-\eta}{1-\eta'}} \frac{(\tau')^{n-1}}{1-\tau'}d\tau' \tag{3.12}$$

and

$$K_{n,B}(\eta,\eta') = \int_{0}^{\frac{1+\eta}{1+\eta'}} \frac{(\tau')^{n-1}}{1-\tau'}d\tau'. \tag{3.13}$$

The kernels for each n can be computed analytically. For instance, the kernels for $n = 1$ are

$$K_{1,A}(\eta,\eta') = -\ln\left(1 - \frac{1-\eta}{1-\eta'}\right) \tag{3.14}$$

and

$$K_{1,B}(\eta,\eta') = -\ln\left(1 - \frac{1+\eta}{1+\eta'}\right). \tag{3.15}$$

Additionally, the kernels for $n = 2$ are

$$K_{2,A}(\eta,\eta') = -\ln\left(1 - \frac{1-\eta}{1-\eta'}\right) - \frac{1-\eta}{1-\eta'} \tag{3.16}$$

and

$$K_{2,B}(\eta,\eta') = -\ln\left(1-\frac{1+\eta}{1+\eta'}\right) - \frac{1+\eta}{1+\eta'}. \tag{3.17}$$

A pattern in the form of the kernels appears likely from the above equations. Indeed, futher computation of the kernels leads to the following:

$$K_{n,A}(\eta,\eta') = -\ln\left(1-\frac{1-\eta}{1-\eta'}\right) - \sum_{i=2}^{n}\frac{1}{i-1}\left(\frac{1-\eta}{1-\eta'}\right)^{i-1} \quad \text{for } n \geq 2 \tag{3.18}$$

and

$$K_{n,B}(\eta,\eta') = -\ln\left(1-\frac{1+\eta}{1+\eta'}\right) - \sum_{i=2}^{n}\frac{1}{i-1}\left(\frac{1+\eta}{1+\eta'}\right)^{i-1} \quad \text{for } n \geq 2. \tag{3.19}$$

The kernels have an interesting property as $n \to \infty$. Note that following property of the summation:

$$\sum_{i=2}^{\infty}\frac{1}{i-1}\left(\frac{1-\eta}{1-\eta'}\right)^{i-1} = -\ln\left(1-\frac{1-\eta}{1-\eta'}\right) \tag{3.20}$$

and

$$\sum_{i=2}^{\infty}\frac{1}{i-1}\left(\frac{1+\eta}{1+\eta'}\right)^{i-1} = -\ln\left(1-\frac{1+\eta}{1+\eta'}\right). \tag{3.21}$$

We see, then, that both the kernels go to zero as $n \to \infty$.

It is possible to compute the first few shape factors analytically. The resulting first and second collided shape factors are [23]:

$$F_1(\eta) = -2\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right) + \left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right] \tag{3.22}$$

and

$$\begin{aligned} F_2(\eta) =& \frac{-\pi^2}{2}\left(1+\eta^2\right) - 6\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right) - 6\left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right) \\ &+ 3\left(\frac{1+\eta}{2}\right)^2\ln\left(\frac{1+\eta}{2}\right)^2 + 3\left(\frac{1-\eta}{2}\right)^2\ln\left(\frac{1-\eta}{2}\right)^2 \\ &+ 6\left(\frac{1+\eta}{2}\right)^2\mathrm{Li}_2\left(\frac{1+\eta}{2}\right) + 6\left(\frac{1-\eta}{2}\right)^2\mathrm{Li}_2\left(\frac{1-\eta}{2}\right) \end{aligned} \tag{3.23}$$

where $\mathrm{Li}_2(z)$ is the dilogarithm function, and is defined as [24]:

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-z)}{z}dz. \tag{3.24}$$

The dilogartihm belongs to the class of functions known as polylogarithms, and is a polylogarithm of order two.

The kernels, equations 3.14, 3.15, 3.18, and 3.19 are singular at the point $\eta' = \eta$. Using the subtraction of singularity method [25], discussed in Appendix A, the integral equation for the shape factor, equation 3.11, can be rewritten as:

$$\begin{aligned} F_n(\eta) =& \frac{n}{2}\left\{F_{n-1}(\eta)\int_{-1}^{\eta} K_{n,A}(\eta,\eta')d\eta' + \int_{-1}^{\eta} K_{n,A}(\eta,\eta')\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta'\right. \\ &\left. + F_{n-1}(\eta)\int_{\eta}^{1} K_{n,B}(\eta,\eta')d\eta' + \int_{\eta}^{1} K_{n,B}(\eta,\eta')\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta'\right\}. \end{aligned} \tag{3.25}$$

The first and third integrals can be performed analytically. The second and fourth integrals must be performed numerically. However, these integrals are equal to zero at the singularity.

Inserting the form of the kernels above into the first and third integrals of equation 3.25 and performing the integration, the following results are found for the first few values of n:

$$K_{1,A}(\eta) = K_{1,B}(\eta) = -2\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right) + \left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right], \tag{3.26}$$

$$K_{2,A}(\eta) = -2\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right)\right], \tag{3.27}$$

$$K_{2,B}(\eta) = -2\left[\left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right], \tag{3.28}$$

$$K_{3,A}(\eta) = -2\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right)\right] - \frac{1-\eta}{2} + \frac{(1-\eta)^2}{4}, \tag{3.29}$$

and

$$K_{3,B}(\eta) = -2\left[\left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right] - \frac{1+\eta}{2} + \frac{(1+\eta)^2}{4}. \tag{3.30}$$

For $n \geq 3$, a pattern emerges for the integration results of the kernels:

$$K_{n,A}(\eta) = -2\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right)\right] + \sum_{i=3}^{n}\left[\frac{-(1-\eta)}{(i-2)(i-1)} + \frac{(1-\eta)^{i-1}}{(i-2)(i-1)2^{i-2}}\right] \tag{3.31}$$

and

$$K_{n,B}(\eta) = -2\left[\left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right] + \sum_{i=3}^{n}\left[\frac{-(1+\eta)}{(i-2)(i-1)} + \frac{(1+\eta)^{i-1}}{(i-2)(i-1)2^{i-2}}\right]. \tag{3.32}$$

3.1.2 Shape Factors

Equation 3.25 now needs to be solved. The first and third integrals can be performed analytically, as shown above, while the second and fourth integrals must be computed numerically. The function to be solved for in equation 3.25, $F_n(\eta)$, appears only on the left-hand side of the equation. Therefore, simple numerical integration methods, such as Gaussian quadrature rules and the Chebyshev Polynomial Expansion method, can be utilized. Each shape factor, F_n, corresponds to the nth collided flux. Shown in Figure 3.1 are the uncollided and first five shape factors.

From this figure, it is evident that, as n increases, the height of the shape factor at $\eta = 0$ increases, while near the wavefront the shape factor goes to zero. It also appears that the area under the curves is conserved. To see if this is the case, the analytic functions for F_1 and F_2 were integrated over the range of η, $[-1, 1]$. When these calculations were performed, it was found that the area under both curves was equal to two. Additionally, some of the higher order shape factors were numerically integrated over η, and the area under these curves was also equal to two. It was

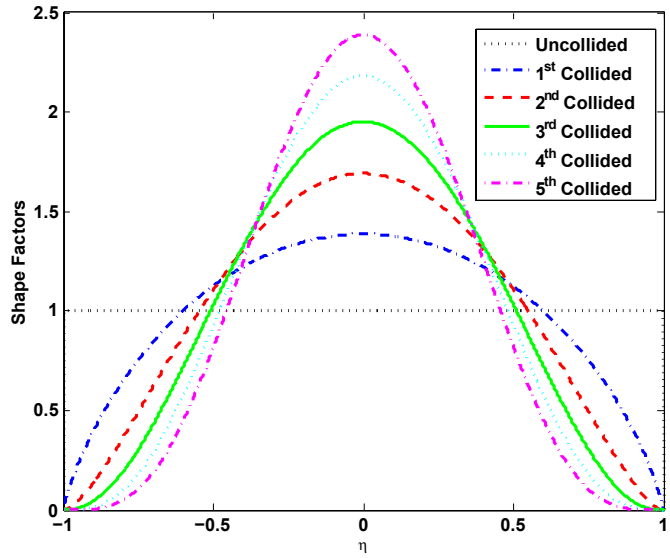

Fig. 3.1.— Infinite Slab Shape Factors

expected that the area under the curves would be constant, since the shape factors only represent the scattering of neutrons. The absorption of neutrons is represented in the exponential decay term of the ansatz, equation 3.8, and is not included in the shape factors. Also, since the calculations were performed for an infinite medium, neutrons would not be lost through leakage.

The points $\eta = \pm 1$ are the wavefronts of the neutrons, and correspond to the points $x = \pm vt$. As expected, only the uncollided shape factor is non-zero at the wavefront. From figure 3.1, we see that as the number of collisions increases, the domain on which the nth shape factor is non-zero decreases. This trend continues, as shown by the $n = 500$ shape factor in figure 3.2. Additionally, the shape factors, for $n \geq 2$ are peaked at the origin, $\eta = 0$. As $n \to \infty$, the shape factors become more sharply peaked, while the nonzero region decreases. In fact, the shape factors form a delta sequence, defined as [26]:

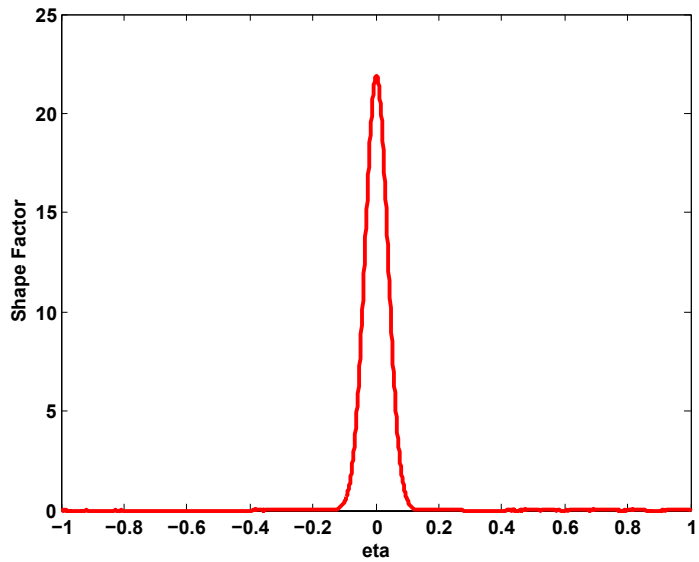

Fig. 3.2.— 500th Shape Factor

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0). \tag{3.33}$$

Given equation 3.33, we would expect the shape factors at large n to approximate the shape of a delta function, with a narrow width and large peak. This is in fact the case, as shown in Fig. 3.2, the $n = 500$ shape factor.

3.1.3 Benchmark Results

The nth collided flux is calculated from the nth shape factor using equation 3.8. The total flux for a pulsed source in time and space in an infinite medium is then calculated as a summation of the individual collided fluxes. For comparison to Ganapol's [11] and Olson's and Henderson's [22] benchmark solutions, the following values were chosen: the source strength, $S_0 = 1$, the neutron velocity, $v = 1$, the total cross section,

$\Sigma = 1$, and the absorption cross section, $\Sigma_a = 0$. For a given mean free time, t, the values for the distances, x_i, are calculated from:

$$x_i = vt\eta_i. \tag{3.34}$$

Equation 3.34 shows that there is a one-to-one correlation between η and x; that is, if 2501 points are used for calculations in η space, then there will be 2501 points in x space. As time increases, the size of the x domain increases, while the size of the η domain remains constant. Therefore, as time increases, the ratio of the length of the x domain to the number x points decreases.

Shown in Figure 3.3 below is the total neutron flux at mean free times of 1, 3, 5, 7, and 9. Figure 3.4 shows the total neutron flux at mean free times of 15, 20, 25, 35, and 45. Notice that the neutrons are always confined behind the wavefronts at $x = \pm vt$. At early times, the neutrons are clustered near the origin, and the flux is highly peaked. As time increases, the neutrons spread out, and the flux peak decreases.

Shown in Table 3.1 below are the results for the infinite slab benchmark at small mean free times, compared to the results obtained by Ganapol, and by Olson and Henderson. Table 3.2 shows the benchmark results, in comparison to the Ganapol and the Olson and Henderson results at later mean free times. Marked with an asterisk are those values that match neither the Ganapol results nor the Olson and Henderson results. Examining the tables, we see that there is very good agreement between the results obtained with the dimensionless integral method and the other benchmark

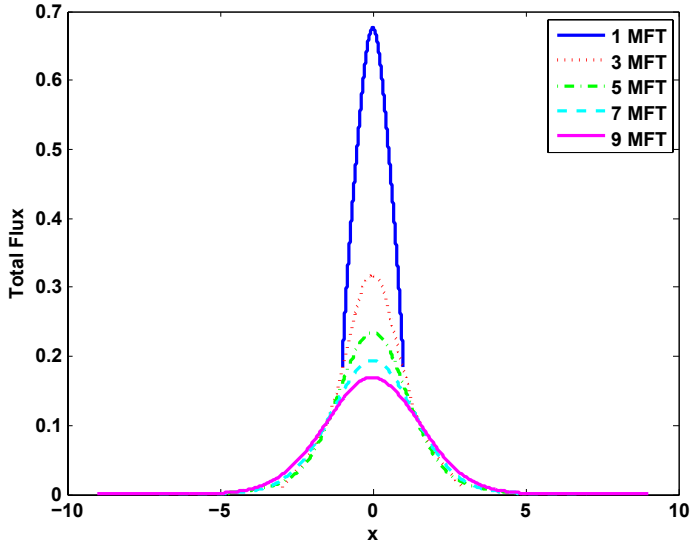

Fig. 3.3.— Total Flux at Various Early Mean Free Times

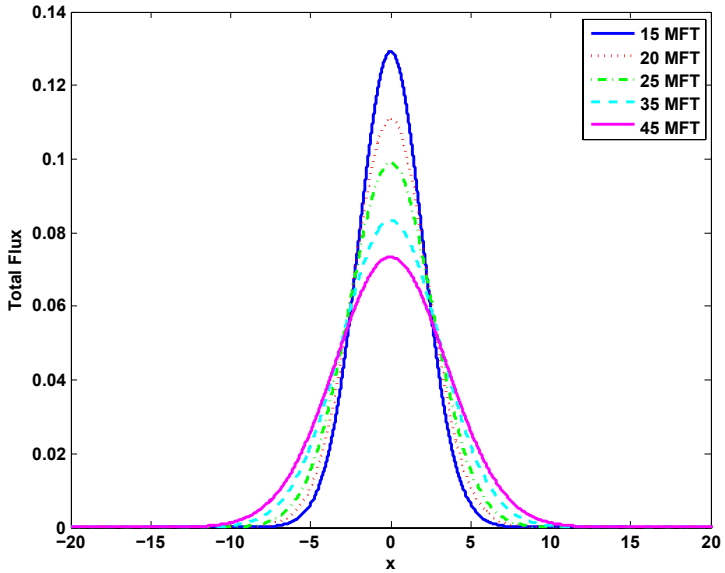

Fig. 3.4.— Total Flux at Various Late Mean Free Times

results. Note that even for those points in disagreement, the discrepancy is in the fourth decimal point.

The total flux can also be compared to PARTISN, as long as the size of the slab is large enough that the neutrons have not reached the boundary. This can be used as a check for PARTISN, to ensure that it is giving accurate results. The total flux for a pulsed source in space and time, at various mean free times, for a slab of half-width of $b = 100$ mean free paths, are given in the figures below. The neutron speed is $v = 1$, so that a neutron will move one mean free path in one mean free time. At mean free times greater than 80, a total of 150 collisions were used to compute the total flux using the dimensionless integral method.

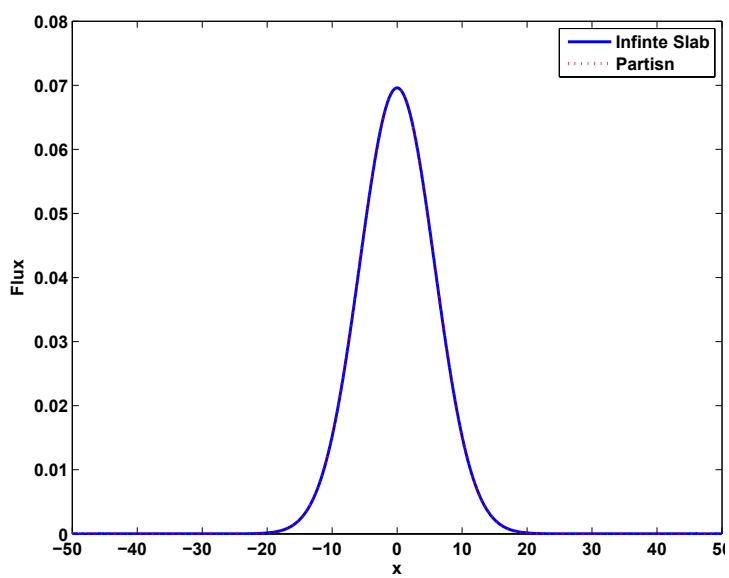

Fig. 3.5.— PARTISN and Infinite Slab Total Flux at 50 MFTs

Table 3.1: Benchmark Results for Infinite Slab at Early Mean Free Times

Time	x	**Ganapol**	**Olson and Henderson**	**Dimensionless Integral**
1	1	1.8394E-01	1.8394E-01	1.8394E-01
1	2	0.0000E+00	0.0000E+00	0.0000E+00
1	3	0.0000E+00	0.0000E+00	0.0000E+00
1	4	0.0000E+00	0.0000E+00	0.0000E+00
1	5	0.0000E+00	0.0000E+00	0.0000E+00
1	6	0.0000E+00	0.0000E+00	0.0000E+00
3	1	2.3942E-01	2.3942E-01	2.3942E-01
3	2	9.3836E-02	9.3835E-02	9.3837E-02*
3	3	8.2978E-03	8.2978E-03	8.2978E-03
3	4	0.0000E+00	0.0000E+00	0.0000E+00
3	5	0.0000E+00	0.0000E+00	0.0000E+00
3	6	0.0000E+00	0.0000E+00	0.0000E+00
5	1	1.9957E-01	1.9957E-01	1.9957E-01
5	2	1.2105E-01	1.2105E-01	1.2105E-01
5	3	4.9595E-02	4.9595E-02	4.9595E-02
5	4	1.1823E-02	1.1823E-02	1.1823E-02
5	5	6.7379E-04	6.7379E-04	6.7379E-04
5	6	0.0000E+00	0.0000E+00	0.0000E+00
7	1	1.7347E-01	1.7347E-01	1.7348E-01*
7	2	1.2293E-01	1.2293E-01	1.2293E-01
7	3	6.8028E-02	6.8028E-02	6.8028E-02
7	4	2.8447E-02	2.8447E-02	2.8447E-02
7	5	8.4158E-03	8.4157E-03	8.4158E-03
7	6	1.5036E-03	1.5036E-03	1.5037E-03*
9	1	1.5528E-01	1.5528E-01	1.5528E-01
9	2	1.1935E-01	1.1935E-01	1.1935E-01
9	3	7.6384E-02	7.6384E-02	7.6385E-02*
9	4	4.0186E-02	4.0186E-02	4.0185E-02*
9	5	1.7004E-02	1.7004E-02	1.7004E-02
9	6	5.5765E-03	5.5764E-03	5.5765E-03

Table 3.2: Benchmark Results for Infinite Slab at Late Mean Free Times

Time	x	**Ganapol**	**Olson and Henderson**	**Dimensionless Integral**
15	1	1.2269E-01	1.2269E-01	1.2269E-01
15	2	1.0514E-01	1.0514E-01	1.0514E-01
15	3	8.1158E-02	8.1159E-02	8.1159E-02
15	4	5.6305E-02	5.6305E-02	5.6305E-02
15	5	3.4985E-02	3.4985E-02	3.4985E-02
15	6	1.9376E-02	1.9376E-02	1.9376E-02
25	1	9.6128E-02	9.6128E-02	9.6129E-02*
25	2	8.7720E-02	8.7720E-02	8.7721E-02*
25	3	7.5287E-02	7.5287E-02	7.5287E-02
25	4	6.0744E-02	6.0744E-02	6.0744E-02
25	5	4.6042E-02	4.6042E-02	4.6042E-02
25	6	3.2757E-02	3.2757E-02	3.2757E-02
35	1	8.1632E-02	8.1632E-02	8.1632E-02
35	2	7.6491E-02	7.6491E-02	7.6491E-02
35	3	6.8624E-02	6.8624E-02	6.8624E-02
35	4	5.8937E-02	5.8937E-02	5.8937E-02
35	5	4.8445E-02	4.8445E-02	4.8444E-02*
35	6	3.8099E-02	3.8099E-02	3.8099E-02
45	1	7.2182E-02	7.2182E-02	7.2182E-02
45	2	6.8630E-02	6.8630E-02	6.8630E-02
45	3	6.3091E-02	6.3091E-02	6.3091E-02
45	4	5.6074E-02	5.6074E-02	5.6073E-02*
45	5	4.8177E-02	4.8177E-02	4.8176E-02*
45	6	4.0007E-02	4.0007E-02	4.0007E-02

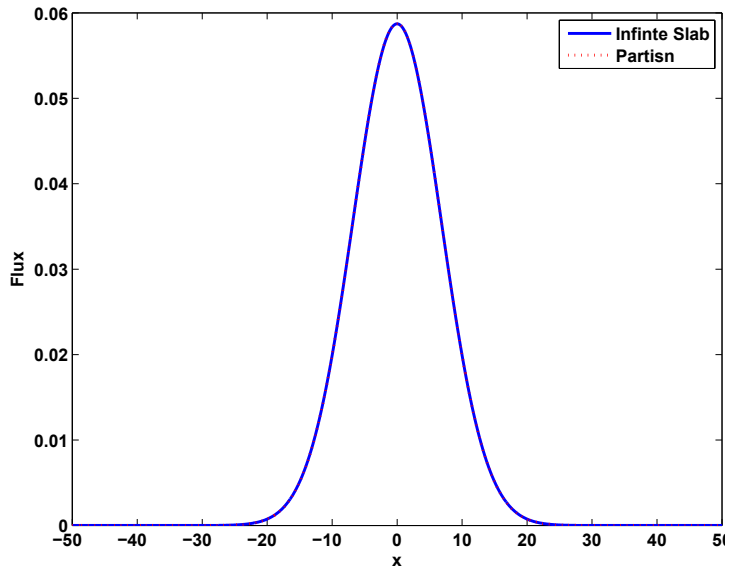

Fig. 3.6.— PARTISN and Infinite Slab Total Flux at 70 MFTs

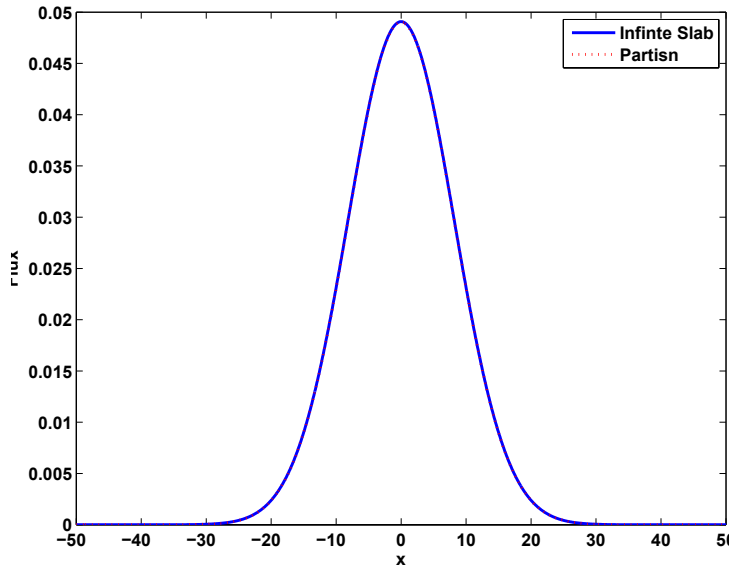

Fig. 3.7.— PARTISN and Infinite Slab Total Flux at 100 MFTs

3.1.3.1 Infinite Slab Error Analysis

The Euclidian norm of the total flux, in comparison to Ganapol's benchmark results, is shown in Table 3.3. Table 3.4 shows the Euclidian norm in comparison to the Olson and Henderson results. The maximum Euclidian norm was set to 10^{-5}, and allowed us to determine how many collisions were necessary for a given mean free time to obtain a total flux that was accurate in comparison to either the Ganapol benchmark results or the Olson and Henderson benchmark results. The Euclidean norm was calculated as:

$$Norm = \sqrt{\frac{\sum_{i=1}^{N} \left(\phi_{n,\mathrm{DI}}(x_i,t) - \phi_{n,\mathrm{Benchmark}}(x_i,t)\right)^2}{N}} \tag{3.35}$$

where $\phi_{n,\mathrm{DI}}$ is the nth collided flux obtained using the dimensionless integral method, and $\phi_{n,\mathrm{Benchmark}}$ is the nth collided flux obtained from either Ganapol's benchmark or Olson and Henderson's benchmark.

Table 3.3: Euclidian Norm and Number of Collisions, Ganapol Results

Time	**Number of Collisions**	**Euclidian Norm**
1	4	2.7941E-07
3	12	6.5686E-06
5	16	9.5323E-06
7	20	2.2152E-06
9	23	7.1675E-06
11	26	7.7016E-06
13	29	6.8604E-06
15	32	8.0801E-06
17	35	6.6292E-06
19	38	7.5120E-06
21	41	6.7145E-06
23	43	9.1038E-06
25	46	7.9991E-06
27	49	6.7690E-06
29	51	9.9345E-06
31	54	7.7301E-06
33	57	5.4631E-06
35	59	8.1612E-06
37	62	5.9891E-06
39	64	8.0880E-06
41	67	5.4323E-06
43	69	7.7964E-06
45	72	5.0386E-06

Table 3.4: Euclidian Norm and Number of Collisions, Olson and Henderson Results

Time	**Number of Collisions**	**Euclidian Norm**
1	4	2.7941E-07
2	10	1.9943E-06
3	12	6.6340E-06
4	14	9.3110E-06
5	16	9.5323E-06
6	18	4.2510E-06
7	20	2.0515E-06
8	21	6.5766E-06
9	23	5.8522E-06
10	24	9.1322E-06
14	30	8.9841E-06
15	32	6.7057E-06
16	33	8.7870E-06
19	38	6.1392E-06
20	39	7.4125E-06
21	41	5.4016E-06
24	45	5.2309E-06
25	46	6.6430E-06
26	47	8.7216E-06
29	51	8.3242E-06
30	53	5.5066E-06
31	54	6.6314E-06
34	58	6.0800E-06
35	59	7.2410E-06
36	60	8.5867E-06
39	64	7.3226E-06
40	65	8.6653E-06
41	66	9.9342E-06
44	70	8.0868E-06
45	71	9.0454E-06

3.2 Infinite Spherical Geometry

3.2.1 Mathmatical Development

To derive the Green's function for the infinite spherical medium case, we begin with the time-dependent differential transport equation in spherical coordinates for a one-dimensional infinite medium with an arbitrary source, $Q(r,t)$:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial r} + \frac{1-\mu^2}{r}\frac{\partial}{\partial \mu} + \Sigma\right)\Psi(r,\mu,t) = \frac{Q(r,t)}{2}. \tag{3.36}$$

As in the slab case, the differential form of the transport equation can be converted to an integral equation for the scalar flux through either the method of characteristics or Laplace transforms [21]. The time-dependent integral equation is then of the form

$$\Phi(r,t) = \int_0^t \int_0^\infty K(r,r';t,t')Q(r',t')dr'dt' \tag{3.37}$$

where $K(r,r';t,t')$ is the time-dependent kernel and $Q(r',t')$ is the time-dependent source. The source $Q(r',t')$ consists of both the external source, $S(r',t')$, and the isotropically scattered source, $\Sigma_s(r')\phi(r',t')$. Inserting the explicit expressions for the spherical shell scalar flux kernel [21] and the time-dependent arbitrary source into equation 3.37, and expanding the integral, one obtains

$$\Phi(r,t) = \Sigma_s \int_0^t \int_0^\infty \frac{e^{-\Sigma v(t-t')}}{8\pi r r'(t-t')} \left[H\left(t - t' - \frac{|r-r'|}{v}\right) - H\left(t - t' - \frac{|r+r'|}{v}\right) \right] \times \Phi(r',t') 4\pi r'^2 dr' dt' + \Phi_0(r,t) \tag{3.38}$$

where $\Phi_0(r,t)$ is the uncollided flux. The uncollided flux is calculated using the point source kernel as:

$$\Phi_0(r,t) = \int_0^t \int_0^\infty \frac{e^{-\Sigma v(t-t')}}{4\pi|\overrightarrow{r} - \overrightarrow{r}'|(t-t')v} \delta\left(t - t' - \frac{|\overrightarrow{r} - \overrightarrow{r}'|}{v}\right) S(r',t') dr' dt'. \tag{3.39}$$

The above equation is applied to the case of a unit point source of pulsed neutrons located at the origin of an infinite medium, $S(r,t) = \frac{S_0}{4\pi r^2}\delta(r)\delta(t)$. Using this source in equation 3.39, the uncollided flux is found to be:

$$\Phi_0(r,t) = \frac{S_0}{4\pi r v t} \left(\frac{e^{-\Sigma v t}}{t}\right) \delta\left(1 - \frac{r}{vt}\right). \tag{3.40}$$

The above solution for the uncollided flux describes an outgoing infinitesimal thin spherical shell of pulsed neutrons that is infinite at the wavefront and zero elsewhere. The uncollided flux has a strong singularity at the wavefront, and the first collided flux will inherit this.

The Neumann series method is used to decompose the time-dependent integral equation, equation 3.38, into a series of equations for the individual collided fluxes. The integral equation for the nth collided flux is

$$\Phi_n(r,t) = \Sigma_s \int_0^t \int_0^\infty \frac{e^{-\Sigma v(t-t')}}{8\pi r r'(t-t')} \left[H\left(t - t' - \frac{|r-r'|}{v}\right) - H\left(t - t' - \frac{|r+r'|}{v}\right) \right] \times \Phi_{n-1}(r',t') 4\pi r'^2 dr' dt' \tag{3.41}$$

for $n \geq 1$.

Equation 3.41 can be used to calculate each collided flux. However, there is a quicker way to calculate the collided fluxes if the planar collided fluxes are known. The following relation allows for the transformation from slab geometry fluxes, $\Phi_{n,pl}$ to spherical geometry fluxes, $\Phi_{n,sp}$, and is a simple way to obtain the spherical fluxes [10]:

$$\Phi_{n,sp}(r,t) = \frac{1}{2\pi r} \frac{\partial}{\partial r} \Phi_{n,pl}(r,t). \tag{3.42}$$

Using this relation and the planar forms for the first and second collided fluxes, the spherical fluxes are

$$\Phi_1(r,t) = \frac{S_0}{4\pi r} \frac{1}{vt} \left(\frac{e^{-\Sigma vt}}{t} \right) H\left(t - \frac{r}{v}\right) \ln\left(\frac{1 + \frac{r}{vt}}{1 - \frac{r}{vt}} \right) \tag{3.43}$$

and

$$\begin{aligned}\Phi_2(r,t) = &\frac{S_0}{4\pi r}\frac{1}{vt}\left(\frac{e^{-\Sigma vt}}{t}\right)\frac{(\Sigma_s vt)^2}{2}H\left(t-\frac{r}{v}\right)\\ &\left[\pi^2\frac{r}{vt}+\frac{3}{2}\left(1-\frac{r}{vt}\right)\ln\left(\frac{1-\frac{r}{vt}}{2}\right)^2-\frac{3}{2}\left(1+\frac{r}{vt}\right)\ln\left(\frac{1+\frac{r}{vt}}{2}\right)^2\right.\\ &\left.+\,3\left(1-\frac{r}{vt}\right)\mathrm{Li}_2\left(\frac{1-\frac{r}{vt}}{2}\right)-3\left(1+\frac{r}{vt}\right)\mathrm{Li}_2\left(\frac{1+\frac{r}{vt}}{2}\right)\right].\end{aligned} \tag{3.44}$$

These solutions agree with the results found in literature [11, 22], but were found through alternate means.

Returning to equation 3.41, the reduced collision equation ansatz for the nth collided flux has the form [27]:

$$\Phi_n(r,t) = \frac{S_0}{2}\left(\frac{e^{-\Sigma vt}}{t}\right)\left(\frac{(\Sigma_s vt)^n}{n!}\right)\frac{1}{(vt)^2}H\left(t-\frac{r}{v}\right)F_n(r,t) \tag{3.45}$$

where

$$F_1(r,t) = \frac{vt}{2\pi r}\ln\left(\frac{1+\frac{r}{vt}}{1-\frac{r}{vt}}\right) \tag{3.46}$$

and

$$\begin{aligned}F_2(r,t) =&\frac{vt}{2\pi r}\left[\pi^2\frac{r}{vt}+\frac{3}{2}\left(1-\frac{r}{vt}\right)\ln\left(\frac{1-\frac{r}{vt}}{2}\right)^2-\frac{3}{2}\left(1+\frac{r}{vt}\right)\ln\left(\frac{1+\frac{r}{vt}}{2}\right)^2\right.\\ &\left.+\,3\left(1-\frac{r}{vt}\right)\mathrm{Li}_2\left(\frac{1-\frac{r}{vt}}{2}\right)-3\left(1+\frac{r}{vt}\right)\mathrm{Li}_2\left(\frac{1+\frac{r}{vt}}{2}\right)\right].\end{aligned} \tag{3.47}$$

Since the first collided flux, equation 3.43 has a singularity at $r = vt$, the numerical calculations must begin at $n = 3$, with the second collided flux as the forcing function.

Inserting the ansatz, equation 3.45, into the Neumann series expansion for the integral form of the time-dependent neutron transport equation, equation 3.41, and simplifying, the following expression is obtained for the nth shape factor:

$$\begin{aligned} F_n(r,t)H\left(t-\frac{r}{v}\right) = \frac{n}{2}\int_0^t\int_0^\infty \frac{dr'dt'}{vt'(t-t')}\left(\frac{t'}{t}\right)^{n-3}\frac{r'}{r}\,H\left(t'-\frac{r'}{v}\right)F_{n-1}(r',t') \\ \times\left[H\left(t-t'-\frac{|r-r'|}{v}\right)-H\left(t-t'-\frac{|r+r'|}{v}\right)\right]. \end{aligned} \tag{3.48}$$

Next the integration variables r' and t' are transformed to the η', τ' domain. The transformed variables are defined as $\eta' = \frac{r'}{vt'}$ and $\tau' = \frac{t'}{t}$. The Jacobian of the transformation is:

$$\left|\frac{\partial(r',t')}{\partial(\eta',\tau')}\right| = vt^2\tau'.$$

Substituting the transformed variables into equation 3.48 and extracting the Heaviside functions, the following equation for the nth shape factor is obtained:

$$\begin{aligned} F_n(\eta) = \frac{n}{2}\Bigg[\int_0^{\frac{1-\eta}{1-\eta'}}\int_0^{\eta}\frac{(\tau')^{n-2}}{1-\tau'}\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta'd\tau' + \int_0^{\frac{1+\eta}{1+\eta'}}\int_{\eta}^{1}\frac{(\tau')^{n-2}}{1-\tau'}\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta'd\tau' \\ -\int_0^{\frac{1-\eta}{1+\eta'}}\int_0^{1}\frac{(\tau')^{n-2}}{1-\tau'}\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta'd\tau'\Bigg]. \end{aligned} \tag{3.49}$$

The notation can be simplified by introducing the concept of kernels:

$$F_n(\eta) = \frac{n}{2}\left[\int_0^{\eta} K_{n,A}(\eta,\eta')\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta' + \int_{\eta}^{1} K_{n,B}(\eta,\eta')\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta' - \int_0^1 K_{n,C}(\eta,\eta')\frac{\eta'}{\eta}F_{n-1}(\eta')d\eta'\right] \tag{3.50}$$

where the kernels are:

$$K_{n,A}(\eta,\eta') = \int_0^{\frac{1-\eta}{1-\eta'}} \frac{(\tau')^{n-2}}{1-\tau'}d\tau', \tag{3.51}$$

$$K_{n,B}(\eta,\eta') = \int_0^{\frac{1+\eta}{1+\eta'}} \frac{(\tau')^{n-2}}{1-\tau'}d\tau', \tag{3.52}$$

and

$$K_{n,C}(\eta,\eta') = \int_0^{\frac{1-\eta}{1+\eta'}} \frac{(\tau')^{n-2}}{1-\tau'}d\tau'. \tag{3.53}$$

The kernels can be evaluated analytically. As in the planar coordinates case, the kernels follow a pattern with n and can be written as:

$$K_{n,A} = \ln\left(1-\frac{1-\eta}{1-\eta'}\right) - \sum_{i=3}^{n}\frac{1}{i-2}\left(\frac{1-\eta}{1-\eta'}\right)^{i-2}, \tag{3.54}$$

$$K_{n,B} = \ln\left(1-\frac{1+\eta}{1+\eta'}\right) - \sum_{i=3}^{n}\frac{1}{i-2}\left(\frac{1+\eta}{1+\eta'}\right)^{i-2}, \tag{3.55}$$

and

$$K_{n,C} = \ln\left(1-\frac{1-\eta}{1+\eta'}\right) - \sum_{i=3}^{n}\frac{1}{i-2}\left(\frac{1-\eta}{1+\eta'}\right)^{i-2}. \tag{3.56}$$

Examining the kernels, we see that $K_{n,A}$ and $K_{n,B}$ have a singularity at $\eta' = \eta$, while $K_{n,C}$ has a singularity at $\eta' = \eta = 0$. The singularities can be handled through the subtraction of singularity method. Using this method, the expression for F_n becomes:

$$\begin{aligned} F_n(\eta) =& \frac{n}{2}\Bigg[\int_0^{\eta} K_{n,A}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^{\eta} K_{n,A}(\eta,\eta')\frac{\eta'}{\eta}d\eta' \\ &+ \int_{\eta}^{1} K_{n,B}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_{\eta}^{1} K_{n,B}(\eta,\eta')\frac{\eta'}{\eta}d\eta' \\ &- \int_0^{1} K_{n,C}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^{1} K_{n,C}(\eta,\eta')\frac{\eta'}{\eta}d\eta'\Bigg]. \end{aligned} \tag{3.57}$$

The second, fourth, and sixth integrals may be evaluated analytically. Carrying out the integration, the following expressions are obtained for $n = 3$:

$$\int_0^{\eta} K_{3,A}(\eta,\eta')\eta' d\eta' = \frac{\eta}{2} - \frac{\eta^2}{2} + \frac{(1-\eta)^2}{2}\ln(1-\eta) - \frac{\eta^2}{2}\ln(\eta), \tag{3.58}$$

$$\int_{\eta}^{1} K_{3,B}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \frac{\eta^2}{2} - \frac{(1+\eta)^2}{2}\ln(1+\eta) - \frac{(1-\eta^2)}{2}\ln(1-\eta) + (1+\eta)\ln 2, \tag{3.59}$$

and

$$\int_0^1 K_{3,C}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \frac{\eta}{2} - \frac{\eta^2}{2}\ln(\eta) - \frac{(1-\eta^2)}{2}\ln(1+\eta) + (1-\eta)\ln 2. \quad (3.60)$$

For $n = 4$, the integrals of the kernels be written as:

$$\int_0^\eta K_{4,A}(\eta,\eta')\eta' d\eta' = -\frac{\eta^2}{2}\ln(\eta), \quad (3.61)$$

$$\int_\eta^1 K_{4,B}(\eta,\eta')\eta' d\eta' = -\frac{1}{4} + \frac{\eta^2}{4} - \frac{(1-\eta^2)}{2}\ln(1-\eta) + \frac{(1-\eta^2)}{2}\ln 2, \quad (3.62)$$

and

$$\int_\eta^1 K_{4,C}(\eta,\eta')\eta' d\eta' = -\frac{1}{4} + \frac{\eta^2}{4} - \frac{\eta^2}{2}\ln(\eta) + \frac{(1-\eta^2)}{2}\ln 2. \quad (3.63)$$

Finally, for $n \geq 5$, a pattern in the integrals of the kernels emerges:

$$\begin{aligned}\int_0^\eta K_{n,A}(\eta,\eta')\eta' d\eta' = \left(1-\eta^2\right)\ln(1-\eta) - \frac{\eta^2}{2}\ln(\eta) + \sum_{i=5}^{n}\Bigg[-\frac{(1-\eta)}{(i-2)(i-3)} \\ + \frac{(1-\eta)^2}{(i-2)(i-4)} - \frac{(1-\eta)^{i-2}}{(i-2)(i-3)(i-4)}\Bigg],\end{aligned} \quad (3.64)$$

$$\begin{aligned}\int_\eta^1 K_{n,B}(\eta,\eta')\eta' d\eta' = \frac{\eta^2}{4} - \frac{1}{4} - \frac{(1-\eta^2)}{2}\ln(1-\eta) + \frac{(1-\eta^2)}{2}\ln 2 \\ + \sum_{i=5}^{n}\left[\frac{(1+\eta)}{(i-2)(i-3)} - \frac{(1+\eta)^2}{(i-2)(i-4)} + \frac{(1+\eta)^{i-2}}{(i-3)(i-4)2^{i-3}}\right],\end{aligned} \quad (3.65)$$

and

$$\int_{\eta}^{1} K_{n,C}(\eta,\eta')\eta' d\eta' = \frac{\eta^2}{4} - \frac{1}{4} + \frac{(1-\eta^2)}{2}\ln(1+\eta) + \frac{(1-\eta^2)}{2}\ln 2 + \sum_{i=5}^{n}\left[\frac{(1-\eta)^{i-2}}{i-2}\left(\frac{2^{4-i}-1}{i-4} - \frac{2^{3-i}}{i-3}\right)\right]. \tag{3.66}$$

The above expressions contain a singularity at the point $\eta = 0$. Appendix B shows the derivation of the expressions that must be solved for the nth shape factor at $\eta = 0$. These expressions are reproduced below:

$$F_3(0) = \frac{3}{2}\left[2F_2(0)\ln 2 + \int_0^1 \left[F_2(\eta') - F_2(0)\right]\eta' \frac{2}{(1+\eta')\eta'} d\eta'\right], \tag{3.67}$$

$$F_4(0) = 2\left[F_3(0) + 2\int_0^1 \left[F_3(\eta') - F_3(0)\right]\eta'\left(\frac{1}{\eta'} - \frac{1}{1+\eta'} - \frac{1}{(1+\eta')^2}\right) d\eta'\right], \tag{3.68}$$

and

$$F_n(0) = \frac{n}{2}\left[F_{n-1}(0)\sum_{i=5}^{n}\left[\frac{(i-2)(16+2^i(i-5))}{(i-3)(i-4)2^i}\right] + 2\int_0^1 \left[F_{n-1}(\eta') - F_{n-1}(0)\right]\eta'\left(\frac{1}{\eta'} - \sum_{i=3}^{n}\frac{1}{(i-2)(1+\eta')^{i-2}}\right) d\eta'\right]. \tag{3.69}$$

3.2.2 Shape Factors

Equation 3.57 now needs to be solved. The second, fourth, and sixth integrals can be evaluated analytically, while the first, third, and fifth integrals must be evaluated numerically. Again, a simple numerical integration method can be implemented. The Clenshaw-Curtis quadrature rule was used for the fifth integral, while the Chebyshev Polynomial Expansion was used for the first and third integrals. At the point $\eta = 0$, equations 3.67, 3.68, and 3.68 must be solved, using either the Clenshaw-Curtis quadrature or the Chebyshev Polynomial Expansion. Shown in figure 3.8 below are the first five collided shape factors. The uncollided flux, equation 3.40, is a delta function at the wavefront.

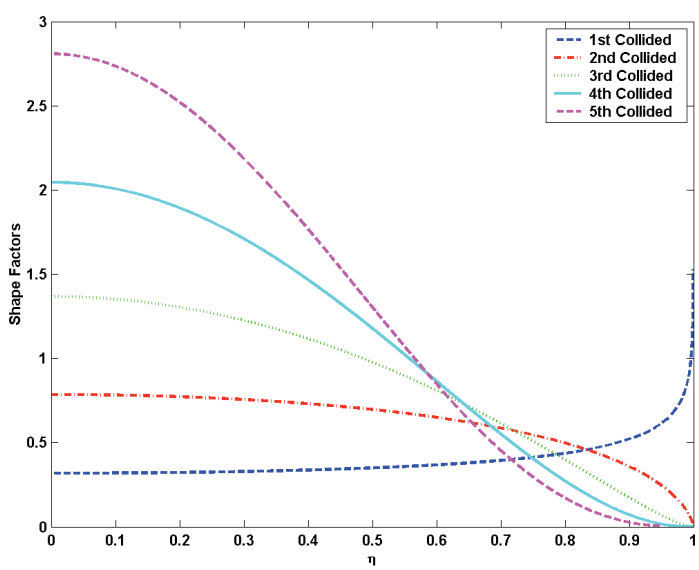

Fig. 3.8.— Infinite Sphere Shape Factors

From this figure, it is apparent that the first collided shape factor has inherited

the singularity from the uncollided shape factor at the wavefront. As in the infinite slab case, the collided shape factors, except for the first collided shape factor, go to zero at the wavefront. It should also be noted that the areas under the curves of the collided shape factors increase, instead of staying constant, as in the slab case. This is because the volume through which the neutrons travel increases as the area of a sphere, $4\pi\eta^2$.

Shown in figure 3.9 below are the first five shape factors multiplied by the factor $4\pi\eta^2$. The area under these curves is equal to the volume integral of the shape factors. Therefore, the area under these curves is conserved, and equals two.

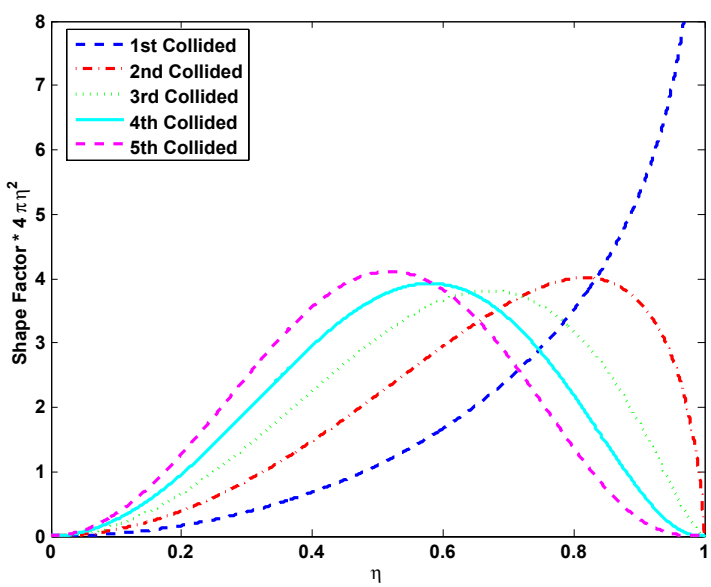

Fig. 3.9.— Infinite Sphere Shape Factors Multiplied by $4\pi\eta^2$

3.2.3 Benchmark Results

The nth collided flux is calculated from the nth shape factor by using equation 3.45. The total flux is again calculated as the sum of the individual collided fluxes. For comparison to Ganapol's [11] and Olson's and Henderson's [22] benchmark solutions, the following values were chosen:

- Source strength, $S_0 = 1$
- Neutron speed, $v = 1$
- Total cross section, $\Sigma = 1$
- Absorption cross section, $\Sigma_a = 0$.

For a given mean free time, t, the values for the distance, r_i, are calculated as:

$$r_i = vt\eta_i \tag{3.70}$$

Again, there is a one-to-one correlation between η and r.

Shown in figures 3.10 and 3.11 below is the total flux at early mean free times. From these figures, it is obvious how quickly the total flux falls off.

The benchmark results, shown in comparison to Ganapol's and Olson's and Henderson's solutions, are given in Tables 3.5 and 3.6. Marked with an asterisk are those values that match neither the Ganapol results nor the Olson and Henderson results.

As with the infinite slab, the total flux can also be compared to PARTISN, as long as the radius of the sphere is large enough that the neutrons have not reached

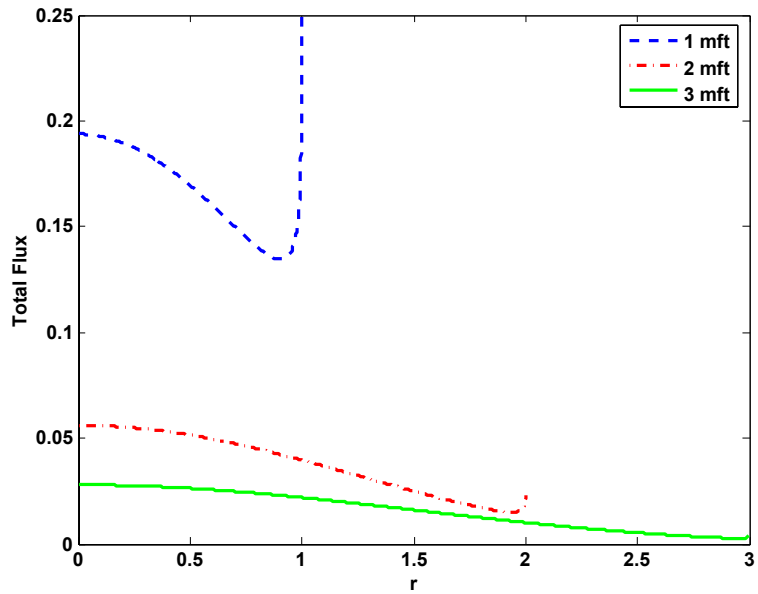

Fig. 3.10.— Infinite Sphere Total Flux at Early Mean Free Times

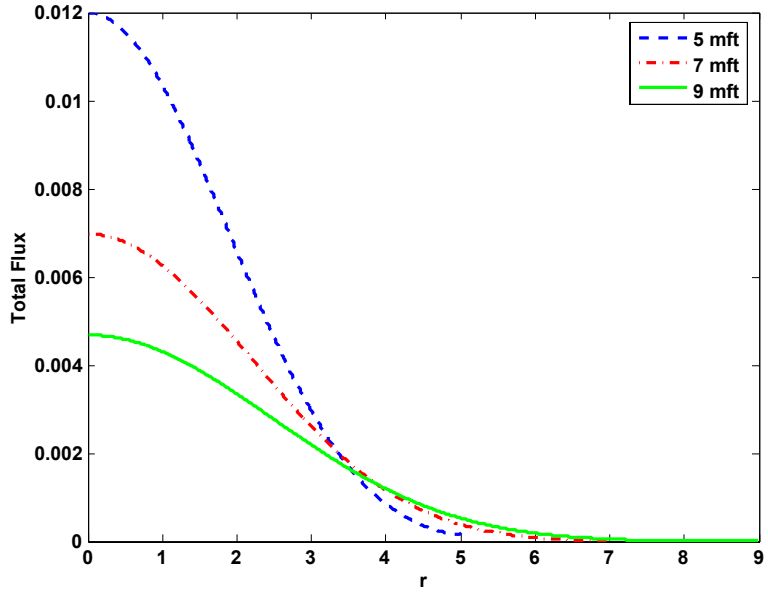

Fig. 3.11.— Infinite Sphere Total Flux at Early Mean Free Times

Table 3.5: Benchmark Results for Infinite Sphere at Early Mean Free Times

Time	r	**Ganapol**	**Olson and Henderson**	**Dimensionless Integral**
1	1	∞	∞	∞
1	2	0.0000E+00	0.0000E+00	0.0000E+00
1	3	0.0000E+00	0.0000E+00	0.0000E+00
1	4	0.0000E+00	0.0000E+00	0.0000E+00
1	5	0.0000E+00	0.0000E+00	0.0000E+00
1	6	0.0000E+00	0.0000E+00	0.0000E+00
3	1	2.2001E-02	2.2001E-02	2.2001E-02
3	2	1.0187E-02	1.0187E-02	1.0187E-02
3	3	∞	∞	∞
3	4	0.0000E+00	0.0000E+00	0.0000E+00
3	5	0.0000E+00	0.0000E+00	0.0000E+00
3	6	0.0000E+00	0.0000E+00	0.0000E+00
5	1	1.0305E-02	1.0305E-02	1.0305E-02
5	2	6.5738E-03	6.5738E-03	6.5739E-03*
5	3	2.9565E-03	2.9565E-03	2.9565E-03
5	4	8.5550E-04	8.5549E-04	8.5550E-04
5	5	∞	∞	∞
5	6	0.0000E+00	0.0000E+00	0.0000E+00
7	1	6.2715E-03	6.2715E-03	6.2715E-03
7	2	4.5417E-03	4.5417E-03	4.5417E-03
7	3	2.6143E-03	2.6143E-03	2.6144E-03*
7	4	1.1654E-03	1.1654E-03	1.1654E-03
7	5	3.8287E-04	3.8287E-04	3.8287E-04
7	6	8.3430E-05	8.3430E-05	8.3430E-05
9	1	4.3089E-03	4.3089E-03	4.3089E-03
9	2	3.3538E-03	3.3538E-03	3.3538E-03
9	3	2.1944E-03	2.1944E-03	2.1944E-03
9	4	1.1937E-03	1.1937E-03	1.1937E-03
9	5	5.3016E-04	5.3016E-04	5.3016E-04
9	6	1.8655E-04	1.8655E-04	1.8655E-04

Table 3.6: Benchmark Results for Infinite Sphere at Late Mean Free Times

Time	r	**Ganapol**	**Olson and Henderson**	**Dimensionless Integral**
15	1	2.0059E-03	2.0059E-03	2.0059E-03
15	2	1.7263E-03	1.7263E-03	1.7263E-03
15	3	1.3423E-03	1.3423E-03	1.3423E-03
15	4	9.4107E-04	9.4106E-04	9.4107E-04
15	5	5.9294E-04	5.9294E-04	5.9295E-04*
15	6	3.3430E-04	3.3430E-04	3.3430E-04
25	1	9.3283E-04	9.3283E-04	9.3284E-04*
25	2	8.5252E-04	8.5252E-04	8.5253E-04*
25	3	7.3353E-04	7.3353E-04	7.3353E-04
25	4	5.9394E-04	5.9394E-04	5.9395E-04*
25	5	4.5228E-04	4.5228E-04	4.5229E-04*
25	6	3.2364E-04	3.2363E-04	3.2364E-04
35	1	5.6323E-04	5.6323E-04	5.6324E-04*
35	2	5.2816E-04	5.2816E-04	5.2816E-04
35	3	4.7444E-04	4.7443E-04	4.7444E-04
35	4	4.0819E-04	4.0819E-04	4.0820E-04*
35	5	3.3630E-04	3.3629E-04	3.3630E-04
35	6	2.6523E-04	2.6523E-04	2.6523E-04
45	1	3.8637E-04	3.8637E-04	3.8637E-04
45	2	3.6752E-04	3.6752E-04	3.6753E-04*
45	3	3.3812E-04	3.3812E-04	3.3812E-04
45	4	3.0083E-04	3.0083E-04	3.0084E-04*
45	5	2.5882E-04	2.5882E-04	2.5882E-04
45	6	2.1530E-04	2.1529E-04	2.1530E-04

the boundary. This can be used as a check for PARTISN, to ensure that it is giving accurate results. The total flux for a pulsed source in space and time, at various mean free times, for a sphere of radius $b = 10$ mean free paths, are given in the figures below. The neutron speed is $v = 1$, so that a neutron will move one mean free path in one mean free time. A smaller value of b was picked than with the infinite slab, so as to illustrate the problems PARTISN has resolving the total flux at early mean free times

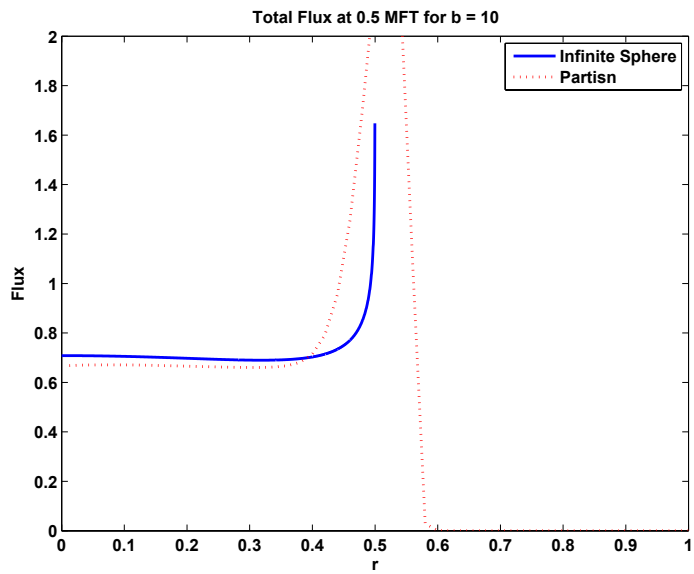

Fig. 3.12.— PARTISN and Infinite Sphere Total Flux at 0.5 MFTs

At early mean free times, PARTISN has trouble resolving the singularity at the wavefront. We see that the PARTISN flux has a more gradual increase in the flux at the wavefront, then has a flux of neutrons in front of the wavefront, where there should be zero flux. At later mean free times the PARTISN and integral transport results show very good agreement.

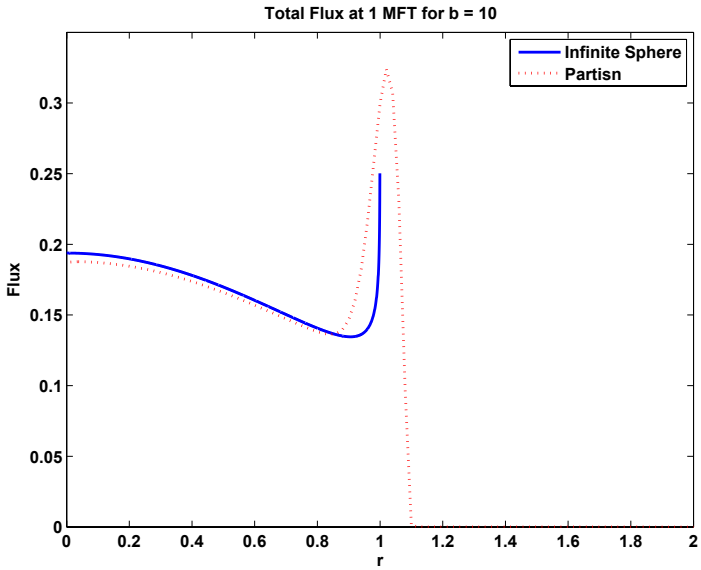

Fig. 3.13.— PARTISN and Infinite Sphere Total Flux at 1 MFTs

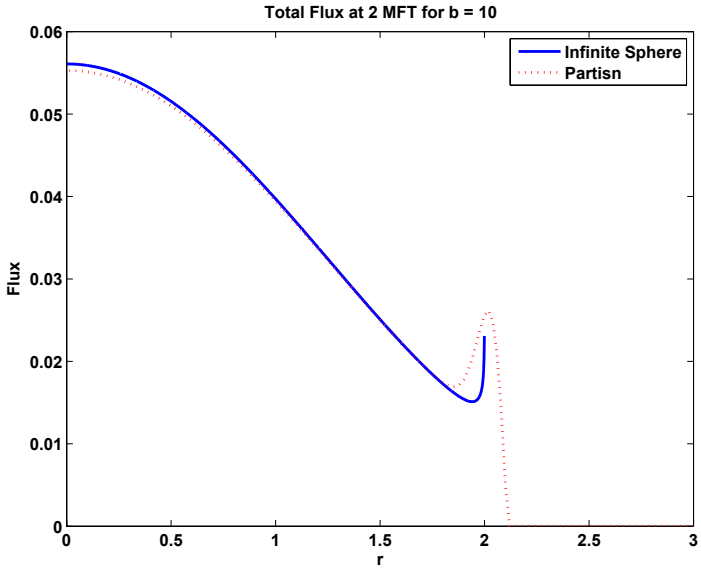

Fig. 3.14.— PARTISN and Infinite Sphere Total Flux at 2 MFTs

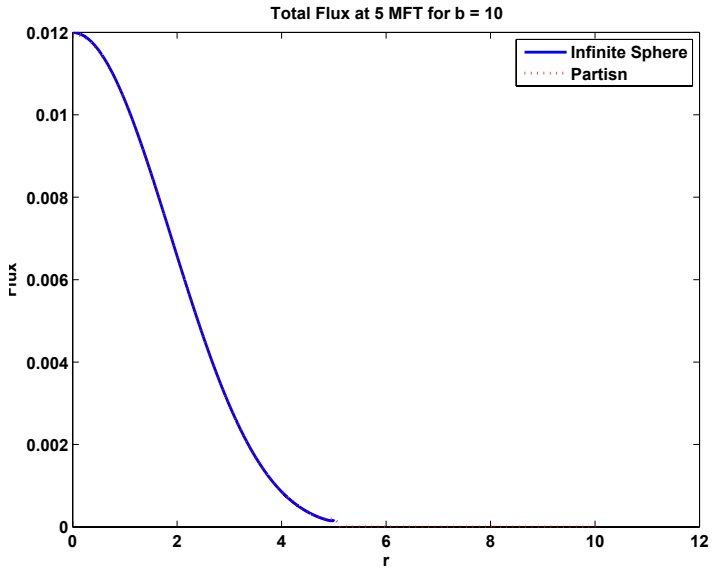

Fig. 3.15.— PARTISN and Infinite Sphere Total Flux at 5 MFTs

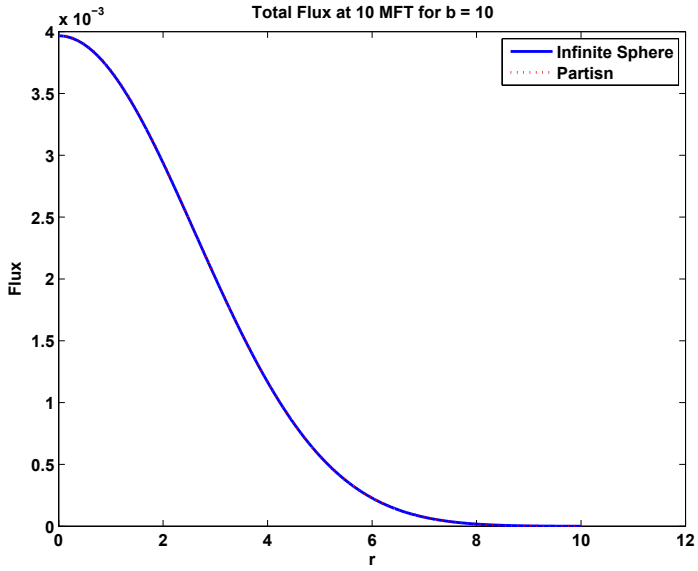

Fig. 3.16.— PARTISN and Infinite Sphere Total Flux at 10 MFTs

3.2.3.1 Infinite Sphere Error Analysis

As with the infinite slab, the Euclidean norm was used to determine how many collisions were necessary for a given mean free time to obtain a total flux that was accurate in comparison to either the Ganapol benchmark results or the Olson and Henderson benchmark results. The Euclidean norm was calculated as:

$$Norm = \sqrt{\frac{\sum_{i=1}^{N} \left(\phi_{n,\mathrm{DI}}(r_i, t) - \phi_{n,\mathrm{Benchmark}}(r_i, t)\right)^2}{N}} \tag{3.71}$$

where $\phi_{n,\mathrm{DI}}$ is the nth collided flux obtained using the dimensionless integral method, and $\phi_{n,\mathrm{Benchmark}}$ is the nth collided flux obtained from either Ganapol's benchmark or Olson and Henderson's benchmark.

The Euclidian norm of the total flux, in comparison to Ganapol's benchmark results, is shown in Table 3.7. Table 3.4 shows the Euclidian norm in comparison to the Olson and Henderson results.

Table 3.7: Euclidian Norm and Number of Collisions, Ganapol Results

Time	**Number of Collisions**	**Euclidian Norm**
1	–	–
3	12	3.4012E-06
5	15	4.5644E-06
7	17	8.6092E-06
9	20	6.4914E-06
11	22	9.7837E-06
13	25	6.7904E-06
15	27	8.8387E-06
17	30	6.1001E-06
19	32	7.4410E-06
21	34	8.6989E-06
23	36	9.8371E-06
25	39	6.9289E-06
27	41	7.7260E-06
29	43	8.4588E-06
31	45	9.1261E-06
33	47	9.7349E-06
35	50	7.0962E-06
37	52	7.5463E-06
39	54	7.9548E-06
41	56	8.3281E-06
43	58	8.6634E-06
45	60	8.9658E-06

Table 3.8: Euclidian Norm and Number of Collisions, Olson and Henderson Results

Time	**Number of Collisions**	**Euclidian Norm**
1	–	–
2	10	3.1735E-06
3	12	3.4012E-06
4	13	8.1046E-06
5	15	4.5644E-06
6	16	6.5469E-06
7	17	8.6092E-06
8	19	4.5405E-06
9	20	5.6217E-06
10	21	6.6429E-06
14	26	6.3916E-06
15	27	7.2163E-06
16	28	8.0174E-06
19	32	6.0770E-06
20	33	6.5933E-06
21	34	7.1052E-06
24	37	8.4951E-06
25	38	8.9221E-06
26	39	9.3257E-06
29	43	6.9396E-06
30	44	7.2264E-06
31	45	7.5031E-06
34	48	8.2601E-06
35	49	8.4874E-06
36	50	8.7067E-06
39	53	9.2933E-06
40	54	9.4670E-06
41	55	9.6334E-06
44	59	7.3757E-06
45	60	7.5108E-06

Chapter 4

Finite Media

This chapter shows how to expand the dimensionless integral formulation to incorporate finite media. Section 4.1 shows how to apply the method to a finite slab of width $2b$, while Sections 4.2.1 and 4.2.2 show two approaches to expanding the formulation to a finite sphere of radius b.

4.1 Finite Slab Medium

The derivation of the finite slab shape factor expression closely follows the derivation for the infinite slab shape factor. The slab now considered will have a width of $2b$, where b is referred to as the slab half-width. Including this width changes the expression for the total flux in Cartesian coordinates, and equation 3.4 becomes:

$$\Phi(x,t) = \Sigma_s \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\Sigma v(t-t')}}{2(t-t')} H\left(t - t' - \frac{|x-x'|}{v}\right) H(b-x')H(b+x')\Phi(x',t')dx'dt' + \Phi_0(x,t), \tag{4.1}$$

where the boundaries of the slab are delimited by the Heaviside functions $H(b-x')$ and $H(b+x')$, and vacuum boundary conditions exist at $x=\pm b$. A pulsed source in space and time is again used, so that the uncollided flux is still given by equation 3.6.

Using the same reduced collision ansatz as for the infinite slab case, equation 3.8, the following expression for the nth shape factor is found:

$$\begin{aligned} F_n(x,t)H(t+\frac{x}{v})H(t-\frac{x}{v}) = \\ \frac{n}{2}\int_0^t\int_{-\infty}^{\infty}\frac{dx'dt'}{(t-t')vt'}\left(\frac{(t')^{n-1}}{t^{n-1}}\right)F_{n-1}(x',t')H\left(t-t'-\frac{|x-x'|}{v}\right) \\ \times H\left(t'+\frac{x'}{v}\right)H\left(t'-\frac{x'}{v}\right)H(b-x')H(b+x'). \end{aligned} \tag{4.2}$$

When deriving the expression for the nth collided shape factor for an infinite slab, the next step was to introduce the dimensionless variables η' and τ'. Here, however, we will first define the integration region in x' and t' space, and then transform the equation to the dimensionless coordinate system. The integration domain delimited by the Heaviside functions can be represented graphically. For the $n=1$ shape factor, the domain is shown in Fig. 4.1 below. The $n=1$ shape factor can be expressed in x' and t' coordinates as:

$$
\begin{aligned}
F_1(x,t) =& \frac{1}{2}\Bigg[\int_{\frac{x-vt}{2}}^{0}\int_{\frac{-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_0(x',t')\frac{dt'dx'}{vt'(t-t')} + \int_{0}^{x}\int_{\frac{x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_0(x',t')\frac{dt'dx'}{vt'(t-t')} \\
&+ \int_{x}^{\frac{x+vt}{2}}\int_{\frac{x'}{v}}^{t-\frac{x'}{v}+\frac{x}{v}} F_0(x',t')\frac{dt'dx'}{vt'(t-t')} \\
&- H(vt-2b-x)\int_{\frac{x-vt}{2}}^{-b}\int_{\frac{-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_0(x',t')\frac{dt'dx'}{vt'(t-t')} \\
&- H(vt-2b+x)\int_{b}^{\frac{x+vt}{2}}\int_{\frac{x'}{v}}^{t-\frac{x'}{v}+\frac{x}{v}} F_0(x',t')\frac{dt'dx'}{vt'(t-t')}\Bigg].
\end{aligned}
\tag{4.3}
$$

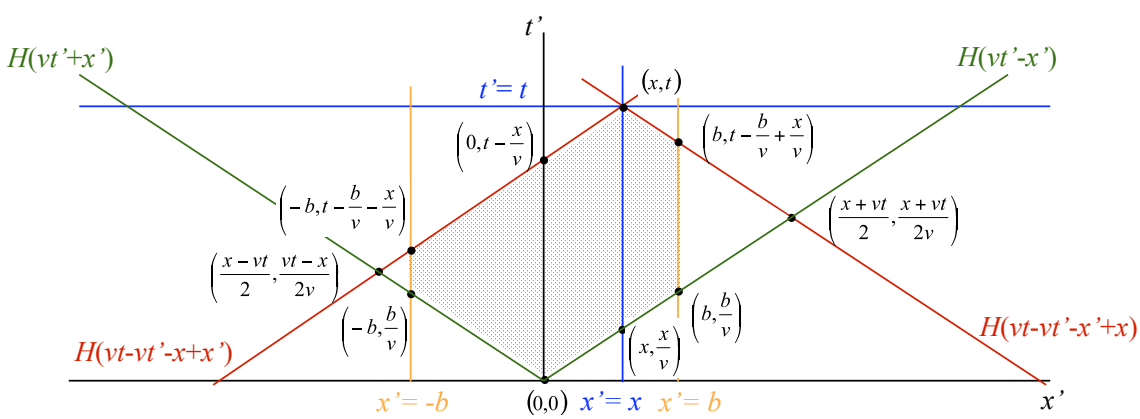

Fig. 4.1.— F_1 Integration Domain in x' and t' Space

Examining equation 4.3, we can identify the first three integrals as the infinite medium solution for the first collided shape factor. The last two integrals correspond to the area subtracted from the infinite medium integration domain as shown in Fig. 4.1 above.

The integration domain for the $n = 1$ shape factor in η' and τ' space is shown graphically in Fig. 4.2 below.

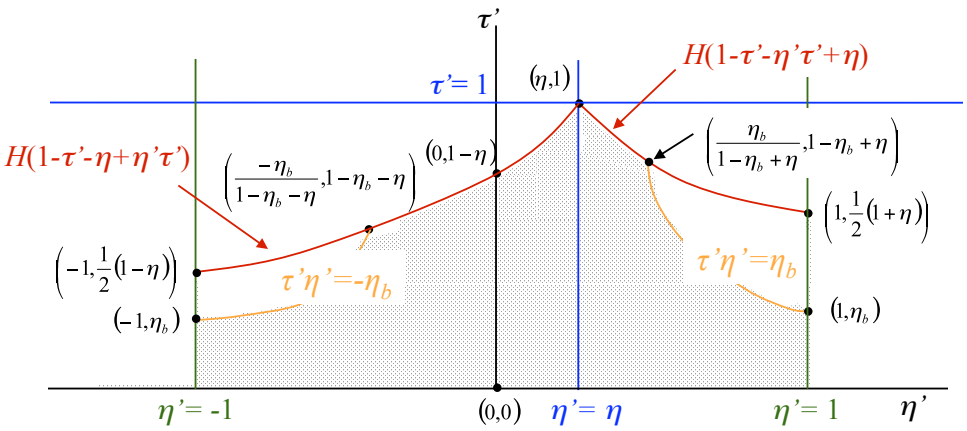

Fig. 4.2.— F_1 Integration domain in η' and τ' Space

The form of the first collided shape factor in η' and τ' space is

$$
\begin{aligned}
F_1(\eta,\tau) =& \frac{1}{2}\Bigg[\int_{-1}^{\eta}\int_{0}^{\frac{1-\eta}{1-\eta'}} F_0(\eta')\frac{d\tau' d\eta'}{1-\tau'} + \int_{\eta}^{1}\int_{0}^{\frac{1+\eta}{1+\eta'}} F_0(\eta')\frac{d\tau' d\eta'}{1-\tau'} \\
&- H(1-2\eta_b-\eta)\int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} F_0(\eta')\frac{d\tau' d\eta'}{1-\tau'} \\
&- H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} F_0(\eta')\frac{d\tau' d\eta'}{1-\tau'}\Bigg]
\end{aligned}
\tag{4.4}
$$

where $\eta_b = \frac{b}{vt}$. Note that η_b has a t in the denominator, making η_b a variable. When a function containing an η_b is written in terms of integration variables, any occurrence of η_b becomes:

$$
\frac{b}{vt'} = \frac{b}{vt'}\frac{t}{t} = \frac{\eta_b}{\tau'}. \tag{4.5}
$$

The first two integrals are simply the infinite medium solution of the first collided shape factor while the last two integrals represent neutrons lost from the medium. The $n = 1$ shape factor can be found analytically as:

$$
\begin{aligned}
F_1(\eta,\tau) = -2 &\left[\left(\frac{1+\eta}{2}\right)\ln\left(\frac{1+\eta}{2}\right) + \left(\frac{1-\eta}{2}\right)\ln\left(\frac{1-\eta}{2}\right)\right] \\
&+ \frac{1}{2}H(1-2\eta_b-\eta)\left[(1+\eta)\ln(1+\eta) + (1-\eta)\ln(1-\eta)\right. \\
&\quad - (1-\eta_b)\ln(1-\eta_b) + (\eta_b+\eta)\ln(\eta_b+\eta) \\
&\quad \left. - (1-\eta_b-\eta)\ln(1-\eta_b-\eta) - \eta_b\ln(\eta_b) - 2\ln 2\right] \\
&+ \frac{1}{2}H(1-2\eta_b+\eta)\left[(1+\eta)\ln(1+\eta) + (1-\eta)\ln(1-\eta)\right. \\
&\quad + (\eta_b-\eta)\ln(\eta_b-\eta) - (1-\eta_b)\ln(1-\eta_b) \\
&\quad \left. - (1-\eta_b+\eta)\ln(1-\eta_b+\eta) - \eta_b\ln(\eta_b) - 2\ln 2\right].
\end{aligned}
\tag{4.6}
$$

The first line is the infinite medium solution. The final terms are depletion waves. These terms represent neutrons that have escaped the medium, but would have reflected back into the slab. The $H(1-2\eta_b-\eta)$ term is the depletion wave coming from the left, or negative half-space, and is referred to as the F_{1L}. The $H(1-2\eta_b+\eta)$ term is the depletion wave coming from the right, or positive half-space, and is referred to as the F_{1R} source. Note that the depletion waves, because of the presence of η_b, are now dependent on both η and τ. This will have implications for the numerical integration scheme.

To obtain the equation for the $n = 2$ shape factor, we insert the $F_1(x', t')$ shape factor into equation 4.2, so that F_1 becomes the source for F_2.

$$
\begin{aligned}
F_2(x,t) &= \int_0^t \int_{-\infty}^{\infty} \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) F_1(x',t') H\left(t - t' - \frac{|x-x'|}{v}\right) H(b-x') H(b+x') \\
&= \int_0^t \int_{-\infty}^{\infty} \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) F_{1,inf}(x',t') H\left(t - t' - \frac{|x-x'|}{v}\right) \\
&\qquad \times H(b-x') H(b+x') H(t' - \frac{x'}{v}) H(t' + \frac{x'}{v}) \\
&+ \int_0^t \int_{-\infty}^{\infty} \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) F_{1L}(x',t') H\left(t - t' - \frac{|x-x'|}{v}\right) \\
&\qquad \times H(b-x') H(b+x') H(vt - 2b - x) \\
&+ \int_0^t \int_{-\infty}^{\infty} \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) F_{1R}(x',t') H\left(t - t' - \frac{|x-x'|}{v}\right) \\
&\qquad \times H(b-x') H(b+x') H(vt - 2b + x).
\end{aligned}
\tag{4.7}
$$

We can see that the additional Heaviside functions from the depletion wave sources, F_{1L} and F_{1R}, will further restrict the integration domain. These integration domains are shown graphically for x' and t' space in Figs. 4.3 and 4.4 below. Note that both F_{1L} and F_{1R} are negative sources, representing neutrons lost from the medium.

As for the $n = 1$ shape factor, the extracted Heaviside functions delimit the integration domain. The integration domain of F_2 from the F_1 infinite medium source, $F_{1,inf}$, is identical to the F_1 integration domain. From Fig. 4.1, we see that the $n = 2$ shape factor calculated from the $F_{1,inf}$ source is:

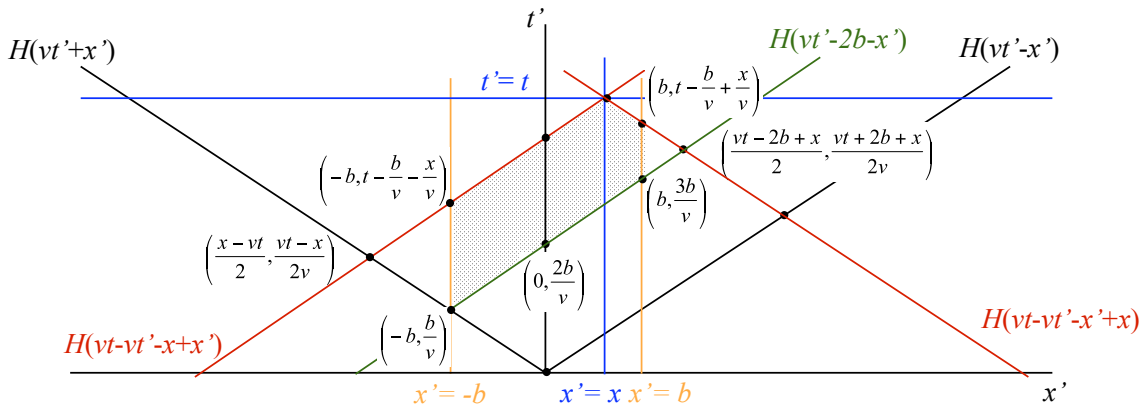

Fig. 4.3.— Integration Domain For F_2 from F_{1L} Source in x' and t' Space

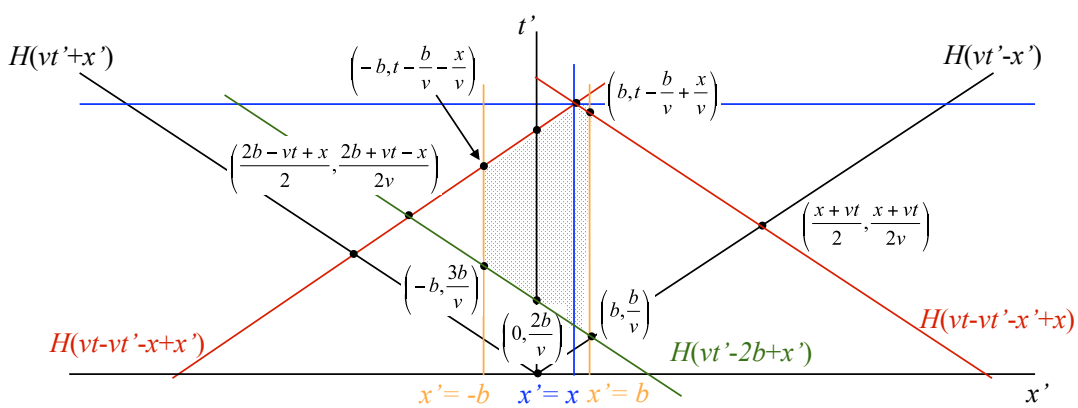

Fig. 4.4.— Integration Domain For F_2 from F_{1R} Source in x' and t' Space

$$
\begin{aligned}
&F_{2,\text{from}F_{1,inf}}(x,t) = \\
&\int\limits_{\frac{x-vt}{2}}^{0} \int\limits_{\frac{-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1inf}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) + \int\limits_{0}^{x} \int\limits_{\frac{x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1inf}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) \\
&+ \int\limits_{x}^{\frac{x+vt}{2}} \int\limits_{\frac{x'}{v}}^{t-\frac{x'}{v}+\frac{x}{v}} F_{1inf}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) \\
&- H(vt-2b-x) \int\limits_{\frac{x-vt}{2}}^{-b} \int\limits_{\frac{-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1inf}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) \\
&- H(vt-2b+x) \int\limits_{b}^{\frac{x+vt}{2}} \int\limits_{\frac{x'}{v}}^{t-\frac{x'}{v}+\frac{x}{v}} F_{1inf}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right).
\end{aligned}
\tag{4.8}
$$

From Fig. 4.3, we find that the $n = 2$ shape factor calculated from the $F_{1,L}$ source is

$$
\begin{aligned}
&F_{2,\text{from}F_{1L}}(x,t) = \\
&H(vt-2b-x) \left[\int\limits_{-b}^{x} \int\limits_{\frac{2b+x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1L}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) \right. \\
&\left. + \int\limits_{x}^{\frac{vt-2b+x}{2}} \int\limits_{\frac{2b+x'}{v}}^{t+\frac{x}{v}-\frac{x'}{v}} F_{1L}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right) \right] \\
&- H(vt-4b+x) \int\limits_{b}^{\frac{vt-2b+x}{2}} \int\limits_{\frac{2b+x'}{v}}^{t+\frac{x}{v}-\frac{x'}{v}} F_{1L}(x',t') \frac{dx'dt'}{vt'(t-t')} \left(\frac{t'}{t}\right),
\end{aligned}
\tag{4.9}
$$

while from Fig. 4.4, the $n = 2$ shape factor calculated from the $F_{1,R}$ source is

$$
\begin{aligned}
&F_{2,\text{from}F_{1R}}(x,t) = \\
&\quad - H(vt - 2b + x)\left[\int_{\frac{2b-vt+x}{2}}^{x}\int_{\frac{2b-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1R}(x',t')\frac{dx'dt'}{vt'(t-t')}\left(\frac{t'}{t}\right)\right. \\
&\quad \left. + \int_{x}^{b}\int_{\frac{2b-x'}{v}}^{t+\frac{x}{v}-\frac{x'}{v}} F_{1R}(x',t')\frac{dx'dt'}{vt'(t-t')}\left(\frac{t'}{t}\right)\right] \\
&\quad - H(vt - 4b - x)\int_{\frac{2b-vt+x}{2}}^{-b}\int_{\frac{2b-x'}{v}}^{t+\frac{x'}{v}-\frac{x}{v}} F_{1R}(x',t')\frac{dx'dt'}{vt'(t-t')}\left(\frac{t'}{t}\right).
\end{aligned}
\tag{4.10}
$$

Then the total $n = 2$ shape factor is found by combining the integrals found in equations 4.8, 4.9, and 4.10.

In η' and τ' space, the integration domains from the F_{1L} and F_{1R} sources are as shown in Figs. 4.5 and 4.6 below. Note that when the depletion waves are converted from x' and t' space to η' and τ', they will contain a τ'. For example:

$$
\begin{aligned}
H(vt' - 2b - x') &= H\left(1 - \frac{2b}{vt'} - \frac{x'}{vt'}\right) = H\left(1 - \frac{2\eta_b}{\tau'} - \eta'\right) \\
&= H(\tau' - 2\eta_b - \eta'\tau') = H((1-\eta')\tau' - 2\eta_b).
\end{aligned}
\tag{4.11}
$$

In η' and τ' space, the expression for $F_{2,\text{from}F_{1,inf}}$ becomes

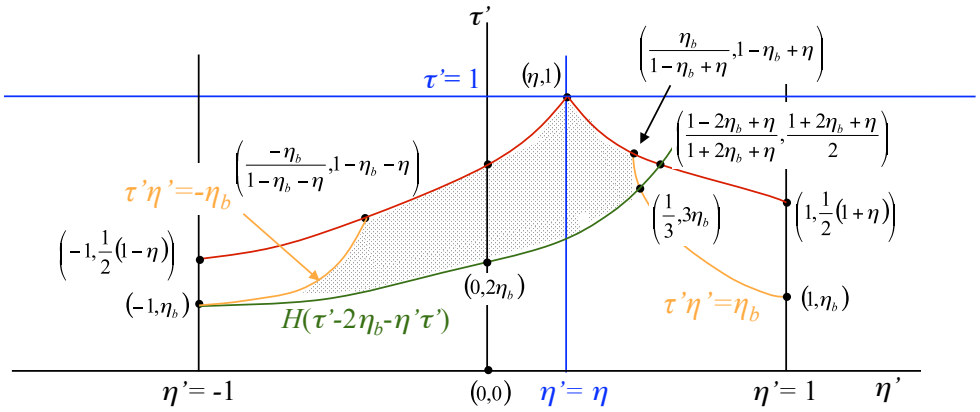

Fig. 4.5.— Integration Domain For F_2 from F_{1L} Source in η' and τ' Space

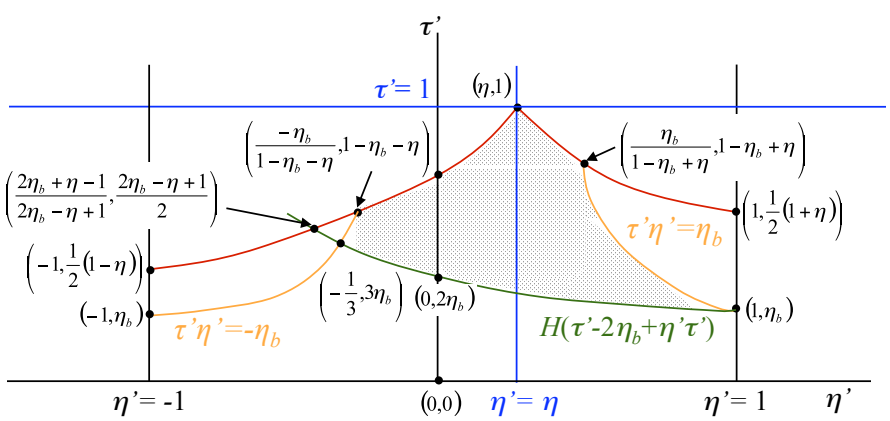

Fig. 4.6.— Integration Domain For F_2 from F_{1R} Source in η' and τ' Space

$$F_{2,\text{from}F_{1,inf}}(\eta,\tau) = \int_{-1}^{\eta}\int_{0}^{\frac{1-\eta}{1-\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_{0}^{\frac{1+\eta}{1+\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta'$$
$$- H(1-2\eta_b-\eta)\int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' \tag{4.12}$$
$$- H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta'$$

while the equation for $F_{2,\text{from}F_{1L}}$ becomes

$$F_{2,\text{from}F_{1,L}}(\eta,\tau) = H(1-2\eta_b-\eta)\left[\int_{-1}^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1L}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right.$$
$$\left. + \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1L}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta' - \int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1L}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right]$$
$$- H(1-4\eta_b+\eta)\left[\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1L}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right.$$
$$\left. + \int_{\frac{1}{3}}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1L}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right]$$

(4.13)

and the equation for $F_{2,\text{from}F_{1R}}$ becomes

$$F_{2,\text{from}F_{1,R}}(\eta,\tau) = H(1-2\eta_b+\eta)\left[\int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1R}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right.$$
$$\left.+\int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1R}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1R}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right]$$
$$-H(1-4\eta_b-\eta)\left[\int_{\frac{-1}{3}}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1R}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right.$$
$$\left.+\int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\frac{-1}{3}}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1R}(\eta',\tau')\frac{\tau'}{1-\tau'}d\tau' d\eta'\right].$$

Since the depletion wave sources are dependent on both η' and τ', we are unable to perform the τ' integration independent of the source. Instead, we apply a variable change from τ' to η_b':

$$\eta_b' = \frac{\eta_b}{\tau'}, \tag{4.14}$$

$$\frac{d\eta_b'}{d\tau'} = \frac{d}{d\tau'}\left(\frac{\eta_b}{\tau'}\right) = \frac{-\eta_b}{(\tau')^2}. \tag{4.15}$$

We can now write F_2 in terms of η_b' and η':

$$
\begin{aligned}
F_2(\eta,\eta_b) = &\int_{-1}^{\eta}\int_{0}^{\frac{1-\eta}{1-\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_{0}^{\frac{1+\eta}{1+\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' \\
&- H(1-2\eta_b-\eta)\int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' \\
&- H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} F_{1inf}(\eta')\frac{\tau'}{1-\tau'}d\tau' d\eta' \\
&+ H(1-2\eta_b-\eta)\left[\int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} F_{1L}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right. \\
&\left.+\int_{-1}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2}} F_{1L}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} - \int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} F_{1L}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right] \\
&- H(1-4\eta_b+\eta)\left[\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} F_{1L}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right. \\
&\left.+\int_{\frac{1}{3}}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} F_{1L}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right] \\
&+ H(1-2\eta_b+\eta)\left[\int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2}} F_{1R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right. \\
&\left.+\int_{\eta}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2}} F_{1R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} F_{1R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b}\right]
\end{aligned}
$$

$$
\begin{aligned}
&- H(1-4\eta_b-\eta)\left[\int_{\frac{-1}{3}}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} F_{1R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right.\\
&\left.+\int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\frac{-1}{3}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2}} F_{1R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^2\frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right].
\end{aligned}
\tag{4.16}
$$

Since the F_{1inf} source is not dependent on τ', the variable change does not need to be performed on the first four integrals.

At this point, we introduce a new problem parameter, m, which we refer to as the reflection number. The reflection number indicates the number of times a neutron has moved across the medium, and varies from $1 \cdots n-1$. By using this parameter, we can write the general expressions for the shape factors. The nth shape factor from the $F_{n-1,m,L}$ source is:

$$
\begin{aligned}
F_{n,\mathrm{from}F_{n-1,m,L}}(\eta,\eta_b) = H(1-2m\eta_b-\eta)\Bigg[& \int_{\eta}^{\frac{1-2m\eta_b+\eta}{1+2m\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} F_{n-1,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b'-\eta_b} \\
& + \int_{\frac{-1}{2m-1}}^{\eta} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} F_{n-1,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b'-\eta_b} \\
& - \int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} F_{n-1,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b'-\eta_b} \Bigg] \\
& - H(1-4m\eta_b+\eta)\Bigg[\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m+1}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} F_{n-1,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b'-\eta_b} \\
& + \int_{\frac{1}{2m+1}}^{\frac{1-2m\eta_b+\eta}{1+2m\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} F_{n-1,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b'-\eta_b} \Bigg] .
\end{aligned}
\tag{4.17}
$$

Similarly, the nth shape factor found from the $F_{n-1,m,R}$ source is:

$$F_{n,\text{from}F_{n-1,m,R}}(\eta,\eta_b) = H(1-2m\eta_b+\eta)\left[\int_{\frac{2m\eta_b+\eta-1}{2m\eta_b-\eta+1}}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} F_{n-1,m,R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right.$$
$$\left. + \int_{\eta}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} F_{n-1,m,R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b} - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} F_{n-1,m,R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right]$$
$$- H(1-4m\eta_b-\eta)\left[\int_{\frac{-1}{2m+1}}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} F_{n-1,m,R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right.$$
$$\left. + \int_{\frac{2m\eta_b+\eta-1}{2m\eta_b-\eta+1}}^{\frac{-1}{2m+1}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} F_{n-1,m,R}(\eta',\eta_b')\left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\,d\eta'}{\eta_b'-\eta_b}\right]. \tag{4.18}$$

The method of evaluating the integrals in equations 4.17 and 4.18 is similar to the infinite medium case, except that there are double integrals to evaluate. However, since there are integration variables in only the η_b' limits of integration, we can evaluate the η_b' integrals first, followed by the η' integrals. Note that the η_b' integrals are singular at the point $\eta_b' = \eta_b$. As such, we will have to perform subtraction of singularity on each of the above integrals. Performing subtraction of singularity on one integral specifically, we find:

$$
\begin{aligned}
&\int\limits_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} F_{1L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} \\
&\quad = \int\limits_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} [F_{1L}(\eta',\eta_b') - F_{1L}(\eta',\eta_b)] \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} \\
&\quad + \int\limits_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} F_{1L}(\eta',\eta_b) \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} \\
&\quad = \int\limits_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} [F_{1L}(\eta',\eta_b') - F_{1L}(\eta',\eta_b)] \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\ d\eta'}{\eta_b'-\eta_b} \\
&\quad + \int\limits_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} F_{1L}(\eta',\eta_b) \left[\ln\left(\frac{(1+\eta')(1-2\eta_b-\eta')}{(1-\eta')(\eta'-\eta)}\right) + \frac{2\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'}\right] d\eta'.
\end{aligned}
\tag{4.19}
$$

We note that the results of the η_b integration are singular at $\eta' = \eta$. Again, we perform subtraction of singularity:

$$
\begin{aligned}
&\int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} [F_{1L}(\eta',\eta_b') - F_{1L}(\eta',\eta_b)] \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b} \\
&\quad + \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} F_{1L}(\eta',\eta_b) \left[\ln\left(\frac{(1+\eta')(1-2\eta_b-\eta')}{(1-\eta')(\eta'-\eta)}\right) + \frac{2\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'}\right] d\eta' \\
= &\int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} [F_{1L}(\eta',\eta_b') - F_{1L}(\eta',\eta_b)] \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b} \\
&\quad + \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} [F_{1L}(\eta',\eta_b) - F_{1L}(\eta,\eta_b)] \left[\ln\left(\frac{(1+\eta')(1-2\eta_b-\eta')}{(1-\eta')(\eta'-\eta)}\right)\right. \\
&\quad \left. + \frac{2\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'}\right] d\eta' \\
&\quad + F_{1L}(\eta,\eta_b) \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \left[\ln\left(\frac{(1+\eta')(1-2\eta_b-\eta')}{(1-\eta')(\eta'-\eta)}\right) + \frac{2\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'}\right] d\eta' \\
= &\int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} [F_{1L}(\eta',\eta_b') - F_{1L}(\eta',\eta_b)] \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b} \\
&\quad + \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}} [F_{1L}(\eta',\eta_b) - F_{1L}(\eta,\eta_b)] \left[\ln\left(\frac{(1+\eta')(1-2\eta_b-\eta')}{(1-\eta')(\eta'-\eta)}\right)\right. \\
&\quad \left. + \frac{2\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'}\right] d\eta' + F_{1L}(\eta,\eta_b)(1-2\eta_b-\eta) \ln\left(\frac{2}{1-\eta}\right).
\end{aligned}
\tag{4.20}
$$

We must use subtraction of singularity on the integrals in equations 4.17 and 4.18. Following the procedure outlined above, and generalizing for the collision number, n and the reflection number m, where m varies from 1 to $n-1$.

We begin with type 1 limits of integration

$$Kt1 = \int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} F_{n,m,L}(\eta', \eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b' \, d\eta'}{\eta_b' - \eta_b}. \tag{4.21}$$

The results of the η_b' integration, without the source:

$$\begin{aligned} &\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b} \\ &\quad = \ln\left(\frac{(1-\eta')(\eta_b+\eta')}{\eta'(\eta-\eta')}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{-\eta_b}{\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]. \end{aligned} \tag{4.22}$$

Equation 4.22 is singular at $\eta' = -\eta_b$. The results of the η' integration:

$$\begin{aligned} &\int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \left\{\ln\left(\frac{(1-\eta')(\eta_b+\eta')}{\eta'(\eta-\eta')}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{-\eta_b}{\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]\right\} d\eta' \\ &\quad = \frac{1}{2m-1} \ln\left(\frac{2m(1+\eta_b-2m\eta_b)}{1-\eta+2m\eta}\right) + \eta \ln\left(\frac{2m(\eta_b+\eta)}{1-\eta+2m\eta}\right) \\ &\quad + \eta_b \ln\left(\frac{\eta_b+\eta}{1+\eta_b-2m\eta_b}\right) + \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)} \Big[\eta_b(1-\eta_b-\eta)^{i-2} \\ &\quad - \eta_b^{i-1}(2m-1)^{i-2} - (1-\eta)(1-\eta_b-\eta)^{i-2} + \left(\frac{2m-1}{2m}\right)^{i-2}(1-\eta)^{i-1}\Big]. \end{aligned} \tag{4.23}$$

Type 2 limits of integration:

$$Kt2 = \int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} F_{n,m,R}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b}. \tag{4.24}$$

The results of the η_b' integration, without the source:

$$\begin{aligned} &\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b} \\ &\quad = \ln\left(\frac{(1+\eta')(\eta-\eta_b')}{\eta'(\eta'-\eta)}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{\eta_b}{\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]. \end{aligned} \tag{4.25}$$

Equation 4.25 is singular at $\eta' = \eta_b$. The results of the η' integration:

$$\begin{aligned} &\int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}} \left\{\ln\left(\frac{(1+\eta')(\eta-\eta_b')}{\eta'(\eta'-\eta)}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{\eta_b}{\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]\right\} d\eta' \\ &\quad = \frac{1}{2m-1}\ln\left(\frac{2m(1+\eta_b-2m\eta_b)}{1+\eta-2m\eta}\right) - \eta\ln\left(\frac{2m(\eta_b-\eta)}{1+\eta-2m\eta}\right) \\ &\quad + \eta_b \ln\left(\frac{\eta_b-\eta}{1+\eta_b-2m\eta_b}\right) + \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)}\left[\eta_b(1-\eta_b+\eta)^{i-2}\right. \\ &\quad \left. - \eta_b^{i-1}(2m-1)^{i-2} - (1+\eta)(1-\eta_b+\eta)^{i-2} + \left(\frac{2m-1}{2m}\right)^{i-2}(1+\eta)^{i-1}\right]. \end{aligned} \tag{4.26}$$

Type 3 limits of integration:

$$Kt3 = \int\limits_{\frac{-1}{2m-1}}^{\eta} \int\limits_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} F_{n,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b}. \tag{4.27}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{1-2m\eta_b-\eta'}{\eta-\eta'}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1-\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]. \tag{4.28}$$

Equation 4.28 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}
&\int_{\frac{-1}{2m-1}}^{\eta} \left\{\ln\left(\frac{1-2m\eta_b-\eta'}{\eta-\eta'}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1-\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]\right\} d\eta' \\
&= \frac{2m}{2m-1}\ln\left(\frac{(1+\eta_b-2m\eta_b)(1-\eta)}{1-2m\eta_b-\eta}\right) - \frac{1}{2m-1}\ln\left(\frac{(1-\eta+2m\eta)(1-\eta)}{2m(1-2m\eta_b-\eta)}\right) \\
&+ \eta\ln\left(\frac{2m(1-2m\eta_b-\eta)}{(1-\eta+2m\eta)(1-\eta)}\right) - 2m\eta_b\ln\left(\frac{(1-\eta)(1+\eta_b-2m\eta_b)}{1-2m\eta_b-\eta}\right) \\
&+ \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)}\left[\frac{(2m\eta_b)^{i-1}}{(1-\eta)^{i-2}} - (2m\eta_b)^{i-1}\left(\frac{2m-1}{2m}\right)^{i-2}\right. \\
&\left. - (1-\eta) + (1-\eta)^{i-1}\left(\frac{2m-1}{2m}\right)^{i-2}\right].
\end{aligned} \tag{4.29}$$

Type 4 limits of integration:

$$Kt4 = \int_{\eta}^{\frac{1}{2m-1}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} F_{n,m,R}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b}. \tag{4.30}$$

The results of the η_b' integration, without the source:

$$\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b}$$
$$= \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1+\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]. \tag{4.31}$$

Equation 4.31 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}
&\int\limits_{\eta}^{\frac{1}{2m-1}} \left\{\ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta'}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1+\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]\right\} d\eta' \\
&= \frac{2m}{2m-1}\ln\left(\frac{(1+\eta_b-2m\eta_b)(1+\eta)}{1-2m\eta_b+\eta}\right) - \frac{1}{2m-1}\ln\left(\frac{(1+\eta-2m\eta)(1+\eta)}{2m(1-2m\eta_b+\eta)}\right) \\
&- \eta\ln\left(\frac{2m(1-2m\eta_b+\eta)}{(1+\eta-2m\eta)(1+\eta)}\right) - 2m\eta_b\ln\left(\frac{(1+\eta)(1+\eta_b-2m\eta_b)}{1-2m\eta_b+\eta}\right) \\
&+ \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)}\left[\frac{(2m\eta_b)^{i-1}}{(1+\eta)^{i-2}} - (2m\eta_b)^{i-1}\left(\frac{2m-1}{2m}\right)^{i-2}\right. \\
&\left. - (1+\eta) + (1+\eta)^{i-1}\left(\frac{2m-1}{2m}\right)^{i-2}\right].
\end{aligned} \tag{4.32}$$

Type 5 limits of integration:

$$Kt5 = \int\limits_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} F_{n,m,L}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b}. \tag{4.33}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b}$$
$$= \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1-\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]. \tag{4.34}$$

Equation 4.34 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}
&\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \left\{\ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)}\right)\right. \\
&\qquad \left. + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1-\eta'}\right)^{i-1} - \left(\frac{1+\eta}{1+\eta'}\right)^{i-1}\right]\right\} d\eta' \\
&= -(1-2m\eta_b-\eta)\ln\left(\frac{1-\eta}{2}\right) \\
&\qquad + \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)}\left[\frac{m\eta_b(1+2m\eta_b+\eta)^{i-2}}{2^{i-3}} - \frac{(2m\eta_b)^{i-1}}{(1-\eta)^{i-2}}\right. \\
&\qquad \left. + (1+\eta)\left(\frac{1+2m\eta_b+\eta}{2}\right)^{i-2} - (1+\eta)\right].
\end{aligned} \tag{4.35}$$

Type 6 limits of integration:

$$Kt6 = \int_{\frac{-(1-\eta-2m\eta_b)}{1-\eta+2m\eta_b}}^{\eta} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} F_{n,m,R}(\eta',\eta_b') \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'\, d\eta'}{\eta_b' - \eta_b}. \tag{4.36}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^n \frac{d\eta_b'}{\eta_b' - \eta_b}$$
$$= \ln\left(\frac{(1-2m\eta_b+\eta')(1-\eta')}{(1+\eta')(\eta-\eta')}\right) + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1+\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]. \tag{4.37}$$

Equation 4.37 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}
&\int_{\frac{-(1-\eta-2m\eta_b)}{1-\eta+2m\eta_b}}^{\eta} \left\{\ln\left(\frac{(1-2m\eta_b+\eta')(1-\eta')}{(1+\eta')(\eta-\eta')}\right)\right. \\
&\qquad \left. + \sum_{i=2}^{n} \frac{1}{i-1}\left[\left(\frac{2m\eta_b}{1+\eta'}\right)^{i-1} - \left(\frac{1-\eta}{1-\eta'}\right)^{i-1}\right]\right\} d\eta' \\
&= -(1-2m\eta_b+\eta)\ln\left(\frac{1+\eta}{2}\right) \\
&\qquad + \sum_{i=3}^{n} \frac{1}{(i-1)(i-2)}\left[\frac{m\eta_b(1+2m\eta_b-\eta)^{i-2}}{2^{i-3}} - \frac{(2m\eta_b)^{i-1}}{(1+\eta)^{i-2}}\right. \\
&\qquad \left. + (1-\eta)\left(\frac{1+2m\eta_b-\eta}{2}\right)^{i-2} - (1-\eta)\right].
\end{aligned} \tag{4.38}$$

4.1.1 Results

Shown in Fig. 4.7 is the total flux for the finite slab benchmark problem at early mean free times. This benchmark problem uses the same parameters as the infinite slab problem, with the addition of a slab half-width of $b = 2$. In the same figure, for reference, the infinite slab total flux is shown. For one mft, the neutrons have not yet reached the boundary, so the finite and infinite medium solutions match. At three mfts, the neutrons have reached the boundary, and some leakage has begun. Equivalently, one could say that near the $x = -b$ boundary, the $H(vt - 2b - x)$ depletion wave has

been activated, while at the $x = +b$ boundary, the $H(vt - 2b + x)$ depletion wave has been activated. Since the Heaviside functions have only been turned on near the boundaries, then only the flux near the boundaries is affected. Near the center of the medium, the infinite medium and finite medium total fluxes match. None of the other depletion waves, such as $H(vt - 4b - x)$ or $H(vt - 4b + x)$, have been activated at three mean free times.

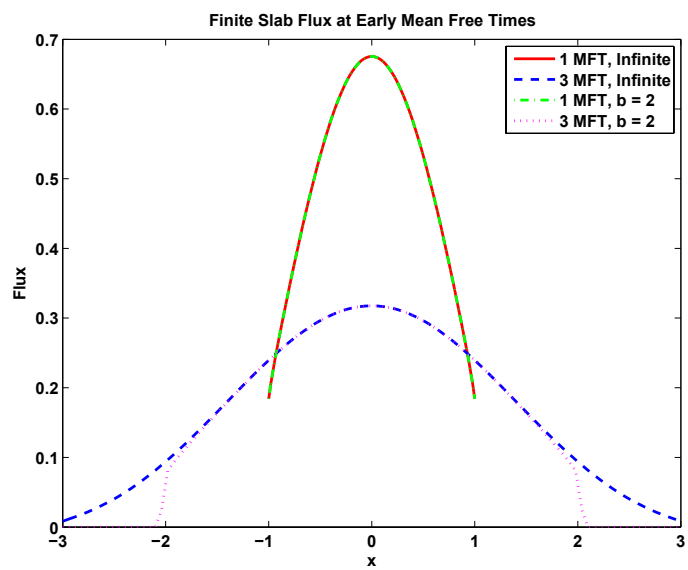

Fig. 4.7.— Total Flux at Early Mean Free Times

Shown in Fig. 4.8 below is the total flux for five and nine mean free time. Again, the infinite medium solution for these same times is shown for comparison. We see that for both these mean free times, the peak finite medium flux at the center of the medium is lower than the infinite medium flux. This means that for every point x, at least one of the depletion waves, $H(vt - 2b - x)$ or $H(vt - 2b + x)$, has passed. At the center of the medium, both the $H(vt - 2b - x)$ and the $H(vt - 2b + x)$ depletion waves

have passed. Equivalently, one could say that the effect of the loss of neutrons from the boundary is felt by the entire medium. At nine mean free times, the $H(vt-2b-x)$ and the $H(vt-2b+x)$ depletion waves have crossed the entire medium. Additionally, for every point x in the medium, at least one of the depletion waves, $H(vt-4b-x)$ or $H(vt-4b+x)$, has passed.

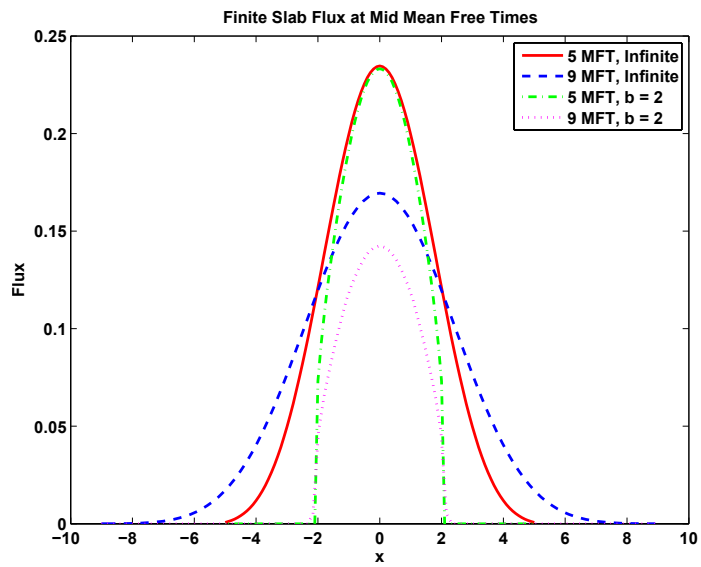

Fig. 4.8.— Total Flux at Midrange Mean Free Times

The results for the pulsed source in space and time benchmark problem were compared to the discrete ordinates code, PARTISN. To model an approximation of a delta function in time in PARTISN, the time-dependent commands *stimes* and *samp* were used. *stimes* lets the user set the time points at which the source is assumed linear. *samp* is the relative source amplitude at each time point [19]. The initial source amplitude was picked to be 10000, which dropped to zero after 0.0001 mean free times passed. To approximate a delta function in space, the source was located in a small

cell centered at $x = 0$. The PARTISN input file was created with the assistance of Dr. Randy Baker at Los Alamos National Laboratory.

Shown in Fig. 4.9 is the total neutron flux at 0.5 MFT for a slab half-width of $b = 5$. The figure shows two integral transport results: one for a total flux calculated using 100 collisions, and one for a total flux calculated using three collisions. By comparing these two fluxes found with the integral transport method, we can see that at early mean free times the first several collisions dominate the total flux. By 15 mfts, the contribution from the first several fluxes has essentially gone to zero, so for later mean free times, the results for the flux calculated using three collisions is not shown.

Note that PARTISN has difficulty resolving the flux at early mean free times, due to the singular nature of the source. By 5 mean free times, the PARTISN result has smoothed out considerably, and the two codes match very well. By 25 MFT, however, the integral transport flux has started to diverge from the PARTISN flux, and by 50 MFT, the two codes give very different results for the total flux.

To determine how many collisions were necessary to accurately obtain the total flux in the finite slab at a given mean free time, a local error was calculated:

$$err = \frac{\phi_n(x, \frac{b}{vt}, t) - \phi_{n-1}(x, \frac{b}{vt}t)}{\phi_n(x, \frac{b}{vt}, t)} \tag{4.39}$$

where $\phi_n(x, \frac{b}{vt}, t)$ is the total flux calculated from n collisions, and $\phi_{n-1}(x, \frac{b}{vt}t)$ is the total flux calculated from $n - 1$ collisions. The maximum local error was set to 10^{-5}. The maximum local error was chosen to ensure that effects of later collisions not

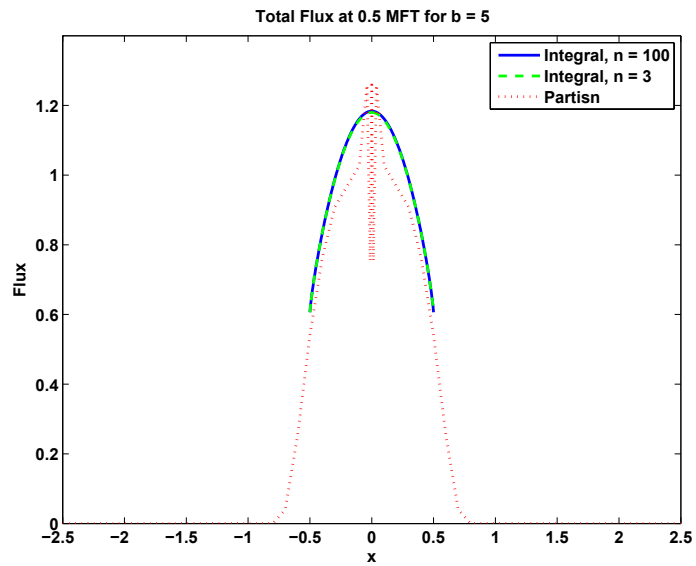

Fig. 4.9.— Total Flux at 0.5 MFT

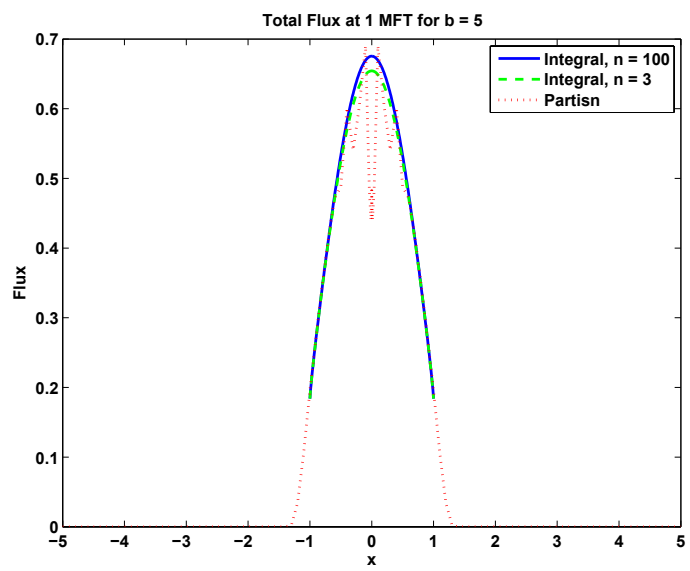

Fig. 4.10.— Total Flux at 1 MFT

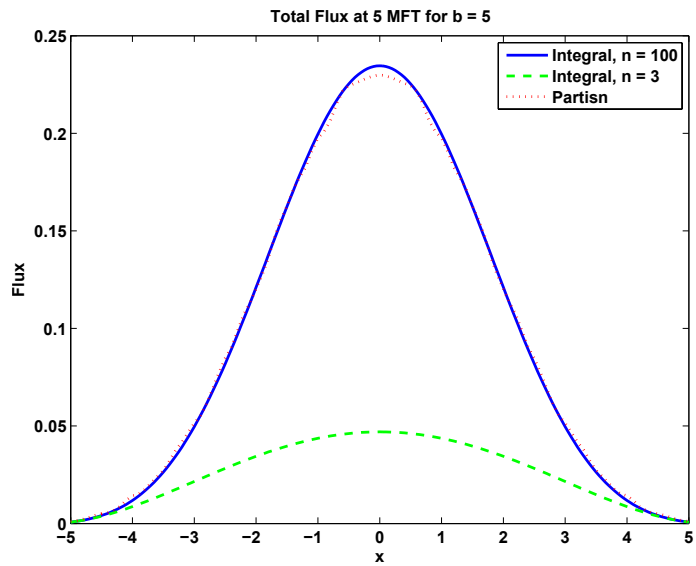

Fig. 4.11.— Total Flux at 5 MFT

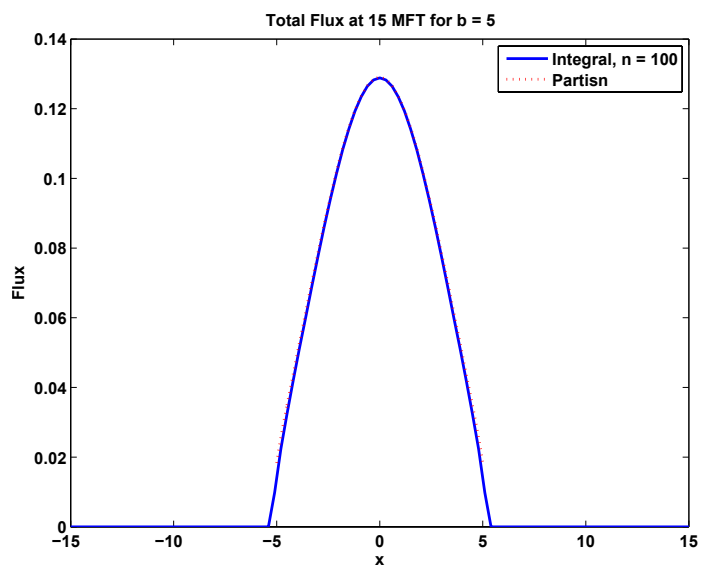

Fig. 4.12.— Total Flux at 15 MFT

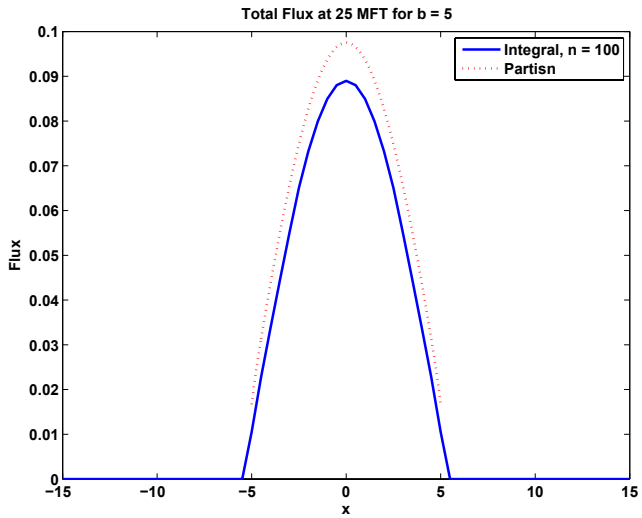

Fig. 4.13.— Total Flux at 25 MFT

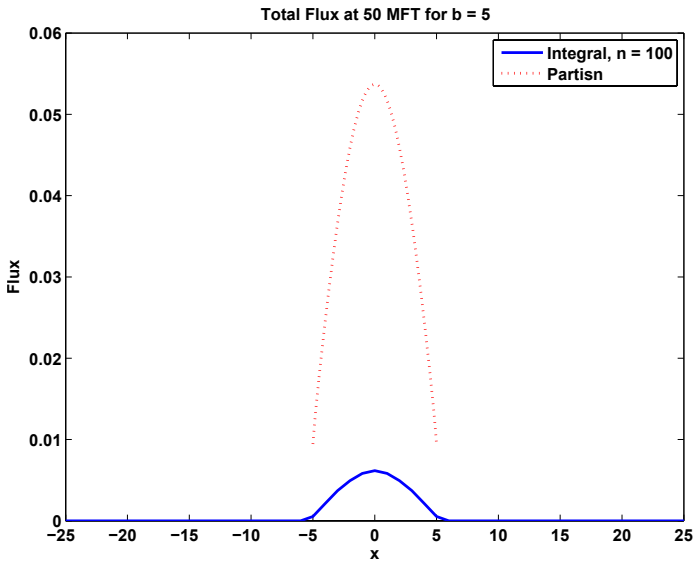

Fig. 4.14.— Total Flux at 50 MFT

included in the total flux calculation would be minimal. Shown in Table 4.1 below are the number of collisions necessary to satisfy the local error requirement at various mean free times.

Table 4.1: Number of Collisions at Various Mean Free Times

Time	**Number of Collisions**
0.5	7
1	9
5	18
15	36
25	50
50	93

Although the results diverge at later times, there was very good agreement between the PARTISN results and the total flux calculated using 100 collisions between 5 and 15 mean free times. Additionally, there was good agreement between the total flux calculated using 100 collisions and the flux calculated using three collisions at early mean free times. This leads to the notion that the integral transport method is especially suited to applications where only a few collisions need be calculated. This is reenforced by the fact that the method becomes increasingly costly, in terms of computer time and memory necessary, to implement as the total number of collisions calculated increases. This is because the total number of integrals that must be evaluated numerically increases with every collision. A timing study was performed, to determine how computation time increases with an increasing number of collisions. The study was performed for a total number of points in η of 101, and the results are shown in Table 4.2.

Table 4.2: Total Time to Calculate Collisions

Number of Collisions	**Total Time [s]**	**Collision Time [s]**
2	7.47	7.47
3	16.84	9.17
5	37.91	10.89
10	98.72	12.80
25	307.26	14.43
100	1446.95	16.08

Specifically, this integral transport method could still be of value to radiation hydrodynamics codes used to model inertial confinement fusion implosions. To understand why this is the case, it is instructive to estimate how many mean free times a neutron is likely to spend in an ICF pellet before escaping. We stated in Section 1.4 that a 14.1 MeV neutron has a speed of approximately 5.2 cm/ns. Given the macroscopic cross-section of highly compressed DT as 39.4 cm^{-1}, the average amount a time before a neutron will experience a reaction in a DT fuel pellet, or the mean free time of a neutron in DT, is approximately 4.9 ps. We also estimated the fuel traversal time of a neutron in a compressed fuel pellet as roughly 4.6 ps, which is on the order of the neutron mean free time. At one mean free time, the total flux is dominated by the first few collisions. We saw that the PARTISN results and the dimensionless integral results were in very good agreement from 5 mean free times through 15 mean free times, giving us confidence in the dimensionless integral results at these times. To determine how many collisions would be necessary to give an accurate total flux for time scales used in inertial confinement fusion problems, we plotted the total flux from 100 collisions and the total flux for three collisions for a slab with a half-width

of $b = 1$ for various early mean free times.

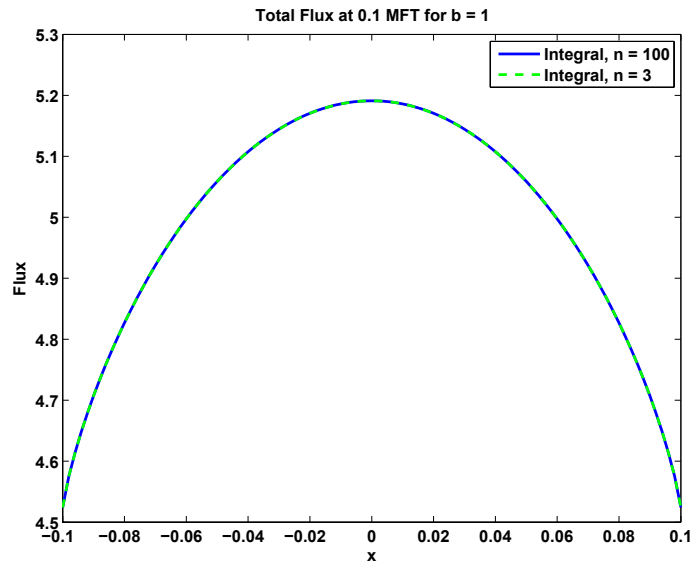

Fig. 4.15.— Total Flux at 0.1 MFT in a Slab of Width 2 MFP

At 0.1 and 0.5 MFTs, the agreement between the two integral transport fluxes is very good. By 1.5 MFTs, the integral transport flux calculated from only three collisions has started to diverge from the flux calculated using 100 collisions, with the peak of the flux calculated using only three collisions off by about 10% from the peak of the flux calculated from 100 collisions. This discrepancy in the flux may not be detrimental to the method's application to a one-dimensional radiation hydrodynamics code, as these codes tend to be used to obtain a range of parameters in which a fusion burn might occur. The parameters then tend to be refined using radiation hydrodynamics codes in two or three dimensions. By 2 MFTs, the $n = 3$ integral transport flux peak is off the $n = 100$ integral transport peak by about 20%. After this time, the discrepancy in the flux calculated from three collisions is too great, and for accurate answers, a greater number of collisions would be required.

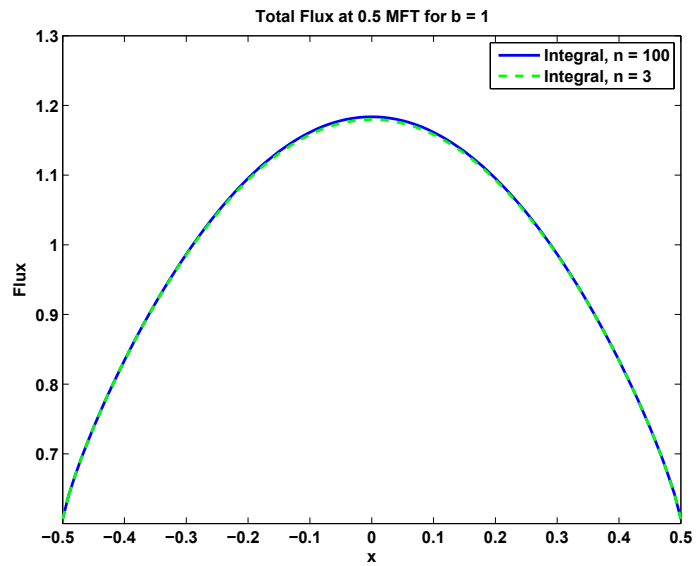

Fig. 4.16.— Total Flux at 0.5 MFT in a Slab of Width 2 MFP

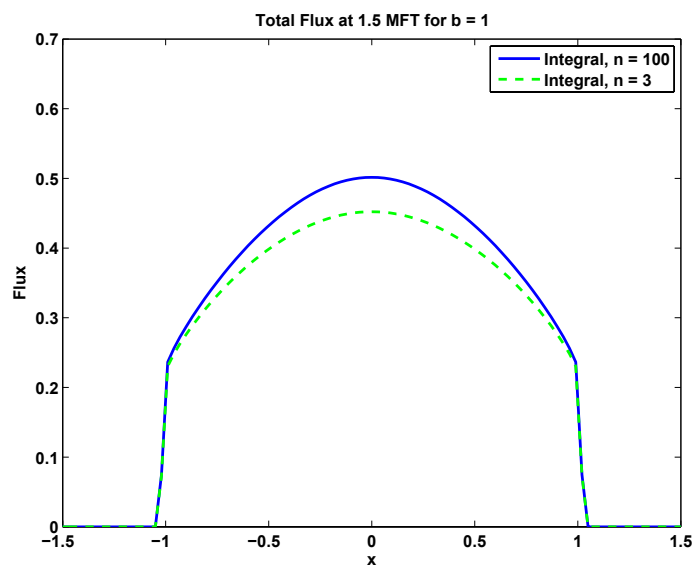

Fig. 4.17.— Total Flux at 1.5 MFT in a Slab of Width 2 MFP

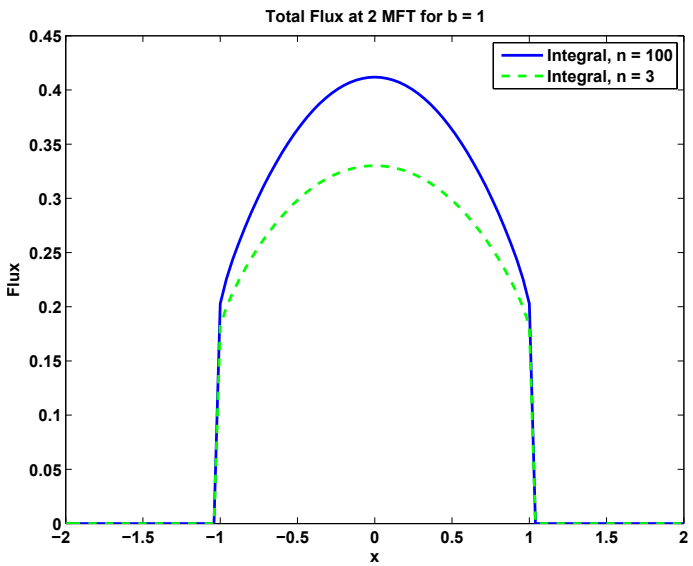

Fig. 4.18.— Total Flux at 2 MFT in a Slab of Width 2 MFP

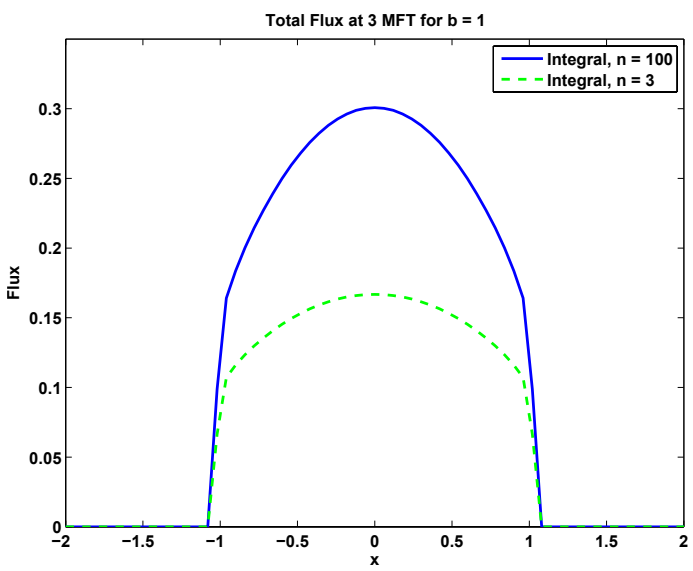

Fig. 4.19.— Total Flux at 3 MFT in a Slab of Width 2 MFP

4.2 Finite Sphere

We now show how to extend this integral neutron transport method to incorporate a finite sphere geometry. We return to the integral form of the time-dependent neutron transport equation, equation 3.38, where $K(r, r'; t, t')$ is the same time-dependent kernel as was used for the infinite sphere. From here, there are two approaches to the problem. Sec. 4.2.1 follows the same method as outlined in Sec. 3.2, using a reflective boundary condition at $r = 0$ and a vacuum boundary condition at $r = b$. Sec. 4.2.2 shows how to reformulate the problem so that the domain of consideration is from $-b \leq r \leq b$, with vacuum boundary conditions at $r = \pm b$.

4.2.1 Finite Sphere Using Reflective Boundary Condition

To obtain the expression for the shape factor for a finite sphere of radius $r = b$, with a vacuum boundary condition at b, we can start with the integral form of the time-dependent neutron transport equation in spherical coordinates:

$$\Phi(r, t) = \int_0^t \int_0^b K(r, r'; t, t') Q(r', t') dr' dt' \tag{4.40}$$

where $K(r, r'; t, t')$ is the time-dependent kernel and $Q(r', t')$ is the time-dependent source. Note that the upper limit of integration for the r' is no longer ∞, but is now the radius of the sphere.

The uncollided flux is the same as for the infinite sphere, and is reproduced below:

$$\Phi_0(r,t) = \frac{S_0}{4\pi r v t}\left(\frac{e^{-\Sigma v t}}{t}\right)\delta\left(1-\frac{r}{vt}\right). \tag{4.41}$$

Inserting the spherical shell kernel into equation 4.40, and using the Neumann series method to decompose the total flux into an infinite sum of the uncollided and collided fluxes leads to the following equation for the nth collided flux:

$$\begin{aligned}\Phi_n(r,t) =& \Sigma_s \int_0^t \int_0^\infty \frac{e^{-\Sigma v(t-t')}}{8\pi r r'(t-t')}\left[H\left(t-t'-\frac{|r-r'|}{v}\right) - H\left(t-t'-\frac{|r+r'|}{v}\right)\right] \\ &\times H(b-r')\Phi_{n-1}(r',t')4\pi r'^2 dr' dt'\end{aligned} \tag{4.42}$$

Note that the boundary at $r = b$ is now denoted with a Heaviside function underneath the integral, rather than as the upper limit of the r' integration.

Substituting the reduced collision ansatz, equation 3.45, into equation 4.42 and simplifying, the following expression for the nth shape factor is found:

$$\begin{aligned}F_n(r,t)H\left(t-\frac{r}{v}\right) =& \frac{n}{2}\int_0^t \int_0^\infty \frac{dr'dt'}{vt'(t-t')}\left(\frac{t'}{t}\right)^{n-3}\frac{r'}{r}\, H\left(t'-\frac{r'}{v}\right)F_{n-1}(r',t') \\ &\times\left[H\left(t-t'-\frac{|r-r'|}{v}\right) - H\left(t-t'-\frac{|r+r'|}{v}\right)\right]H(b-r').\end{aligned} \tag{4.43}$$

When deriving the equation for the nth collided shape factor for the infinite sphere case, the next step was to introduce a change of variables to a dimensionless integration space. However, in the finite sphere case, it is instructive to examine the integration domain in r' and t' space, in addition to η' and τ' space. Extracting the

Heaviside functions in equation 4.43 will give the integration domain for the $n = 1$ shape factor in r' and t' space. The integration domain is shown graphically in Fig. 4.20 below.

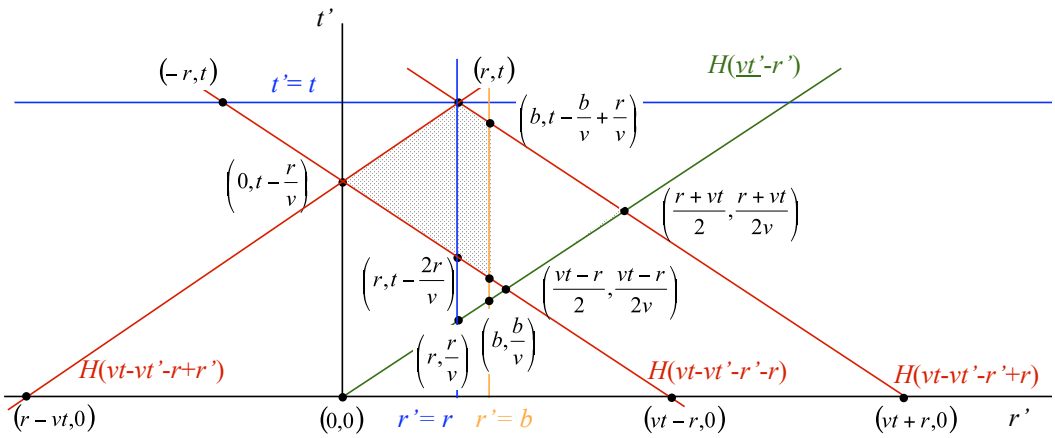

Fig. 4.20.— Finite sphere integration domain for $n = 1$ shape factor in r' and t' space.

Upon examining the integration domain in Fig. 4.20, we find that the expression for the $n = 1$ shape factor in r' and t' space can be written as:

$$
\begin{aligned}
F_1(r,t) = &\int_0^r \int_{\frac{r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')} dr'dt' + \int_r^{\frac{vt+r}{2}} \int_{\frac{r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')} dr'dt' \\
&- \int_0^{\frac{vt-r}{2}} \int_{\frac{r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')} dr'dt' \\
&- H(vt+r-2b) \int_b^{\frac{vt+r}{2}} \int_{\frac{r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')} dr'dt' \\
&+ H(vt-r-2b) \int_b^{\frac{vt-r}{2}} \int_{\frac{r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')} dr'dt'.
\end{aligned}
\tag{4.44}
$$

To obtain the integration domain in η' and τ' space, we must first perform a change of variables on equation 4.43. As before, the dimensionless variables are defined as $\tau' = \frac{t'}{t}$ and $\eta' = \frac{r'}{vt'}$. In addition to these variables, there is a third parameter, to indicate the radius of the sphere in dimensionless space: $\eta_b = \frac{b}{vt}$. The general expression for the nth shape factor in η' and τ' space is then:

$$
\begin{aligned}
F_n(\eta,\tau) = &\int_0^1 \int_0^\infty \frac{(\tau')^{n-2}}{1-\tau'} \frac{\eta'}{\eta} F_{n-1}(\eta',\tau') H(\eta_b - \tau'\eta') H(1-\eta') \\
&\times \left[H(1-\tau'-|\eta-\eta'\tau'|) - H(1-\tau'-|\eta+\eta'\tau'|)\right] d\eta' d\tau'.
\end{aligned}
\tag{4.45}
$$

To determine the integration domain in η' and τ' space, we must extract the Heaviside functions in equation 4.45. The integration domain is shown in Fig. 4.21 below. The boundary at $r' = b$ has become a curved line in η' and τ' space, where

$\eta_b = \frac{b}{vt}$.

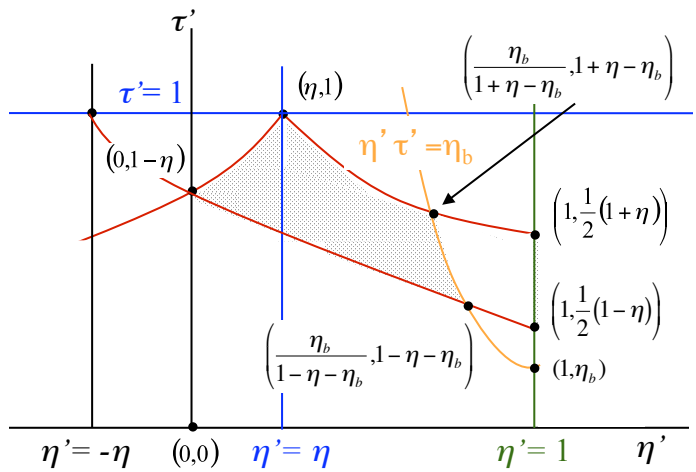

Fig. 4.21.— Finite sphere integration domain for $n = 1$ shape factor in η' and τ' space.

The limits of integration indicated by Fig. 4.21 give the following expression for the $n = 1$ shape factor:

$$
\begin{aligned}
F_1(\eta,\eta_b) = &\int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{1}{\tau'(1-\tau')}\frac{\eta'}{\eta}F_0(\eta)d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')}\frac{\eta'}{\eta}F_0(\eta)d\tau' d\eta' \\
&- \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')}\frac{\eta'}{\eta}F_0(\eta)d\tau' d\eta' \\
&- H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')}\frac{\eta'}{\eta}F_0(\eta)d\tau' d\eta' \\
&+ H(1-2\eta_b-\eta)\int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')}\frac{\eta'}{\eta}F_0(\eta)d\tau' d\eta'.
\end{aligned}
\tag{4.46}
$$

The first three integrals correspond to the expression for the $n = 1$ infinite sphere

shape factor. The last two integrals are depletion waves, negative sources of neutrons that move across the medium, depleting neutrons. Solving equation 4.46 analytically, the following expression for the $n = 1$ shape factor is obtained:

$$\begin{aligned} F_1(\eta, \eta_b) =& \frac{1}{4\pi\eta} \left\{ 2\ln\left(\frac{1+\eta}{1-\eta}\right) - H(1-2\eta_b-\eta)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) - \ln(1-\eta_b) + \ln(\eta_b)\right] \right. \\ & \left. - H(1-2\eta_b+\eta)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln(1-\eta_b) - \ln(\eta_b)\right] \right\}. \end{aligned} \tag{4.47}$$

Note that the expression for the $n = 1$ shape factor is dependent on τ', unlike for the infinite sphere. This is due to the definition of $\eta_b = \frac{b}{vt}$. When $F_1(\eta, \eta_b)$ is used as the source for the $n = 2$ shape factor, we will need to rewrite the source to explicitly include τ':

$$\begin{aligned} F_1(\eta,' \tau') =& \frac{1}{4\pi\eta'} \left\{ 2\ln\left(\frac{1+\eta'}{1-\eta'}\right) \right. \\ & - H\left(1 - \frac{2\eta_b}{\tau'} - \eta'\right)\left[\ln\left(\frac{1+\eta'}{1-\eta'}\right) - \ln\left(1-\frac{\eta_b}{\tau'}\right) + \ln\left(\frac{\eta_b}{\tau'}\right)\right] \\ & \left. - H\left(1 - \frac{2\eta_b}{\tau'} + \eta'\right)\left[\ln\left(\frac{1+\eta'}{1-\eta'}\right) + \ln\left(1-\frac{\eta_b}{\tau'}\right) - \ln\left(\frac{\eta_b}{\tau'}\right)\right] \right\}. \end{aligned} \tag{4.48}$$

We can write the general expression for the $n = 2$ shape factor in r' and t' space in terms of the $n = 1$ shape factor as

$$\begin{aligned} F_2(r,t)H\left(t - \frac{r}{v}\right) =& \int_0^t \int_0^\infty \left(\frac{t'}{t}\right)^{-1} \frac{r'}{r} H\left(t' - \frac{r'}{v}\right) H(b-r') \frac{F_1(r',t')}{vt'(t-t')} \\ & \times \left[H\left(t - t' - \frac{|r-r'|}{v}\right) - H\left(t - t' - \frac{|r+r'|}{v}\right) \right] dr'dt'. \end{aligned} \tag{4.49}$$

However, there are three distinct sources that make up the total $n = 1$ shape factor. These are the infinite medium source; the left depletion wave source, $H(vt - 2b - r)$; and the right depletion wave source, $H(vt - 2b + r)$. Inserting the three sources into equation 4.49, the following expression for the $n = 2$ shape factor is obtained:

$$
\begin{aligned}
F_2(r,t) = & \int_0^t \int_0^\infty \left(\frac{t'}{t}\right)^{-1} \frac{r'}{r} H\left(t' - \frac{r'}{v}\right) H(b - r') \frac{F_{1,inf}(r',t')}{vt'(t-t')} \\
& \times \left[H\left(t - t' - \frac{|r - r'|}{v}\right) - H\left(t - t' - \frac{|r + r'|}{v}\right) \right] dr'dt' \\
& + \int_0^t \int_0^\infty \left(\frac{t'}{t}\right)^{-1} \frac{r'}{r} H\left(t' - \frac{2b}{v} - \frac{r'}{v}\right) H(b - r') \frac{F_{1,L}(r',t')}{vt'(t-t')} \\
& \times \left[H\left(t - t' - \frac{|r - r'|}{v}\right) - H\left(t - t' - \frac{|r + r'|}{v}\right) \right] dr'dt' \\
& + \int_0^t \int_0^\infty \left(\frac{t'}{t}\right)^{-1} \frac{r'}{r} H\left(t' - \frac{2b}{v} + \frac{r'}{v}\right) H(b - r') \frac{F_{1,R}(r',t')}{vt'(t-t')} \\
& \times \left[H\left(t - t' - \frac{|r - r'|}{v}\right) - H\left(t - t' - \frac{|r + r'|}{v}\right) \right] dr'dt'.
\end{aligned}
\tag{4.50}
$$

The first integral, with the $F_{1,inf}$ source underneath, has the same Heaviside functions as the $n = 1$ case, so the integration domain for that source is as shown in Fig. 4.21. For the $F_{1,L}$ source, the integration domain is shown in Fig. 4.22 below.

Referring to Fig. 4.22 for the limits of integration, the expression for the $n = 2$ shape factor from the $F_{1,L}$ source is:

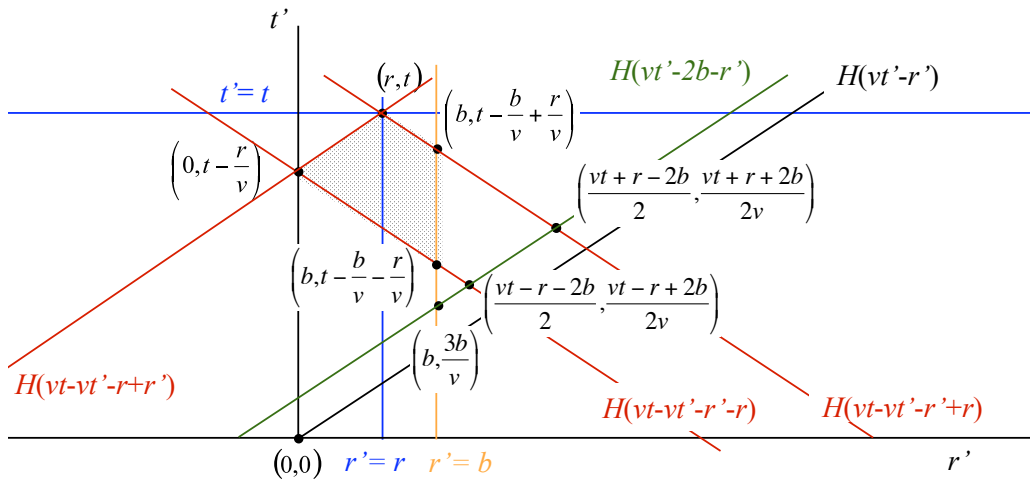

Fig. 4.22.— Finite sphere integration domain for $n = 2$ shape factor from $F_{1,L}$ source in r' and t' space.

$$
\begin{aligned}
F_{2,\text{from}F_{1,L}}(r,t) = H\left(vt - r - 2b\right) \Bigg[& \int_0^{r} \int_{\frac{2b+r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')} dt'dr' \\
& + \int_r^{\frac{vt+r-2b}{2}} \int_{\frac{2b+r'}{v}}^{t+\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')} dt'dr' - \int_0^{\frac{vt-r-2b}{2}} \int_{\frac{2b+r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')} dt'dr' \Bigg] \\
& - H(vt + r - 4b) \int_b^{\frac{vt+r-2b}{2}} \int_{\frac{2b+r'}{v}}^{t+\frac{r}{v}-r'v} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')} dt'dr' \\
& + H(vt - r - 4b) \int_b^{\frac{vt-r-2b}{2}} \int_{\frac{2b+r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')} dt'dr'.
\end{aligned}
\tag{4.51}
$$

Transforming this to η' and τ' space, the following expression is obtained:

$$F_{2,\mathrm{from}F_{1,L}}(\eta,\tau) = H(1-\eta-2\eta_b)\left[\int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right.$$
$$\left. + \int_{\eta}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{0}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right]$$
$$- H(1+\eta-4\eta_b)\left[\int_{\frac{1}{3}}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right]$$
$$+ H(1-\eta-4\eta_b)\left[\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{1}{3}}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right]. \tag{4.52}$$

The integration domain for the $n = 2$ shape factor from the $F_{1,L}$ source in η' and τ' space is shown in Fig. 4.23 below.

The integration domain for the $n = 2$ shape factor from the $F_{1,R}$ source in r' and t' space is shown in Fig. 4.24. Referring to this figure to obtain the limits of integration, we find that the form of the shape factor from the $F_{1,R}$ source is:

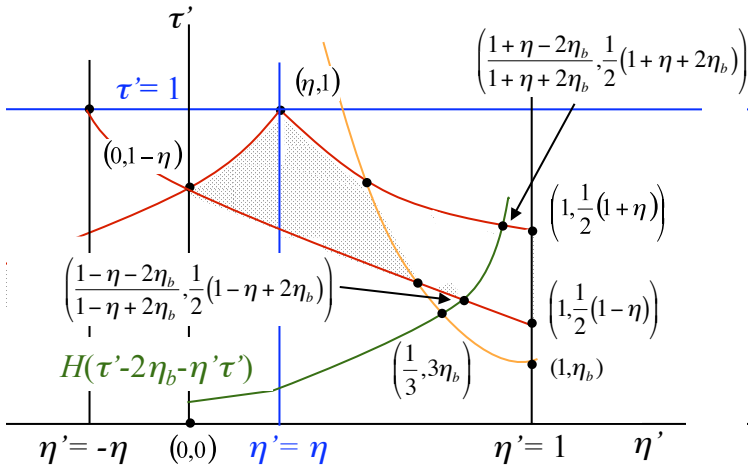

Fig. 4.23.— Finite sphere integration domain for $n = 2$ shape factor from $F_{1,L}$ source in r' and t' space.

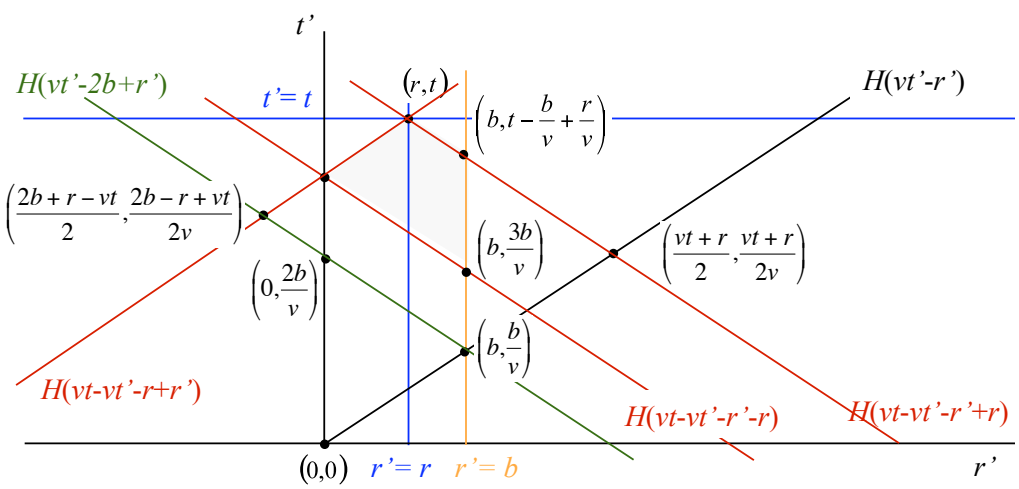

Fig. 4.24.— Finite sphere integration domain for $n = 2$ shape factor from $F_{1,R}$ source in r' and t' space.

$$
\begin{aligned}
F_{2,\text{from}F_{1,R}}(r,t) = H(vt-2b+r) &\left[\int_{\frac{r+2b-vt}{2}}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr' \right.\\
&\left. + \int_{b}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'\right] \\
&+ H(vt-2b-r)\int_{0}^{b}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'.
\end{aligned}
\tag{4.53}
$$

Examining the lower limit of integration of the first integral, we see that this limit would give a negative value for r'. Since our domain is restricted to positive values of r', we need to perform a change of variables on this integral from r' to $-r'$:

$$
\begin{aligned}
&\int_{\frac{r+2b-vt}{2}}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr' \\
&= \int_{\frac{r+2b-vt}{2}}^{0}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr' + \int_{0}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr' \\
&= \int_{\frac{vt-r-2b}{2}}^{0}\int_{\frac{2b+r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\left(\frac{-r'}{r}\right)\frac{F_{1,R}(-r',t')}{vt'(t-t')}dt'(-dr') + \int_{0}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr' \\
&= -\int_{0}^{\frac{vt-r-2b}{2}}\int_{\frac{2b+r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,L}(r',t')}{vt'(t-t')}dt'dr' + \int_{0}^{r}\int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'.
\end{aligned}
\tag{4.54}
$$

Note that we have taken advantage of the fact that $F_{1,L}(r',t') = F_{1,R}(-r',t')$:

$$
\begin{aligned}
F_{1,R}(-r',t) &= \frac{-vt'}{4\pi(-r')}\left[\ln\left(\frac{1+\frac{-r'}{vt'}}{1-\frac{-r'}{vt'}}\right)+\ln\left(1-\frac{b}{vt'}\right)-\ln\left(\frac{b}{vt'}\right)\right] \\
&= \frac{-vt'}{4\pi(-r')}\left[\ln\left(\frac{1-\frac{r'}{vt'}}{1+\frac{r'}{vt'}}\right)+\ln\left(1-\frac{b}{vt'}\right)-\ln\left(\frac{b}{vt'}\right)\right] \\
&= \frac{-vt'}{4\pi r'}\left[\ln\left(\frac{1+\frac{r'}{vt'}}{1-\frac{r'}{vt'}}\right)-\ln\left(1-\frac{b}{vt'}\right)+\ln\left(\frac{b}{vt'}\right)\right] \\
&= F_{1,L}(r',t').
\end{aligned}
\tag{4.55}
$$

We can now rewrite the equation for $F_{2,\text{from}F_{1,R}}$ as:

$$
\begin{aligned}
F_{2,\text{from}F_{1,R}}(r,t) = H(vt-2b+r)&\left[\int_0^r \int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}+\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'\right. \\
&\left.+\int_b^r \int_{\frac{2b-r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'\right] \\
&+H(vt-2b-r)\int_0^b \int_{\frac{2b-r'}{v}}^{t-\frac{r}{v}-\frac{r'}{v}} \frac{t}{t'}\frac{r'}{r}\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'.
\end{aligned}
\tag{4.56}
$$

In η' and τ' space, the integration domain for F_2 from the $F_{1,R}$ is shown in Fig. 4.25 below. Examining Fig. 4.25 we see that the integration domain is:

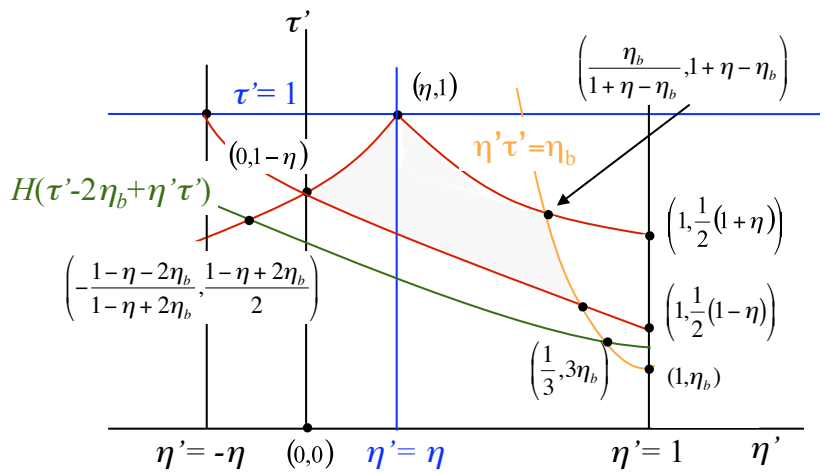

Fig. 4.25.— Finite sphere integration domain for $n = 2$ shape factor from $F_{1,R}$ source in η' and τ' space.

$$
\begin{aligned}
F_{2,\text{from}F_{1,R}}(\eta,\tau) = H(1+\eta-2\eta_b)&\left[\int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right.\\
&\left. - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{-\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right]\\
&+ H(1-\eta-2\eta_b)\left[-\int_{0}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right.\\
&\left. + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{-\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{0}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right].
\end{aligned}
\tag{4.57}
$$

Note that we have two integrals with limits of integration that move into $-\eta'$ space. Since our domain is restricted to positive values of η', we perform a change of

variables from η' to $-\eta'$.

$$
\begin{aligned}
&-\int_{-\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{0}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&=-\int_{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{0}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{-\eta'}{\eta}\frac{F_{1,R}(-\eta',\tau')}{1-\tau'}d\tau'(-d\eta') \\
&=\int_{0}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'.
\end{aligned}
\tag{4.58}
$$

Note that the result of the transformation of variables on this integral is equal to the negative of the third integral in equation 4.52. Since these two integrals will cancel each other out, they may be ignored.

$$
\begin{aligned}
&\int_{-\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&=\int_{-\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{0}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta'+\int_{0}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&=\int_{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}^{0}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{-\eta'}{\eta}\frac{F_{1,R}(-\eta',\tau')}{1-\tau'}d\tau'(-d\eta')+\int_{0}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&=-\int_{0}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'+\int_{0}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta'.
\end{aligned}
\tag{4.59}
$$

Now the equation for the $n = 2$ shape factor from the $F_{1,R}$ source can be written as:

$$
\begin{aligned}
F_{2,\mathrm{from}F_{1,R}}(\eta,\tau) = H(1+\eta-2\eta_b) &\left[\int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right.\\
&\left. - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{0}^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right]\\
&+ H(1-\eta-2\eta_b)\left[-\int_{0}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right.\\
&\left. + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \right]
\end{aligned}
\tag{4.60}
$$

while the equation for the $n = 2$ shape factor from the $F_{1,L}$ source becomes:

$$F_{2,\text{from}F_{1,L}}(\eta,\tau) = H(1-\eta-2\eta_b)\left[\int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right.$$
$$\left.+\int_{\eta}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right] - H(1+\eta-2\eta_b)\int_0^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'$$
$$-H(1+\eta-4\eta_b)\left[\int_{\frac{1}{3}}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right]$$
$$+H(1-\eta-4\eta_b)\left[\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{1}{3}}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}}\frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\right]. \tag{4.61}$$

Examining the expression for the $n = 2$ shape factor from the $F_{1,L}$ source, equations 4.51 and 4.61, we see that the original Heaviside functions, corresponding to the depletion wave source, is present, along with two new Heaviside functions. The expressions multiplying the two new Heaviside functions will become additional depletion wave sources, so that the $n = 3$ shape factor will have a total of five sources: $F_{2,inf}$, $F_{2,H(1-\eta-2\eta_b)}$, $F_{2,H(1+\eta-2\eta_b)}$, $F_{2,H(1-\eta-4\eta_b)}$, and $F_{2,H(1+\eta-4\eta_b)}$.

As was the case for the finite slab, all the shape factors, excluding the infinite medium pieces, have a dependence on τ'. To make this dependence more explicit, we will again perform a change of variables from τ' to η_b':

$$\eta_b' = \frac{\eta_b}{\tau'} \tag{4.62}$$

and

$$\frac{d\eta_b'}{d\tau'} = \frac{d}{d\tau'}\left(\frac{\eta_b}{\tau'}\right) = \frac{-\eta_b}{(\tau')^2}. \tag{4.63}$$

We can now write $F_{2,\text{from}F_{1L}}$ in terms of η_b' and η':

$$\begin{aligned}
F_{2,\text{from}F_{1,L}}(\eta,\eta_b) = H(1-\eta-2\eta_b)&\left[\int_0^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.\\
&\left.+\int_{\eta}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]\\
&-H(1+\eta-2\eta_b)\int_0^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\\
&-H(1+\eta-4\eta_b)\left[\int_{\frac{1}{3}}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.\\
&\left.+\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]\\
&+H(1-\eta-4\eta_b)\left[\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.\\
&\left.+\int_{\frac{1}{3}}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta_b'}\frac{F_{1,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]
\end{aligned} \tag{4.64}$$

and $F_{2,fromF_{1R}}$ becomes:

$$
\begin{aligned}
F_{2,\mathrm{from}F_{1,R}}(\eta,\eta_b) = H(1+\eta-2\eta_b) &\left[\int_{\eta}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1,R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right.\\
&\left. -\int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1,R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' + \int_{0}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1,R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right]\\
&+ H(1-\eta-2\eta_b)\left[-\int_{0}^{1}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1,R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right.\\
&\left. +\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1,R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right].
\end{aligned}
\tag{4.65}
$$

The above integrals can be evaluated analytically to obtain a closed form solution for the $n = 2$ shape factor. The analytic expression for the $n = 2$ shape factor is reproduced in Appendix C.

At this point, we introduce a new problem parameter, m, which we refer to as the reflection number. The reflection number indicates the number of times a neutron has moved across the medium, and varies from $1 \cdots n-1$. By using this parameter, we can rewrite the expressions for the shape factors. The nth shape factor from the $F_{n-1,H(1-\eta-2m\eta_b)}$ source is:

$$
\begin{aligned}
F_{n,\text{from}F_{n-1,m,L}}(\eta,\eta_b) = H(1-\eta-2m\eta_b) &\Bigg[\int_0^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta' \\
&+\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\Bigg] \\
&- H(1+\eta-2m\eta_b)\int_0^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta' \\
&- H(1+\eta-2(m+1)\eta_b)\Bigg[\int_{\frac{1}{2m+1}}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta' \\
&+\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m+1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\Bigg] \\
&+ H(1-\eta-2(m+1)\eta_b)\Bigg[\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta' \\
&+\int_{\frac{1}{2m+1}}^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\Bigg].
\end{aligned}
\tag{4.66}
$$

Similarly, using the reflection number, m, we can write a general expression for n^{th} shape factor from the $F_{n-1,H(1+\eta-2m\eta_b)}$ source:

$$F_{n,\text{from}F_{n-1,m,R}}(\eta,\eta_b) = H(1+\eta-2m\eta_b)\left[\int_{\eta}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.$$
$$-\int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'$$
$$\left.+\int_{0}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]$$
$$+H(1-\eta-2m\eta_b)\left[-\int_{0}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.$$
$$\left.+\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]. \tag{4.67}$$

Examining equations 4.66 and 4.67, we see that there is a singularity at $\eta_b' = \eta_b$. For the specific case of $n = 3$, the subtraction of singularity method for each of the integrals in the above equations is shown below:

$$Kt1 = \int_{0}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{2}\frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'. \tag{4.68}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{1-2m\eta_b-\eta'}{\eta-\eta'}\right) - \frac{1-\eta}{1-\eta'} + \frac{2m\eta_b}{1-\eta'}. \tag{4.69}$$

Equation 4.69 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}\int_0^{\eta} &\left\{\ln\left(\frac{1-2m\eta_b-\eta'}{\eta-\eta'}\right) - \frac{1-\eta}{1-\eta'} + \frac{2m\eta_b}{1-\eta'}\right\}\eta' d\eta' \\ &= \frac{\eta\,(1-2m\eta_b-\eta)}{2} + \frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b}{1-2m\eta_b-\eta}\right) \\ &+ (1-2m\eta_b-\eta)\ln(1-\eta) - \frac{\eta^2}{2}\ln\left(\frac{y}{1-2m\eta_b-\eta}\right).\end{aligned} \tag{4.70}$$

The type 2 limits of integration:

$$Kt2 = \int_0^{\eta} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'. \tag{4.71}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{(1-2m\eta_b-\eta')(1-\eta')}{(1+\eta')(\eta-\eta')}\right) - \frac{1-\eta}{1-\eta'} + \frac{2m\eta_b}{1+\eta'}. \tag{4.72}$$

Equation 4.72 is in fact singular for a range of values of η', and the subtraction of singularity method cannot be used on this particular integral.

The type 3 limits of integration:

$$Kt3 = \int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'. \tag{4.73}$$

The results of the η_b' integration, without the source:

$$\begin{aligned} &\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b'-\eta_b} \\ &\qquad = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}. \end{aligned} \tag{4.74}$$

Equation 4.74 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned} &\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \left\{\ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}\right\} \eta' d\eta' \\ &\qquad = -\frac{1-\eta^2}{2} + m\eta_b(1+\eta) - (1-2m\eta_b+\eta)\ln\left(\frac{1-\eta}{2}\right) \\ &\qquad - 2(m\eta_b)^2 \ln\left(\frac{2m\eta_b}{1+2m\eta_b+\eta}\right) + \frac{(1+\eta)^2}{2}\ln\left(\frac{1-\eta}{1+2m\eta_b+\eta}\right). \end{aligned} \tag{4.75}$$

The type 4 limits of integration:

$$Kt4 = \int_{\eta}^{\frac{1}{2m-1}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'. \tag{4.76}$$

The results of the η_b' integration, without the source:

$$\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}. \tag{4.77}$$

Equation 4.77 is singular at $\eta' = \eta$. The results of the η' integration:

$$\begin{aligned}
&\int_{\eta}^{\frac{1}{2m-1}} \left\{\ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}\right\} \eta' d\eta' \\
&= -\frac{(1-2m\eta_b+\eta)(1+\eta-2m\eta)}{2(2m-1)} + \frac{1+2\eta}{2}\ln(2m) + 2(m\eta_b)^2 \ln\left(\frac{2m-1}{2m}\right) \\
&- \frac{(1+\eta)^2}{2}\ln(2m-1) + \frac{1}{2(2m-1)^2}\ln\left(\frac{2m(1+\eta_b-2m\eta_b)}{1+\eta-2m\eta}\right) \\
&- (1-2m\eta-b-\eta)\ln(1+\eta) + \frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b+\eta}{1+\eta_b-2m\eta_b}\right) \\
&+ \frac{\eta^2}{2}\ln\left(\frac{1+\eta-2m\eta}{1-2m\eta_b+\eta}\right).
\end{aligned} \tag{4.78}$$

The type 5 limits of integration:

$$Kt5 = \int_{0}^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}} \int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'. \tag{4.79}$$

The results of the η_b' integration, without the source:

$$\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'+\eta)}\right) - \frac{1-\eta}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}. \tag{4.80}$$

There are no singularities in equation 4.80, given the η' limits of integration.

The type 6 limits of integration:

$$Kt6 = \int\limits_{0}^{\frac{1}{2m-1}} \int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b' - \eta_b} d\eta_b' d\eta'. \tag{4.81}$$

The results of the η_b' integration, without the source:

$$\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'+\eta}\right) - \frac{1-\eta}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}. \tag{4.82}$$

There are no singularities in equation 4.82, given the η' limits of integration.

The type 7 limits of integration:

$$Kt7 = \int\limits_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m-1}} \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \frac{\eta'}{\eta} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b' - \eta_b} d\eta_b' d\eta'. \tag{4.83}$$

The results of the η_b' integration, without the source:

$$\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{(\eta_b - \eta')(1+\eta')}{\eta'(\eta - \eta')}\right) - \frac{1+\eta}{1+\eta'} + \frac{\eta_b}{\eta'}. \tag{4.84}$$

Equation 4.84 is singular at $\eta' = \eta_b$. The results of the η' integration:

$$\begin{aligned}
&\int\limits_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m-1}} \left\{\ln\left(\frac{(1+\eta')(\eta_b - \eta')}{\eta'(\eta - \eta')}\right) - \frac{1+\eta}{1+\eta'} + \frac{\eta_b}{\eta'}\right\} \eta' d\eta' \\
&= -\frac{1 - 2m\eta_b + \eta}{2(2m-1)} + \frac{1+2\eta}{2}\ln(2m) + \frac{\eta_b^2}{2}\ln\left(\frac{\eta_b(2m-1)(\eta_b - \eta)}{(1+\eta_b - 2m\eta_b)(1-\eta_b+\eta)}\right) \\
&+ \frac{1}{2(2m-1)^2}\ln\left(\frac{2m(1+\eta_b - 2m\eta_b)}{1+\eta-2m\eta}\right) + \frac{(1+\eta)^2}{2}\ln\left(\frac{1-\eta_b+\eta}{(2m-1)(1+\eta)}\right) \\
&+ \frac{\eta^2}{2}\ln\left(\frac{1+\eta-2m\eta}{\eta_b - \eta}\right).
\end{aligned} \tag{4.85}$$

The type 8 limits of integration:

$$Kt8 = \int\limits_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m-1}} \int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta', \eta_b')}{\eta_b' - \eta_b} d\eta_b' d\eta'. \tag{4.86}$$

The results of the η_b' integration, without the source:

$$\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} = \ln\left(\frac{(\eta' - \eta_b)(1+\eta')}{\eta'(\eta + \eta')}\right) - \frac{1-\eta}{1+\eta'} + \frac{\eta_b}{\eta'}. \tag{4.87}$$

Equation 4.87 is singular at $\eta' = \eta_b$. The results of the η' integration:

$$\begin{aligned}
&\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m-1}} \left\{ \ln\left(\frac{(\eta'-\eta_b)(1+\eta')}{\eta'(\eta+\eta')} \right) - \frac{1-\eta}{1+\eta'} + \frac{\eta_b}{\eta'} \right\} \eta' d\eta' \\
&\quad = -\frac{1-2m\eta_b-\eta}{2(2m-1)} + \frac{1-2\eta}{2}\ln(2m) + \frac{\eta_b^2}{2}\ln\left(\frac{\eta_b(2m-1)(\eta_b+\eta)}{(1+\eta_b-2m\eta_b)(1-\eta_b-\eta)} \right) \\
&\quad + \frac{1}{2(2m-1)^2}\ln\left(\frac{2m(1+\eta_b-2m\eta_b)}{1-\eta-2m\eta} \right) + \frac{(1-\eta)^2}{2}\ln\left(\frac{1-\eta_b-\eta}{(2m-1)(1-\eta)} \right) \\
&\quad + \frac{\eta^2}{2}\ln\left(\frac{1-\eta+2m\eta}{\eta_b+\eta} \right).
\end{aligned} \tag{4.88}$$

4.2.2 Finite Sphere Without Reflective Boundary Condition

This method is based off the following derivation in [16], for steady-state neutron transport in a finite sphere. The integral form of the neutron transport equation can be written as:

$$r\phi(r,E) = \frac{1}{2}\int_0^a r' q(r',E) \left\{ E_1\left[\sigma(E)|r-r'|\right] - E_1\left[\sigma(E)(r+r')\right] \right\} dr' \tag{4.89}$$

where a is the radius of the sphere, E is the energy, E_1 is the exponential integral, $q(r,E)$ is the external source and σ is the cross section. Note that the source, $q(r,E)$ is symmetric around $r=0$, that is, $q(-r,E) = q(r,E)$. Using this fact, we can rewrite equation 4.89 to expand the domain from $-a \le r \le a$ as:

$$r\phi(r,E) = \frac{1}{2}\int_{-a}^{a} r'q(r',E)E_1\left(\sigma(E)|r-r'|\right)dr'. \tag{4.90}$$

Of particular importance to this method is that we have replaced the spherical form of the integration kernel, $E_1\left[\sigma(E)|r-r'|\right] - E_1\left[\sigma(E)(r+r')\right]$, with the slab form of the integration kernel, $E_1\left(\sigma(E)|r-r'|\right)$.

Now we will extend this method to incorporate a time-dependent kernel. The time-dependent neutron transport equation for the nth collided flux in a finite sphere can be written as:

$$\begin{aligned} r\phi_n(r,t) =& \Sigma_s \int_0^t \int_0^b \frac{e^{-\Sigma v(t-t')}}{8\pi r'(t-t')}\left[H\left(t-t'-\frac{|r-r'|}{v}\right) - H\left(t-t'-\frac{|r+r'|}{v}\right)\right] \\ &\times 4\pi r'^2\phi_{n-1}(r',t')dr'dt' \end{aligned} \tag{4.91}$$

where b is the radius of the sphere. By analogy with the steady-state sphere, we can rewrite the nth collided flux over the domain $-b \leq r \leq b$ using the slab form of the integration kernel Heaviside functions:

$$r\phi_n(r,t) = \Sigma_s \int_0^t \int_{-b}^b \frac{e^{-\Sigma v(t-t')}}{8\pi r'(t-t')} H\left(t-t'-\frac{|r-r'|}{v}\right) 4\pi r'^2\phi_{n-1}(r',t')dr'dt' \tag{4.92}$$

or, alternatively, we can write the above as

$$\phi_n(r,t) = \Sigma_s \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\Sigma v(t-t')}}{8\pi r r'(t-t')} H\left(t - t' - \frac{|r-r'|}{v}\right) \times H(b-r')H(b+r')4\pi r'^2 \phi_{n-1}(r',t')dr'dt'. \tag{4.93}$$

Note that the Heavisides in equation 4.93 are identical to those in the finite slab equation, equation 4.1. The advantage of this approach is now apparent. No further work in extracting the Heaviside functions to determine the limits of integration is necessary. We can use the same limits of integration as derived for the finite slab case.

The next step is to determine the uncollided flux from the external source. The external source is the same as for the infinite sphere, and is a pulsed source in space and time. The uncollided flux, too, is the same as for the infinite sphere, and is given by equation 3.40.

Applying the spherical coordinates ansatz, equation 3.45 to the expression for the nth collided flux, equation 4.93, results in the following equation for the nth shape factor:

$$\begin{aligned} &H\left(t + \frac{r}{v}\right) H\left(t - \frac{r}{v}\right) F_n(r,t) \\ &\quad = \frac{n}{2} \int_0^t \int_{-b}^{b} \frac{r'}{r} \left(\frac{t'}{t}\right)^{n-3} \frac{1}{vt'(t-t')} H\left(t' - \frac{r'}{v}\right) H\left(t' + \frac{r'}{v}\right) H\left(t - t' - \frac{|r-r'|}{v}\right) \\ &\quad \times F_{n-1}(r',t')dr'dt'. \end{aligned} \tag{4.94}$$

To determine the integration domain for the $n = 1$ shape factor in r' and t' space, extract the Heaviside functions in equation 4.94. The integration domain is as

shown in Fig. 4.1. The expression for the $n = 1$ is then:

$$F_1(r,t) = \frac{1}{2}\left[\int_{\frac{r-vt}{2}}^{0}\int_{\frac{-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')}dt'dr' + \int_{0}^{r}\int_{\frac{r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')}dt'dr'\right.$$
$$+ \int_{r}^{\frac{r+vt}{2}}\int_{\frac{r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')}dt'dr'$$
$$- H(vt-2b-r)\int_{\frac{r-vt}{2}}^{-b}\int_{\frac{-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')}dt'dr'$$
$$\left. - H(vt-2b+r)\int_{b}^{\frac{r+vt}{2}}\int_{\frac{r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)^2 \frac{F_0(r',t')}{vt'(t-t')}dt'dr'\right]. \tag{4.95}$$

Solving this analytically, the following expression for the $n = 1$ shape factor is found:

$$F_1(r,t) = \frac{vt}{4\pi r}\left\{2\ln\left(\frac{1+\frac{r}{vt}}{1-\frac{r}{vt}}\right) - H(vt+r-2b)\left[\left(\frac{1+\frac{r}{vt}}{1-\frac{r}{vt}}\right) + \ln\left(1-\frac{b}{vt}\right)\right.\right.$$
$$\left.\left. - \ln\left(\frac{b}{vt}\right)\right] - H(vt-r-2b)\left[\ln\left(\frac{1+\frac{r}{vt}}{1-\frac{r}{vt}}\right) - \ln\left(1-\frac{b}{vt}\right) + \ln\left(\frac{b}{vt}\right)\right]\right\}. \tag{4.96}$$

This result was previously published in [21].

To obtain the equation for the $n = 1$ shape factor in η' and τ'space, we first need to change variables in equation 4.94 from r' and t' space to η' and τ' space. Doing so, we obtain the general expression for the nth shape factor:

$$F_n(\eta,\tau) = \int_0^1 \int_{-\infty}^{\infty} \frac{(\tau')^{n-2}}{1-\tau'} \frac{\eta'}{\eta} F_{n-1}(\eta',\tau') H(\eta_b - \tau'\eta') H(\eta_b + \tau'\eta') H(1-\eta') H(1+\eta')$$
$$\times H(1-\tau' - |\eta - \eta'\tau'|) d\eta' d\tau'. \tag{4.97}$$

In η' and τ' space, the integration domain is found from Fig. 4.2, and the equation for the $n = 1$ shape factor can be written as

$$F_1(\eta,\tau) = \frac{1}{2}\left[\int_{-1}^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{1}{\tau'(1-\tau')} \frac{\eta'}{\eta} F_0(\eta) d\tau' d\eta' + \int_{\eta}^{1} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')} \frac{\eta'}{\eta} F_0(\eta) d\tau' d\eta'\right.$$
$$- H(1-2\eta_b-\eta) \int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}} \int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{1}{\tau'(1-\tau')} \frac{\eta'}{\eta} F_0(\eta) d\tau' d\eta'$$
$$\left. - H(1-2\eta_b+\eta) \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{1}{\tau'(1-\tau')} \frac{\eta'}{\eta} F_0(\eta) d\tau' d\eta'\right]. \tag{4.98}$$

Evaluating the above integrals analytically gives the same result for the $n = 1$ shape factor as was found using the reflective boundary condition, and is given in equation 4.47.

To obtain the $n = 2$ shape factor, we use the $n = 1$ shape factor as the source. The $n = 1$ shape factor can be broken into three separate sources: the infinite medium source, the left depletion wave and the right depletion wave. The integration domain found from the infinite medium source is identical to the integration domain for the $n = 1$ shape factor. The integration domain from the left depletion wave, in r' and t'

space is shown in Fig 4.3, while the integration domain from the right depletion wave is shown in Fig. 4.4. The expression for the $n = 2$ shape factor from the $n = 1$ infinite medium source is:

$$
\begin{aligned}
F_{2,fromF_{1,inf}}(r,t) = \; & \frac{1}{2}\Bigg[\int_{\frac{r-vt}{2}}^{0}\int_{\frac{-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,inf}(r',t')}{vt'(t-t')}dt'dr' \\
& + \int_{0}^{r}\int_{\frac{r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,inf}(r',t')}{vt'(t-t')}dt'dr' + \int_{r}^{\frac{r+vt}{2}}\int_{\frac{r'}{v}}^{t-\frac{r'}{v}+\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,inf}(r',t')}{vt'(t-t')}dt'dr' \\
& - H(vt-2b-r)\int_{\frac{r-vt}{2}}^{-b}\int_{\frac{-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,inf}(r',t')}{vt'(t-t')}dt'dr' \\
& - H(vt-2b+r)\int_{b}^{\frac{r+vt}{2}}\int_{\frac{r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}} \frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,inf}(r',t')}{vt'(t-t')}dt'dr'\Bigg].
\end{aligned}
\tag{4.99}
$$

The expression for the $n = 2$ shape factor from the $n = 1$ left depletion wave is:

$$F_{2,fromF_{1L}}(r,t) =$$

$$H(vt-2b-r)\left[\int_{-b}^{r}\int_{\frac{2b+r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,L}(r',t')}{vt'(t-t')}dt'dr'\right.$$

$$\left.+\int_{r}^{\frac{vt-2b+r}{2}}\int_{\frac{2b+r'}{v}}^{t+\frac{r}{v}-\frac{r'}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,L}(r',t')}{vt'(t-t')}dt'dr'\right] \tag{4.100}$$

$$-H(vt-4b+r)\int_{b}^{\frac{vt-2b+r}{2}}\int_{\frac{2b+r'}{v}}^{t+\frac{r}{v}-\frac{r'}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,L}(r',t')}{vt'(t-t')}dt'dr'$$

and the expression for the $n=2$ shape factor from the $n=1$ right depletion wave is:

$$F_{2,fromF_{1R}}(r,t) =$$

$$-H(vt-2b+r)\left[\int_{\frac{2b-vt+r}{2}}^{r}\int_{\frac{2b-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'\right.$$

$$\left.+\int_{r}^{b}\int_{\frac{2b-r'}{v}}^{t+\frac{r}{v}-\frac{r'}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'\right] \tag{4.101}$$

$$-H(vt-4b-r)\int_{\frac{2b-vt+r}{2}}^{-b}\int_{\frac{2b-r'}{v}}^{t+\frac{r'}{v}-\frac{r}{v}}\frac{r'}{r}\left(\frac{t}{t'}\right)\frac{F_{1,R}(r',t')}{vt'(t-t')}dt'dr'.$$

The integration domain for $F_{2,fromF_{1L}}$ and $F_{2,fromF_{1R}}$ in $\eta'$and τ'space are shown in Figs. 4.5 and 4.6, respectively. In $\eta'$and τ'space, the equations for $F_{2,fromF_{1,inf}}$, $F_{2,fromF_{1L}}$ and $F_{2,fromF_{1R}}$ are:

$$
\begin{aligned}
F_{2,fromF_{1,inf}}(\eta,\tau) = & \int_{-1}^{\eta}\int_{0}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1inf}(\eta')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_{0}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1inf}(\eta')}{1-\tau'}d\tau' d\eta' \\
& - H(1-2\eta_b-\eta)\int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1inf}(\eta')}{1-\tau'}d\tau' d\eta' \\
& - H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1inf}(\eta')}{1-\tau'}d\tau' d\eta',
\end{aligned}
\tag{4.102}
$$

$$
\begin{aligned}
F_{2,fromF_{1,L}}(\eta,\tau) = & H(1-2\eta_b-\eta)\Bigg[\int_{-1}^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
& + \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\Bigg] \\
& - H(1-4\eta_b+\eta)\Bigg[\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
& + \int_{\frac{1}{3}}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'\Bigg]
\end{aligned}
\tag{4.103}
$$

and

$$
\begin{aligned}
F_{2,fromF_{1,R}}(\eta,\tau) = H(1-2\eta_b+\eta) \Bigg[& \int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\eta} \int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{F_{1R}(\eta',\tau')}{1-\tau'} d\tau' d\eta' \\
& + \int_{\eta}^{1} \int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{F_{1R}(\eta',\tau')}{1-\tau'} d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{F_{1R}(\eta',\tau')}{1-\tau'} d\tau' d\eta' \Bigg] \\
& - H(1-4\eta_b-\eta) \Bigg[\int_{\frac{-1}{3}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{F_{1R}(\eta',\tau')}{1-\tau'} d\tau' d\eta' \\
& + \int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\frac{-1}{3}} \int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{F_{1R}(\eta',\tau')}{1-\tau'} d\tau' d\eta' \Bigg]
\end{aligned}
\tag{4.104}
$$

respectively.

Introducing the variable change from $\tau'$to η'_b allows us to rewrite equations 4.103 and 4.104 as:

$$F_{2,fromF_{1,L}}(\eta,\eta_b) = H(1-2\eta_b-\eta)\left[\int_{-1}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1L}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right.$$
$$+ \int_{\eta}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1L}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' - \left.\int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1L}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta'\right]$$
$$- H(1-4\eta_b+\eta)\left[\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1L}(\eta',\eta'_b)}{\eta_b-\eta'_b}d\eta'_b d\eta' \right.$$
$$\left. + \int_{\frac{1}{3}}^{\frac{1-2\eta_b+\eta}{1+2\eta_b+\eta}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1L}(\eta',\eta'_b)}{\eta_b-\eta'_b}d\eta'_b d\eta' \right] \tag{4.105}$$

and

$$F_{2,fromF_{1,R}}(\eta,\eta_b) = H(1-2\eta_b+\eta)\left[\int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta'}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right.$$
$$\left. + \int_{\eta}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right]$$
$$- H(1-4\eta_b-\eta)\left[\int_{\frac{-1}{3}}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right.$$
$$\left. + \int_{\frac{2\eta_b+\eta-1}{2\eta_b-\eta+1}}^{\frac{-1}{3}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2}} \frac{\eta'}{\eta}\frac{\eta_b}{\eta'_b}\frac{F_{1R}(\eta',\eta'_b)}{\eta'_b-\eta_b}d\eta'_b d\eta' \right]. \tag{4.106}$$

By using the reflection number, m, we can write the above equations more generally as:

$$
\begin{aligned}
F_{n,fromF_{n-1,m,L}}(\eta,\eta_b) = H(1-2m\eta_b-\eta)\Bigg[& \int_{\frac{-1}{2m-1}}^{\eta} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1} \frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
+ & \int_{\eta}^{\frac{1-2m\eta_b+\eta}{1+2m\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1} \frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
- & \int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1} \frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \Bigg] \\
- H(1-2(m+1)\eta_b-\eta)\Bigg[& \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m+1}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1} \frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b-\eta_b'} d\eta_b' d\eta' \\
+ & \int_{\frac{1}{2m+1}}^{\frac{1-2m\eta_b+\eta}{1+2w\eta_b+\eta}} \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1} \frac{F_{n-1,m,L}(\eta',\eta_b')}{\eta_b-\eta_b'} d\eta_b' d\eta' \Bigg]
\end{aligned}
\tag{4.107}
$$

and

$$F_{n,fromF_{n-1,m,R}}(\eta,\eta_b) = H(1-2m\eta_b+\eta)\left[\int_{\eta}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.$$

$$+\int_{\frac{2m\eta_b+\eta-1}{2m\eta_b-\eta+1}}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta'}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'$$

$$\left.-\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]$$

$$-H(1-2(m+1)\eta_b+\eta)\left[\int_{\frac{-1}{2m+1}}^{\frac{-\eta_b}{1-\eta_b-\eta}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{-\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right.$$

$$\left.+\int_{\frac{2m\eta_b+\eta-1}{2m\eta_b-\eta+1}}^{\frac{-1}{2m+1}}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^{n-1}\frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b}d\eta_b'd\eta'\right]. \tag{4.108}$$

The integrals in the above two equations are singular at $\eta_b' = \eta_b$. Subtraction of singularity will have to be performed on each of the above integrals. The results of the η_b' integrations are shown in equations 4.22, 4.25, 4.28, 4.31, 4.34, and 4.37, given in Section 4.1, where n is replaced by $n-1$. Because of the factor of η' in each of the integrals, the results of the η' integration are not the same as for the finite slab case. The results of the η_b' integration, for the specific case of $n = 3$ are shown below.

Beginning with the type 1:

$$\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}} \left\{ \ln\left(\frac{(1+\eta')(\eta'-\eta_b)}{\eta'(\eta'-\eta)}\right) + \frac{\eta_b}{\eta'} - \frac{1+\eta}{1+\eta'} \right\} \eta' d\eta'$$

$$= \frac{1-2m\eta_b-\eta}{2(2m-1)} - \frac{1-2\eta}{2}\ln(2m) + \frac{(1-\eta)^2}{2}\ln\left(\frac{(1-\eta)(2m-1)}{1-\eta_b-\eta}\right) \tag{4.109}$$

$$+ \frac{\eta_b^2}{2}\ln\left(\frac{(1-\eta_b-\eta)(1+\eta_b-2m\eta_b)}{\eta_b(2m-1)(\eta_b+\eta)}\right) + \frac{\eta^2}{2}\ln\left(\frac{\eta_b+\eta}{1-\eta+2m\eta}\right)$$

$$+ \frac{1}{2(2m-1)^2}\ln\left(\frac{1-\eta+2m\eta}{2m(1+\eta_b-2m\eta_b)}\right),$$

Type 2:

$$\int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \left\{ \ln\left(\frac{(1-\eta')(\eta'+\eta_b)}{\eta'(\eta-\eta')}\right) + \frac{-\eta_b}{\eta'} - \frac{1-\eta}{1-\eta'} \right\} \eta' d\eta'$$

$$= -\frac{1-2m\eta_b+\eta}{2(2m-1)} + \frac{1+2\eta}{2}\ln(2m) - \frac{(1+\eta)^2}{2}\ln\left(\frac{(1+\eta)(2m-1)}{1-\eta_b+\eta}\right) \tag{4.110}$$

$$- \frac{\eta_b^2}{2}\ln\left(\frac{(1-\eta_b+\eta)(1+\eta_b-2m\eta_b)}{\eta_b(2m-1)(\eta_b-\eta)}\right) - \frac{\eta^2}{2}\ln\left(\frac{\eta_b-\eta}{1+\eta-2m\eta}\right)$$

$$- \frac{1}{2(2m-1)^2}\ln\left(\frac{1+\eta-2m\eta}{2m(1+\eta_b-2m\eta_b)}\right),$$

Type 3:

$$\int_{\frac{-1}{2m-1}}^{\eta} \left\{ \ln\left(\frac{1-2m\eta_b-\eta'}{\eta-\eta'}\right) + \frac{2m\eta_b}{1-\eta'} - \frac{1-\eta}{1-\eta'} \right\} \eta' d\eta'$$
$$= \frac{(1-2m\eta_b-\eta)(1-\eta+2m\eta)}{2(2m-1)} - \frac{1-2\eta}{2}\ln(2m) - 2(m\eta_b)^2 \ln\left(\frac{2m-1}{2m}\right)$$
$$+ \frac{(1-\eta)^2}{2}\ln(2m-1) + (1-2m\eta_b-\eta)\ln(1-\eta)$$
$$- \frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b-\eta}{1+\eta_b-2m\eta_b}\right) + \frac{\eta^2}{2}\ln\left(\frac{1-2m\eta_b-\eta}{1-\eta+2m\eta}\right)$$
$$+ \frac{1}{2(2m-1)^2}\ln\left(\frac{1-\eta+2m\eta}{2m(1+\eta_b-2m\eta_b)}\right), \tag{4.111}$$

Type 4:

$$\int_{\eta}^{\frac{1}{2m-1}} \left\{ \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta}\right) + \frac{2m\eta_b}{1+\eta'} - \frac{1+\eta}{1+\eta'} \right\} \eta' d\eta'$$
$$= -\frac{(1-2m\eta_b+\eta)(1+\eta-2m\eta)}{2(2m-1)} + \frac{1+2\eta}{2}\ln(2m) + 2(m\eta_b)^2 \ln\left(\frac{2m-1}{2m}\right)$$
$$- \frac{(1+\eta)^2}{2}\ln(2m-1) - (1-2m\eta_b+\eta)\ln(1+\eta)$$
$$+ \frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b+\eta}{1+\eta_b-2m\eta_b}\right) - \frac{\eta^2}{2}\ln\left(\frac{1-2m\eta_b+\eta}{1+\eta-2m\eta}\right)$$
$$- \frac{1}{2(2m-1)^2}\ln\left(\frac{1+\eta-2m\eta}{2m(1+\eta_b-2m\eta_b)}\right), \tag{4.112}$$

Type 5:

$$\begin{aligned}
&\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \left\{ \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)} \right) + \frac{2m\eta_b}{1-\eta'} - \frac{1+\eta}{1+\eta'} \right\} \eta' d\eta' \\
&\quad = -\frac{(1+\eta)(1-2m\eta_b-\eta)}{2} - (1-2m\eta_b+\eta)\ln\left(\frac{1-\eta}{2}\right) \\
&\quad - 2(m\eta_b)^2 \ln\left(\frac{2m\eta_b}{1+2m\eta_b+\eta}\right) - \frac{(1+\eta)^2}{2}\ln\left(\frac{1+2m\eta_b+\eta}{1-\eta}\right),
\end{aligned} \tag{4.113}$$

and

Type 6:

$$\begin{aligned}
&\int_{\frac{-(1-\eta-2m\eta_b)}{1-\eta+2m\eta_b}}^{\eta} \left\{ \ln\left(\frac{(1-2m\eta_b+\eta')(1-\eta')}{(1+\eta')(\eta-\eta')} \right) + \frac{2m\eta_b}{1+\eta'} - \frac{1-\eta}{1-\eta'} \right\} \eta' d\eta' \\
&\quad = \frac{(1-\eta)(1-2m\eta_b+\eta)}{2} + (1-2m\eta_b-\eta)\ln\left(\frac{1+\eta}{2}\right) \\
&\quad + 2(m\eta_b)^2 \ln\left(\frac{2m\eta_b}{1+2m\eta_b-\eta}\right) + \frac{(1-\eta)^2}{2}\ln\left(\frac{1+2m\eta_b-\eta}{1+\eta}\right).
\end{aligned} \tag{4.114}$$

4.2.3 Comparison of the Two Finite Sphere Methods

To ensure that the finite sphere derivations are consistent, there are a number of checks that may be performed. This section describes many of these checks, in an effort to determine if and where any errors in the method may have occurred.

For a finite sphere, it is possible to determine at what time, in terms of mean free paths, that the flux for a given collision goes to zero. For instance, in a sphere of radius b, the $n = 1$ flux must go to zero at time $t = \frac{3b}{v}$. A neutron that has not collided can travel, at most, the radius of the sphere. Once the neutron has reached the boundary, it must either exit the sphere or collide. Once the neutron has collided, it moves into the $n = 1$ collision. At this point, the maximum distance the neutron

can travel is an additional $2b$, where b is the radius of the sphere, for a total distance of $3b$. At this point, the neutron is at the boundary and must either exit the sphere, or collide, where it will contribute to the $n = 2$ collisional flux. Therefore, to verify that the $n = 1$ collided shape factor is behaving as expected, we need to ensure that when $t = \frac{3b}{v}$, the shape factor is zero. In terms of η_b, the $n = 1$ shape factor must go to zero when $\eta_b = \frac{1}{3}$, and must stay zero for smaller values of η_b. Plugging $\eta_b = \frac{1}{3}$ into equation 4.47

$$
\begin{aligned}
F_1\left(\eta, \eta_b = \frac{1}{3}\right) &= \frac{1}{4\pi\eta}\left\{2\ln\left(\frac{1+\eta}{1-\eta}\right)\right. \\
&- H\left(1 - \frac{2}{3} - \eta\right)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) - \ln\left(1 - \frac{1}{3}\right) + \ln\left(\frac{1}{3}\right)\right] \\
&\left. - H\left(1 - \frac{2}{3} + \eta\right)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln\left(1 - \frac{1}{3}\right) - \ln\left(\frac{1}{3}\right)\right]\right\} \\
&= \frac{1}{4\pi\eta}\left\{2\ln\left(\frac{1+\eta}{1-\eta}\right)\right. \\
&- H\left(\frac{1}{3} - \eta\right)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) - \ln\left(\frac{2}{3}\right) + \ln\left(\frac{1}{3}\right)\right] \\
&\left. - H\left(\frac{1}{3} + \eta\right)\left[\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\right)\right]\right\} \\
&= \frac{1}{4\pi\eta}\left\{2\ln\left(\frac{1+\eta}{1-\eta}\right)\right. \\
&- \left[\ln\left(\frac{1+\eta}{1-\eta}\right) - \ln\left(\frac{2}{3}\right) + \ln\left(\frac{1}{3}\right)\right] \\
&\left. - \left[\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{3}\right)\right]\right\} \\
&= 0.
\end{aligned}
\tag{4.115}
$$

Here, we have taken advantage of the fact that $\eta \leq \eta_b$, which allows us to determine that the $H\left(\frac{1}{3} - \eta\right)$ Heaviside function is always turned on. Additionally, in the formulation utilizing the reflective boundary at $\eta = 0$, η is defined to be strictly

positive, therefore the $H\left(\frac{1}{3}+\eta\right)$ Heaviside function is always turned on. Alternatively, in the formulation not utilizing the reflective boundary condition, we note that $\eta_b \geq |\eta|$. This means that $-\eta \geq \eta_b$, and the $H\left(\frac{1}{3}-\eta\right)$ is always turned on.

Another way to utilize the fact that the shape factors must go to zero is to examine the limits of integration. We can insert the source into the integrals, and then ensure that, by matching limits of integration, the result is equal to zero when all the depletion wave Heaviside functions are turned on. We do this for the $n = 2$ shape factor in the reflective boundary condition formulation in Appendix D. This same check was performed on the $n = 2$ shape factor formulation that does not utilize the reflective boundary condition, though it is not reproduced in this work. In both cases, it was found that the $n = 2$ shape factor went to zero after all the Heaviside functions were turned on.

An additional check, utilizing the time at which the shape factor goes to zero, may be performed on the $n = 2$ shape factor. Since the maximum distance a neutron contributing to the second collided flux may travel is $t = \frac{5b}{v}$, we can check that after this time, the $n = 2$ flux goes to zero. Since we have an analytic expression for the $n = 2$ shape factor, we check this graphically. Figs. 4.26 through 4.29 show the $n = 2$ shape factor, found from both the formulation using the reflective boundary condition, and the formulation without the reflective boundary condition, for various values of η_b. Here we have provided two checks of the finite sphere formulation. We have found that both formulations give the same result for the $n = 2$ shape factor, and that the $n = 2$ shape factor goes to zero when $\eta_b = 0.2$.

To ensure that the derivation of the finite sphere shape factors in both formulations are consistent, we can match integrals for the shape factors. Table 4.3 shows

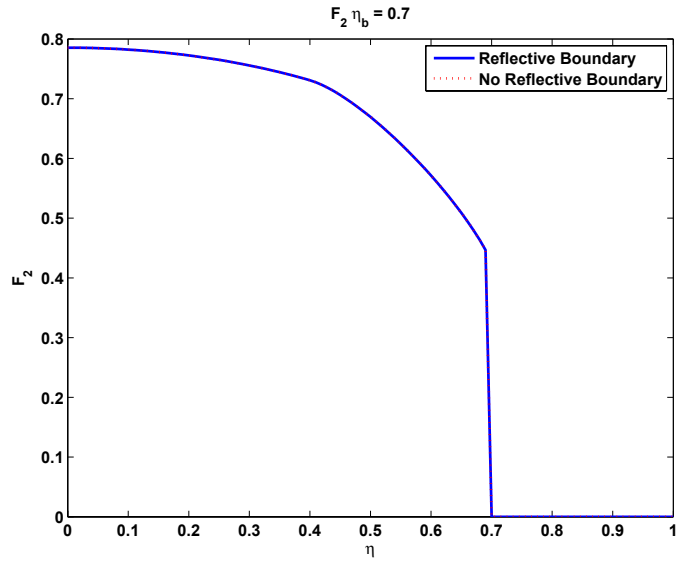

Fig. 4.26.— $n = 2$ Finite Sphere Shape Factor for $\eta_b = 0.7$.

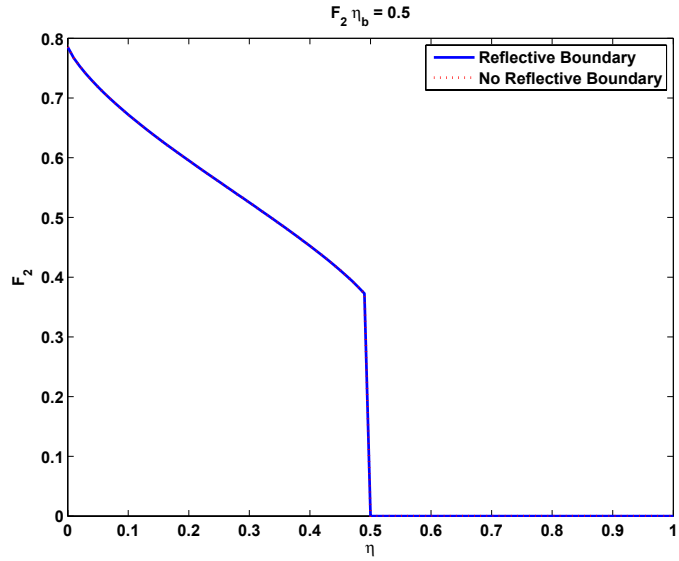

Fig. 4.27.— $n = 2$ Finite Sphere Shape Factor for $\eta_b = 0.5$.

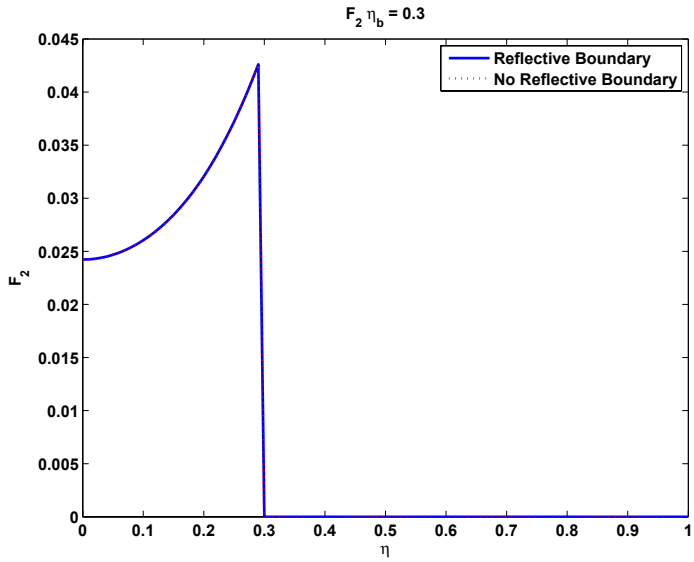

Fig. 4.28.— $n = 2$ Finite Sphere Shape Factor for $\eta_b = 0.3$.

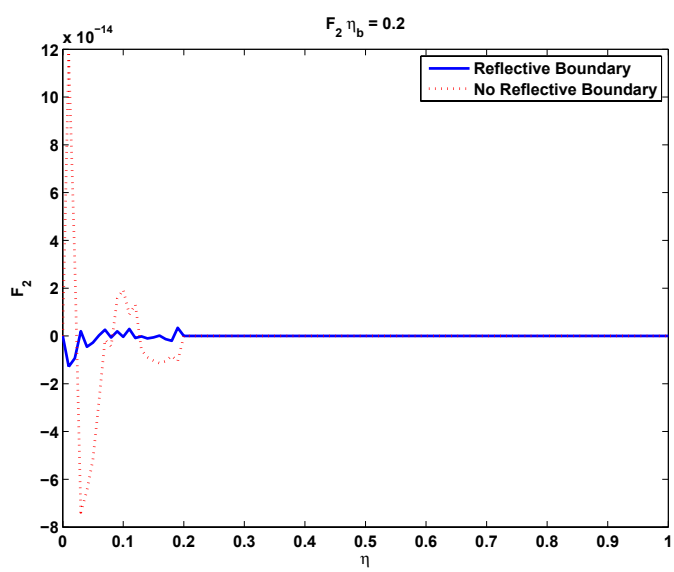

Fig. 4.29.— $n = 2$ Finite Sphere Shape Factor for $\eta_b = 0.2$.

the $n = 3$ integrals in both formulations.

Table 4.3:: Matching Integrals From Both Formulations,

$n = 3$ Shape Factor

Reflective Boundary	**No Reflective Boundary**
$\int_0^{\frac{1-\eta}{1-\eta'}} \int_0^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau' - \int_0^{\frac{1-\eta}{1+\eta'}} \int_0^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$	$\int_0^{\frac{1-\eta}{1-\eta'}} \int_{-1}^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$
$\int_0^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$	$\int_0^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$
$\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$	$- \int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{-1}^{\frac{-\eta_b}{1-\eta_b-\eta}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$
$\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$	$\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,inf}(\eta') d\eta' d\tau'$
$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,L}(\eta', \tau') d\eta' d\tau'$	$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,L}(\eta', \tau') d\eta' d\tau'$
$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_0^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,L}(\eta', \tau') d\eta' d\tau' - \int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_0^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$	$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{\frac{-1}{2m-1}}^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$
$\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$	$- \int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{\frac{-1}{2m-1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,L}(\eta', \tau') d\eta' d\tau'$
$\int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_0^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$	$\int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{\frac{-(1-\eta-2m\eta_b)}{1-\eta+2m\eta_b}}^{\eta} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$
$- \int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_0^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,L}(\eta', \tau') d\eta' d\tau'$	
$\int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$	$\int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\eta}^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$
$- \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$	$\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m-1}} \frac{\eta'}{\eta} \frac{\tau'}{1-\tau'} F_{2,m,R}(\eta', \tau') d\eta' d\tau'$

Table 4.3 – Continued

Reflective Boundary	**No Reflective Boundary**
$\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$	$-\int_{\frac{-\eta_b}{\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{\frac{-1}{2m+1}}^{\frac{-\eta_b}{1-\eta_b-\eta}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,R}(\eta',\tau')d\eta' d\tau'$
$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \int_{\frac{1}{2m+1}}^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$	$-\int_{\frac{2m\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \int_{\frac{-(1-\eta-2m\eta_b)}{1-\eta+2m\eta_b}}^{\frac{-1}{2m+1}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,R}(\eta',\tau')d\eta' d\tau'$
$\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m+1}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$	$\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m+1}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$
$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{1}{2m+1}}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$	$\int_{\frac{2m\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \int_{\frac{1}{2m+1}}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} \frac{\eta'}{\eta}\frac{\tau'}{1-\tau'} F_{2,m,L}(\eta',\tau')d\eta' d\tau'$

4.2.4 Finite Sphere Results

Although the finite sphere dimensionless integral method was never successfully implemented numerically, the first two collisions were calculated analytically. At early mean free times, or for a sphere with a very small radius with respect to the neutron speed, we would expect the early collisions to dominate the total flux. To verify this, we plot the finite sphere total flux for a sphere of radius $b = 1$, and compare it to the infinite medium flux at early mean free times, and the PARTISN results at later mean free times.

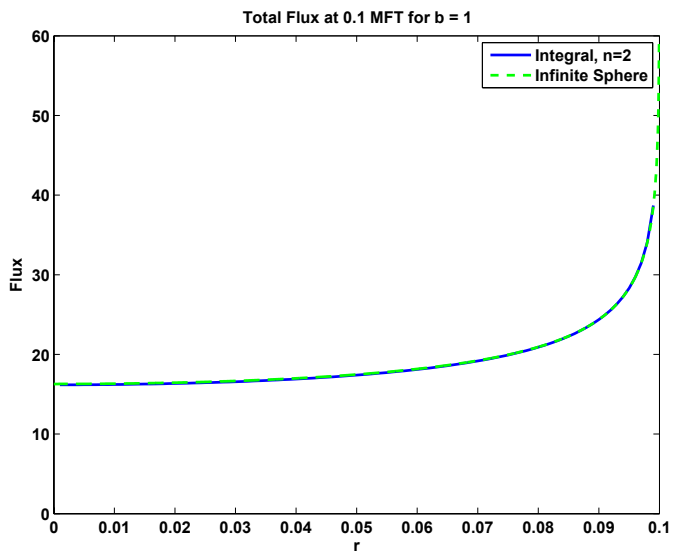

Fig. 4.30.— Total Flux at 0.1 MFT in a Sphere of Radius 1 MFP

At 0.1 MFT, the infinite sphere and the finite sphere utilizing two collisions match very well, indicating that the two first collisions are dominating the flux. At 0.5 MFTs, we see that the finite sphere result has already started to diverge from the

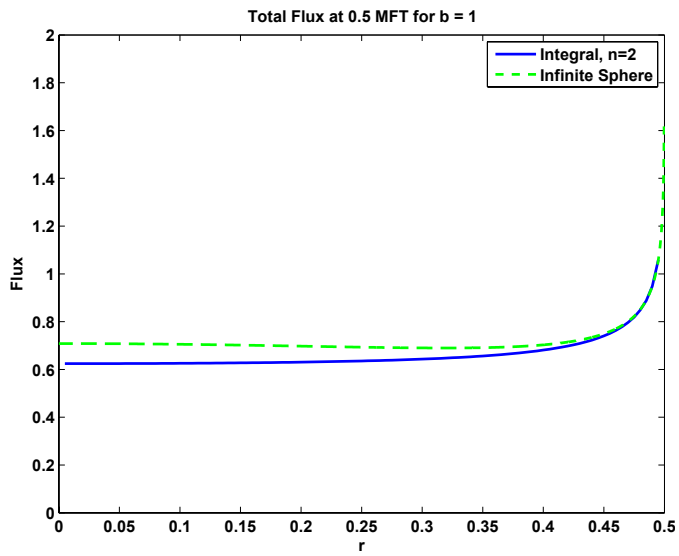

Fig. 4.31.— Total Flux at 0.5 MFT in a Sphere of Radius 1 MFP

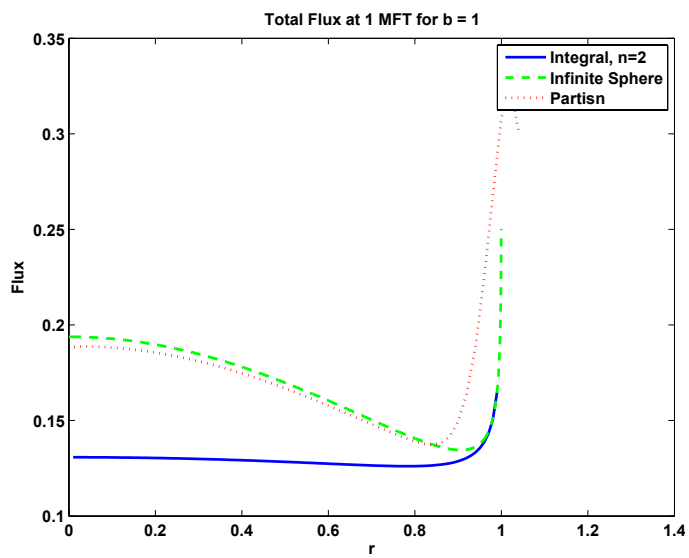

Fig. 4.32.— Total Flux at 1 MFT in a Sphere of Radius 1 MFP

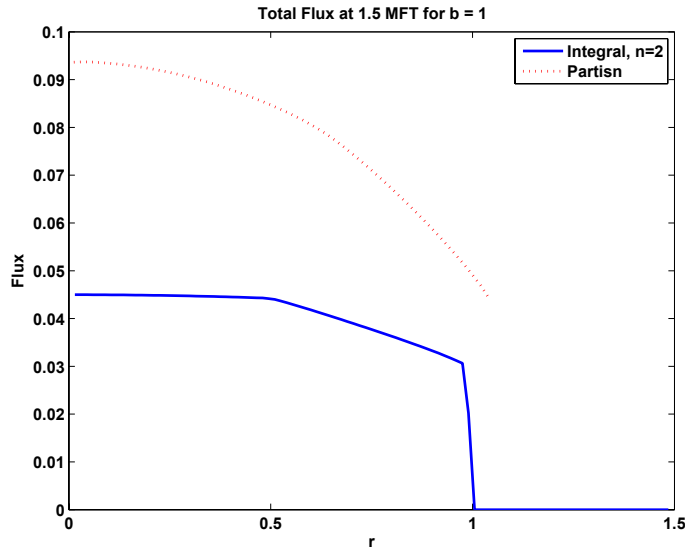

Fig. 4.33.— Total Flux at 1.5 MFT in a Sphere of Radius 1 MFP

infinite medium result, indicating that some of the later collisions are now affecting the total infinite sphere flux. By 1 MFT, the flux from two collisions is about 30% smaller than the infinite sphere flux, indicating that more collisions are necessary to obtain an accurate total neutron flux.

4.3 Discussion

The above sections showed how to expand the dimensionless integral transport method to incorporate finite media, and showed some results for the total flux. This section will discuss the limitations and possible applications of this method.

The finite slab total flux was compared to the total flux as calculated by PARTISN for a pulsed source in space and time. This comparison showed that the total flux from the integral transport method and the PARTISN results matched very well for early mean free times, but that the results diverged at later mean free times. The

total flux from the integral transport method was also compared with a flux calculated from only a few collisions. This comparison showed that at early mean free times, the first several collisions dominate the total flux results.

The finite sphere flux was calculated using only two collisions, as the method was never successfully implemented numerically. Numerical calculations for the third collided flux gave a negative flux value. The flux was compared to PARTISN results for a pulsed source in space and time, as well as the infinite sphere results. Comparing the infinite sphere results to the PARTISN results showed that PARTISN had trouble resolving the singularity in the flux at the wavefront. At later mean free times, after the singularity was no longer as pronounced, the infinite sphere and PARTISN flux results matched very well. Comparing the infinite sphere flux to the finite sphere flux showed good agreement before 1 MFT. This indicates again that at early mean free times the total flux is dominated by the first several collided fluxes. However, by 1 MFT, the finite sphere flux from two collisions had diverged from the infinite sphere results, indicating that more collisions are necessary after this time.

Also noted above is the fact that the dimensionless integral method coupled with the Neumann series method becomes unwieldy as the number of collisions increases. This is due to the fact that as the number of collisions increases, the total number of integrals that must be evaluated numerically also increases. This fact, coupled with the method's failure at accurately calculating the total flux at later mean free times, leads to the conclusion that the method is not appropriate for problems in which neutrons will suffer many collisions, or problems that may need results at late mean free times.

However, the method was successful at calculating the first several collisions,

numerically in the finite slab case, and analytically in the finite sphere case. This means that the method still holds potential for being incorporated into a radiation-hydrodynamics code. As was discussed in Section 1.4, the majority of neutrons born in an Inertial Confinement Fusion implosion will suffer few, if any, collisions before escaping the target. This was a prime motivating factor for choosing a Multiple Collision approach to evaluating the neutron transport equation: only a few collisions would need to be calculated to obtain an accurate total flux.

Chapter 5

Conclusions

We have introduced a new method for solving the one-dimensional, one-speed, time-dependent neutron transport equation for homogenous media. This method was applied to both infinite and finite media. This method involved utilizing the Multiple Collisions method in conjunction with dimensionless variables to solve the integral form of the transport equation. While the integral form of the neutron transport equation has been previously used to solve a variety of transport problems in both finite and infinite media [22], this is the first time that the integral form of the equation has been coupled with a dimensionless integration space.

The infinite media results were compared to benchmark problems previously published in the literature [11, 22]. The benchmark problem was for a pulsed source in space and time, in both infinite slabs and spheres. The media were purely scattering with a scattering cross section of one, and the neutron speed was set to one. For both the infinite slab and infinite sphere formulations, there was excellent agreement between the dimensionless integral transport method presented and the previous methods.

The finite media chapter showed how to expand the dimensionless integral formulation to incorporate either a finite slab or a finite sphere. Since there are no benchmark problems in the literature for a finite slab or sphere with a pulsed source in space or time, the finite media results were compared to PARTISN. The PARTISN runs approximated a pulsed source in time using a source of large relative amplitude that went to zero after a very short amount of time. The localized nature of the source was approximated by locating the source within a cell of very small size.

For the finite slab case, the PARTISN results were compared to both the integral transport results for a full run of 100 collisions and for a total flux calculated with only three collisions. For the sphere case, the PARTISN results were compared to both the infinite medium flux at early times, and the finite flux calculated with only two collisions.

For the slab case, it was found that at early mean free times, there was very good agreement between the total flux calculated using three collisions and the total flux calculated using 100 collisions. This led us to the conclusion that at early mean free times, a method that only calculates a few collisions could be adequate. At later mean free times, the PARTISN results and the integral transport method results diverged. Assuming that the problem lay in the integral transport method would lead us to conclude that the method is not well suited to calculate the total flux in situations where the effects of the later collisions dominate. A further limitation of the neutron transport method, in which ever more integrals must be numerically evaluated for each successive collision, reinforces this conclusion.

For the finite sphere case, the infinite sphere flux and the finite sphere flux calculated from the first two collisions were compared at early mean free times. There

was good agreement between the infinite medium results and the finite medium results calculated from the first two collisions at mean free times less than one. This again bodes well for using this method in specific applications where only a few collisions need be calculated, or where the total flux is only needed for a few time steps, such as ICF. However, to make this method appropriate for radiation-hydrodynamics codes, several features would need to be added to the neutron transport method. A 14.1 MeV neutron will suffer only glancing collisions with the DT fuel, therefore, anisotropic scattering would need to be incorporated. Additionally, the composition of the target is not homogeneous, and heterogeneous media would need to be incorporated. The time-dependent, heterogeneous media integration is derived in Appendix E.

Given the unwieldy nature of the Multiple Collision Method coupled with the dimensionless integral method, future work in this area might best be pursued along a course that does not include the Multiple Collision Method, but still retains the novel dimensionless integral approach. By solving for the total flux, rather than a large number of collided fluxes, some of the issues encountered in this method, namely the ever increasing number of integrals that must be evaluated as the number of collisions increases, would not be encountered. But by keeping the dimensionless variables, a major limitation of previous work in integral neutron transport, namely the increasing size of the time domain, is not encountered.

Appendix A

Subtraction of Singularity

Consider an integral with a singularity at some point x_0. The idea of subtraction of singularity is to extract the singular part of the integrand. This is done by subtracting, from the integrand, an expression integrable in closed form, which eliminates the singularity and yields an integral which can be evaluated numerically [25]. For instance, consider

$$I(q) = \int_0^q \frac{e^{-x}dx}{1-x} \quad 0 \leq q \leq 1. \tag{A.1}$$

The integrand has a singularity at $x = 1$, and $I(1) = \infty$. However, we can subtract the singularity in the following manner:

$$\begin{aligned} I(q) = \int_0^q \frac{e^{-x}dx}{1-x} &= e^{-1}\int_0^q \frac{dx}{1-x} + \int_0^q \left(\frac{e^{-x}}{1-x} - \frac{e^{-1}}{1-x}\right) dx \\ &= -\,e^{-1}\ln(1-q) + \int_0^q \left(\frac{e^{-x}}{1-x} - \frac{e^{-1}}{1-x}\right) dx. \end{aligned} \tag{A.2}$$

The first integral is evaluated analytically. The second integral has no singularity,

since it equals zero at $x = 1$, and can be evaluated numerically over the whole range.

Now consider the infinite slab geometry case. The expression for the nth shape factor, in terms of the $(n-1)^{th}$ shape factor is:

$$F_n(\eta) = \frac{n}{2}\left[\int_{-1}^{\eta} K_{A,n}(\eta,\eta')F_{n-1}(\eta')d\eta' + \int_{\eta}^{1} K_{B,n}(\eta,\eta')F_{n-1}(\eta')d\eta'\right] \tag{A.3}$$

where $K_{A,n}(\eta,\eta')$ and $K_{B,n}(\eta,\eta')$ are the kernels, and are calculated as:

$$K_{A,n}(\eta,\eta') = \int_{0}^{\frac{1-\eta}{1-\eta'}} \frac{(\tau')^{n-1}}{1-\tau'}d\tau' = -\ln\left(1-\frac{1-\eta}{1-\eta'}\right) - \sum_{i=2}^{n}\frac{1}{i-1}\left(\frac{1-\eta}{1-\eta'}\right)^{i-1} \tag{A.4}$$

and

$$K_{B,n}(\eta,\eta') = \int_{0}^{\frac{1+\eta}{1+\eta'}} \frac{(\tau')^{n-1}}{1-\tau'}d\tau' = -\ln\left(1-\frac{1+\eta}{1+\eta'}\right) - \sum_{i=2}^{n}\frac{1}{i-1}\left(\frac{1+\eta}{1+\eta'}\right)^{i-1}. \tag{A.5}$$

The kernels have a singularity at the point $\eta' = \eta$. To apply the subtraction of singularity method, we need to ensure that the form of the integrand that is extracted is integrable in closed form, and that the integral calculated numerically is equal to zero at $\eta' = \eta$. Rewriting equation A.3 as

$$\begin{aligned} F_n(\eta) =& \frac{n}{2}\left\{F_{n-1}(\eta)\int_{-1}^{\eta} K_{A,n}(\eta,\eta')d\eta' + \int_{-1}^{\eta} K_{A,n}(\eta,\eta')\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta'\right. \\ &\left. + F_{n-1}(\eta)\int_{\eta}^{1} K_{B,n}(\eta,\eta')d\eta' + \int_{\eta}^{1} K_{B,n}(\eta,\eta')\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta'\right\} \end{aligned} \tag{A.6}$$

fulfills the above requirements. The first and third integrals, which contain a singularity at $\eta' = \eta$, can be evaluated analytically. Meanwhile, the second and fourth integrals, which cannot be evaluated analytically, are equal to zero at the point $\eta' = \eta$.

Appendix B

Derivation of Shape Factor Equations at $\eta = 0$ for Spherical Medium

This appendix gives the detailed derivation of the shape factor equations at the point $\eta = 0$ for the infinite spherical medium with isotropic scattering problem.

B.1 Infinite Sphere Shape Factors

We begin the discussion with the equation for the nth collided shape factor, equation 3.57 in section 3.2.1:

$$\begin{aligned} F_n(\eta) =& \frac{n}{2}\Bigg[\int_0^{\eta} K_{A,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^{\eta} K_{A,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta' \\ &+ \int_{\eta}^{1} K_{B,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_{\eta}^{1} K_{B,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta' \\ &- \int_0^{1} K_{C,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^{1} K_{C,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta'\Bigg] \end{aligned} \tag{B.1}$$

where the kernels are given by

$$K_{A,n}(\eta,\eta') = \int_0^{\frac{1-\eta}{1-\eta'}} \frac{(\tau')^{n-2}}{1-\tau'} d\tau' = \ln\left(1 - \frac{1-\eta}{1-\eta'}\right) - \sum_{i=3}^{n} \frac{1}{i-2}\left(\frac{1-\eta}{1-\eta'}\right)^{i-2}, \tag{B.2}$$

$$K_{B,n}(\eta,\eta') = \int_0^{\frac{1+\eta}{1+\eta'}} \frac{(\tau')^{n-2}}{1-\tau'} d\tau' = \ln\left(1 - \frac{1+\eta}{1+\eta'}\right) - \sum_{i=3}^{n} \frac{1}{i-2}\left(\frac{1+\eta}{1+\eta'}\right)^{i-2}, \tag{B.3}$$

and

$$K_{C,n}(\eta,\eta') = \int_0^{\frac{1-\eta}{1+\eta'}} \frac{(\tau')^{n-2}}{1-\tau'} d\tau' = \ln\left(1 - \frac{1-\eta}{1+\eta'}\right) - \sum_{i=3}^{n} \frac{1}{i-2}\left(\frac{1-\eta}{1+\eta'}\right)^{i-2}, \tag{B.4}$$

and where the integrals of the kernels are given by

$$\int_0^{\eta} K_{A,3}(\eta,\eta')\eta' d\eta' = \frac{\eta}{2} - \frac{\eta^2}{2} + \frac{(1-\eta)^2}{2}\ln(1-\eta) - \frac{\eta^2}{2}\ln(\eta), \tag{B.5}$$

$$\int_{\eta}^{1} K_{B,3}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \frac{\eta^2}{2} - \frac{(1+\eta)^2}{2}\ln(1+\eta) - \frac{(1-\eta^2)}{2}\ln(1-\eta) + (1+\eta)\ln 2, \tag{B.6}$$

$$\int_0^{1} K_{C,3}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \frac{\eta}{2} - \frac{\eta^2}{2}\ln(\eta) - \frac{(1-\eta^2)}{2}\ln(1+\eta) + (1-\eta)\ln 2, \tag{B.7}$$

for $n = 3$, and

$$\int_0^\eta K_{A,n}(\eta,\eta')\eta' d\eta' = \left(1-\eta^2\right)\ln(1-\eta) - \frac{\eta^2}{2}\ln(\eta) + \sum_{i=5}^{n}\left[-\frac{(1-\eta)}{(i-2)(i-3)} + \frac{(1-\eta)^2}{(i-2)(i-4)} - \frac{(1-\eta)^{i-2}}{(i-2)(i-3)(i-4)}\right], \tag{B.8}$$

$$\int_\eta^1 K_{B,n}(\eta,\eta')\eta' d\eta' = \frac{\eta^2}{4} - \frac{1}{4} - \frac{(1-\eta^2)}{2}\ln(1-\eta) + \frac{(1-\eta^2)}{2}\ln 2 + \sum_{i=5}^{n}\left[\frac{(1+\eta)}{(i-2)(i-3)} - \frac{(1+\eta)^2}{(i-2)(i-4)} + \frac{(1+\eta)^{i-2}}{(i-3)(i-4)2^{i-3}}\right], \tag{B.9}$$

$$\int_\eta^1 K_{C,n}(\eta,\eta')\eta' d\eta' = \frac{\eta^2}{4} - \frac{1}{4} - \frac{(1-\eta^2)}{2}\ln(1+\eta) + \frac{(1-\eta^2)}{2}\ln 2 + \sum_{i=5}^{n}\left[\frac{(1-\eta)^{i-2}}{i-2}\left(\frac{2^{4-i}-1}{i-4} - \frac{2^{3-i}}{i-3}\right)\right] \tag{B.10}$$

for $n \geq 4$.

The above derived expressions are not enough to numerically solve the spherical infinite medium problem because there is a singularity at $\eta = 0$. Therefore, the limit of F_n as $\eta \to 0$ must be computed. Plugging in $\eta = 0$ for the $n = 1$ shape factor results in $\frac{0}{0}$, or indeterminate. Applying L'Hospital's rule gives:

$$F_1(0) = \lim_{\eta\to 0}\left[\frac{1}{2\pi\eta}\left(\frac{1+\eta}{1-\eta}\right)\right] = \frac{\frac{d}{d\eta}\ln\left(\frac{1+\eta}{1-\eta}\right)}{2\pi\frac{d}{d\eta}\eta} = \frac{1}{\pi}. \tag{B.11}$$

Again, with the $n = 2$ shape factor, plugging in $\eta = 0$ results in the indeterminate relation. Applying L'Hospital's rule yields

$$
\begin{aligned}
F_2(0) =& \lim_{\eta\to 0}\left\{\frac{1}{2\pi\eta}\left[\pi^2\eta + \frac{3}{2}(1-\eta)\ln\left(\frac{1-\eta}{2}\right)^2 - \frac{3}{2}(1+\eta)\ln\left(\frac{1+\eta}{2}\right)^2\right.\right.\\
&\left.\left. + 3(1-\eta)\mathrm{Li}_2\left(\frac{1-\eta}{2}\right) - 3(1+\eta)\mathrm{Li}_2\left(\frac{1+\eta}{2}\right)\right]\right\}\\
=& \frac{1}{2\pi\frac{d}{d\eta}\eta}\frac{d}{d\eta}\left[\pi^2\eta + \frac{3}{2}(1-\eta)\ln\left(\frac{1-\eta}{2}\right)^2 \frac{3}{2}(1+\eta)\ln\left(\frac{1+\eta}{2}\right)^2\right.\\
&\left. + 3(1-\eta)\mathrm{Li}_2\left(\frac{1-\eta}{2}\right) - 3(1+\eta)\mathrm{Li}_2\left(\frac{1+\eta}{2}\right)\right] = \frac{\pi}{4}.
\end{aligned}
\tag{B.12}
$$

Next, the equation for the shape factors for $n \geq 3$ at $\eta = 0$ will be determined. The derivation begins by taking the limit of equation B.1:

$$
\begin{aligned}
&\lim_{\eta\to 0}\{F_n(\eta)\\
&=\frac{n}{2}\left[\int_0^\eta K_{A,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^\eta K_{A,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta'\right.\\
&+\int_\eta^1 K_{B,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_\eta^1 K_{B,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta'\\
&\left.-\int_0^1 K_{C,n}(\eta,\eta')\frac{\eta'}{\eta}\left[F_{n-1}(\eta') - F_{n-1}(\eta)\right]d\eta' + F_{n-1}(\eta)\int_0^1 K_{C,n}(\eta,\eta')\frac{\eta'}{\eta}d\eta'\right]\Bigg\}.
\end{aligned}
\tag{B.13}
$$

From equation B.13, it is apparent that the first and second integrals are equal to zero in the limit at $\eta \to 0$. Therefore, in the limit that $\eta \to 0$, equation B.13 simplifies to

$$
\begin{aligned}
&F_n(0)\\
&=\frac{n}{2}\left\{\int_0^1 [F_{n-1}(\eta')-F_{n-1}(0)]\lim_{\eta\to 0}\frac{K_{B,n}(\eta,\eta')}{\eta}\eta' d\eta' + F_{n-1}(0)\lim_{\eta\to 0}\int_\eta^1 \frac{K_{B,n}(\eta,\eta')}{\eta}\eta' d\eta'\right.\\
&\left.-\int_0^1 [F_{n-1}(\eta')-F_{n-1}(0)]\lim_{\eta\to 0}\frac{K_{C,n}(\eta,\eta')}{\eta}\eta' d\eta' + F_{n-1}(0)\lim_{\eta\to 0}\int_\eta^1 \frac{K_{C,n}(\eta,\eta')}{\eta}\eta' d\eta'\right\}.
\end{aligned}
\tag{B.14}
$$

For the specific case where $n = 3$, the kernels at $\eta = 0$ are

$$
K_{B,3}(0,\eta') = \ln(1+\eta') - \ln(\eta') - \frac{1}{1+\eta'} \tag{B.15}
$$

and

$$
K_{C,3}(0,\eta') = \ln(1+\eta') - \ln(\eta') - \frac{1}{1+\eta'}. \tag{B.16}
$$

At $\eta = 0$, the kernels are equal. Therefore, subtracting the third integral from the first integral in equation B.14 results in the indeterminate relation. Once again, L'Hospital's rule can be used and gives

$$
\begin{aligned}
&\int_0^1 [F_2(\eta')-F_2(0)]\,\eta'\lim_{\eta\to 0}\frac{[K_{B,3}(\eta,\eta')-K_{C,3}(\eta,\eta')]}{\eta}d\eta'\\
&=\int_0^1 [F_2(\eta')-F_2(0)]\,\eta'\lim_{\eta\to 0}\frac{\frac{d}{d\eta}[K_{B,3}(\eta,\eta')-K_{C,3}(\eta,\eta')]}{\frac{d}{d\eta}\eta}d\eta'\\
&=\int_0^1 [F_2(\eta')-F_2(0)]\,\eta'\frac{2}{(1+\eta')\eta'}d\eta'.
\end{aligned}
\tag{B.17}
$$

The integrated kernels in the limit that $\eta \to 0$, for $n = 3$ are

$$\lim_{\eta\to 0} \int_{\eta}^{1} K_{B,3}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \ln(2) \tag{B.18}$$

and

$$\lim_{\eta\to 0} \int_{0}^{1} K_{C,3}(\eta,\eta')\eta' d\eta' = -\frac{1}{2} + \ln(2). \tag{B.19}$$

Again, the integrals of the kernels are equal. Therefore, subtracting the fourth integral from the second integral in equation B.14 results in an indeterminate. Once again, L'Hospital's rule is used:

$$\begin{aligned} F_2(0) \lim_{\eta\to 0} \frac{1}{\eta} &\left[\int_{\eta}^{1} K_{B,3}(\eta,\eta')\eta' d\eta' - \int_{0}^{1} K_{C,3}(\eta,\eta')\eta' d\eta' \right] \\ =& F_2(0) \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[\int_{\eta}^{1} K_{B,3}(\eta,\eta')\eta' d\eta' - \int_{0}^{1} K_{C,3}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta} \\ =& 2F_2(0)\ln 2. \end{aligned} \tag{B.20}$$

Therefore, the equation for the $n = 3$ shape factor at $\eta = 0$ is

$$F_3(0) = \frac{3}{2}\left\{ 2F_2(0)\ln 2 + \int_{0}^{1} \left[F_2(\eta') - F_2(0)\right] \eta' \frac{2}{(1+\eta')\eta'} d\eta' \right\}. \tag{B.21}$$

These same steps can be applied to determine the equation to be solved for $F_4(0)$. Start again by examining the kernels for $n = 4$ when $\eta = 0$.

$$K_{B,4}(0,\eta') = \ln(1+\eta') - \ln(\eta') - \frac{1}{1+\eta'} - \frac{1}{(1+\eta')^2} \tag{B.22}$$

and

$$K_{C,4}(0,\eta') = \ln(1+\eta') - \ln(\eta') - \frac{1}{1+\eta'} - \frac{1}{(1+\eta')^2}. \tag{B.23}$$

Again, subtracting the third integral from the first integral results in the indeterminate relation. Applying L'Hospital's rule yields

$$\begin{aligned}
&\int_0^1 [F_3(\eta') - F_3(0)]\,\eta' \lim_{\eta\to 0} \frac{[K_{B,4}(\eta,\eta') - K_{C,4}(\eta,\eta')]}{\eta} d\eta' \\
&= \int_0^1 [F_3(\eta') - F_3(0)]\,\eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}[K_{B,4}(\eta,\eta') - K_{C,4}(\eta,\eta')]}{\frac{d}{d\eta}\eta} d\eta' \\
&= \int_0^1 [F_3(\eta') - F_3(0)]\,\eta' \left[\frac{2}{\eta'} - \frac{2}{1+\eta'} - \frac{2}{(1+\eta')^2}\right] d\eta'.
\end{aligned} \tag{B.24}$$

Taking the limits of the integrals of the kernels as $\eta \to 0$,

$$\lim_{\eta\to 0} \int_\eta^1 K_{B,4}(\eta,\eta')\eta' d\eta' = -\frac{1}{4} + \frac{\ln(2)}{2} \tag{B.25}$$

and

$$\lim_{\eta\to 0}\int_0^1 K_{C,4}(\eta,\eta')\eta' d\eta' = -\frac{1}{4}+\frac{\ln(2)}{2}. \tag{B.26}$$

As in the $n=3$ case, the integrals of the kernels are equal to each other at $\eta=0$. Subtracting the fourth integral from the second integral again results in the indeterminate relation. Once again, L'Hospital's rule may be applied, and results in

$$\begin{aligned} F_3(0)\lim_{\eta\to 0}\frac{1}{\eta}&\left[\int_\eta^1 K_{B,4}(\eta,\eta')\eta' d\eta' - \int_0^1 K_{C,4}(\eta,\eta')\eta' d\eta'\right] \\ =&F_3(0)\lim_{\eta\to 0}\frac{\frac{d}{d\eta}\left[\int_\eta^1 K_{B,4}(\eta,\eta')\eta' d\eta' - \int_0^1 K_{C,4}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta} \\ =&F_3(0). \end{aligned} \tag{B.27}$$

Finally, the equation to be solved for $n=4$ is

$$F_4(0) = 2\left\{F_3(0) + 2\int_0^1 \left[F_3(\eta')-F_3(0)\right]\eta'\left[\frac{1}{\eta'}-\frac{1}{1+\eta'}-\frac{1}{(1+\eta')^2}\right]d\eta'\right\}. \tag{B.28}$$

The same procedure can be applied to find the equation that must be solved at $\eta=0$ for $n\geq 5$. First, look at the expressions for the kernels at $\eta=0$:

$$K_{B,n}(0,\eta') = \ln(1+\eta')-\ln(\eta')-\sum_{i=3}^{n}\frac{1}{(i-2)\,(1+\eta')^{i-2}} \tag{B.29}$$

and

$$K_{C,n}(0,\eta') = \ln(1+\eta') - \ln(\eta') - \sum_{i=3}^{n} \frac{1}{(i-2)\,(1+\eta')^{i-2}}. \tag{B.30}$$

As in the cases with $n = 3$ and $n = 4$, the kernels are equal to each other at $\eta = 0$. Subtracting the third integral from the first in equation B.14 results in the indeterminate relation. Once again, apply L'Hospital's rule to obtain

$$\begin{aligned}
&\int_0^1 [F_{n-1}(\eta') - F_{n-1}(0)]\,\eta' \lim_{\eta\to 0} \frac{[K_{B,n}(\eta,\eta') - K_{C,n}(\eta,\eta')]}{\eta} d\eta' \\
&= \int_0^1 [F_{n-1}(\eta') - F_{n-1}(0)]\,\eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}[K_{B,n}(\eta,\eta') - K_{C,n}(\eta,\eta')]}{\frac{d}{d\eta}\eta} d\eta' \\
&= 2\int_0^1 [F_{n-1}(\eta') - F_{n-1}(0)]\,\eta' \left[\frac{1}{\eta'} - \sum_{i=3}^{n} \frac{1}{(i-2)\,(1+\eta')^{i-2}}\right] d\eta'.
\end{aligned} \tag{B.31}$$

The final step is to examine the integrals of the kernels at $\eta = 0$.

$$\lim_{\eta\to 0} \int_{\eta}^{1} K_{B,n}(\eta,\eta')\eta' d\eta' = -\frac{1}{4} + \frac{\ln(2)}{2} + \sum_{i=5}^{n} \left[\frac{i-2-2^{i-3}}{(i-2)(i-3)(i-4)2^{i-3}}\right] \tag{B.32}$$

and

$$\lim_{\eta\to 0} \int_{0}^{1} K_{C,n}(\eta,\eta')\eta' d\eta' = -\frac{1}{4} + \frac{\ln(2)}{2} + \sum_{i=5}^{n} \left[\frac{i-2-2^{i-3}}{(i-2)(i-3)(i-4)2^{i-3}}\right]. \tag{B.33}$$

Once again, the integrals of the kernels are equal to each other in the limit that $\eta = 0$.

Subtracting the fourth integral from the second in equation B.14 again results in the indeterminate relation. Applying L'Hospital's rule yields

$$\begin{aligned} F_{n-1}(0) \lim_{\eta\to 0}\frac{1}{\eta}&\left[\int_{\eta}^{1} K_{B,n}(\eta,\eta')\eta' d\eta' - \int_{0}^{1} K_{C,n}(\eta,\eta')\eta' d\eta'\right] \\ =&F_{n-1}(0)\lim_{\eta\to 0}\frac{\frac{d}{d\eta}\left[\int_{\eta}^{1} K_{B,n}(\eta,\eta')\eta' d\eta' - \int_{0}^{1} K_{C,n}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta} \\ =&F_{n-1}(0)\sum_{i-5}^{n}\left[\frac{(i-2)\left(16+2^{i}(i-5)\right)}{(i-3)(i-4)2^{i}}\right]. \end{aligned} \tag{B.34}$$

Finally, the equation that must be solved at $\eta = 0$ for the case of $n \geq 5$ is

$$\begin{aligned} F_n(0) =&\frac{n}{2}\left\{F_{n-1}(0)\sum_{i-5}^{n}\left[\frac{(i-2)\left(16+2^{i}(i-5)\right)}{(i-3)(i-4)2^{i}}\right]\right. \\ &\left. + 2\int_{0}^{1}\left[F_{n-1}(\eta') - F_{n-1}(0)\right]\eta'\left[\frac{1}{\eta'} - \sum_{i=3}^{n}\frac{1}{(i-2)\left(1+\eta'\right)^{i-2}}\right]d\eta'\right\}. \end{aligned} \tag{B.35}$$

B.2 Finite Sphere Shape Factors

The finite sphere shape factors equations were derived using two formulations, one using a reflective boundary condition at $\eta = 0$, and one without the reflective boundary condition. The following sections show how to derive the equations at $\eta = 0$ for the specific case of $n = 3$ in the formulation utilizing the reflective boundary condition. This same method may be applied to obtain the equations at $\eta = 0$ for the formulation not utilizing the reflective boundary condition.

Recall from Section 4.2 that the equations for the $n = 3$ finite sphere shape factor from the various $n = 2$ sources are

$$F_{3,\mathrm{from}F_{2,inf}}(\eta,\eta_b) = \int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\tau'}{(1-\tau')}\frac{\eta'}{\eta}F_{2,inf}(\eta')d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')}\frac{\eta'}{\eta}F_{2,inf}(\eta')d\tau' d\eta'$$
$$- \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')}\frac{\eta'}{\eta}F_{2,inf}(\eta')d\tau' d\eta'$$
$$- H(1-2\eta_b+\eta)\int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')}\frac{\eta'}{\eta}F_{2,inf}(\eta')d\tau' d\eta'$$
$$+ H(1-2\eta_b-\eta)\int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')}\frac{\eta'}{\eta}F_{2,inf}(\eta')d\tau' d\eta'. \tag{B.36}$$

$$
\begin{aligned}
F_{3,\mathrm{from}F_{2,m,L}}(\eta,\eta_b) = H(1-\eta-2m\eta_b) &\Bigg[\int\limits_0^{\eta}\int\limits_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
&+ \int\limits_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}}\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \Bigg] \\
&- H(1+\eta-2m\eta_b) \int\limits_0^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}}\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
&- H(1+\eta-2(m+1)\eta_b) \Bigg[\int\limits_{\frac{1}{2m+1}}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}}\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
&+ \int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{2m+1}}\int\limits_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \Bigg] \\
&+ H(1-\eta-2(m+1)\eta_b) \Bigg[\int\limits_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}}\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \\
&+ \int\limits_{\frac{1}{2m+1}}^{\frac{1-\eta-2m\eta_b}{1-\eta+2m\eta_b}}\int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,L}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta' \Bigg] .
\end{aligned}
\tag{B.37}
$$

$$F_{3,\text{from}F_{2,m,R}}(\eta,\eta_b) = H(1+\eta-2m\eta_b)\left[\int_{\eta}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'\right.$$

$$-\int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1+\eta'}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'$$

$$\left.+\int_{0}^{\eta}\int_{\frac{\eta_b(1-\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{n-1,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'\right]$$

$$+H(1-\eta-2m\eta_b)\left[-\int_{0}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'\right.$$

$$\left.+\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m-1}}\int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \frac{\eta'}{\eta}\left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{F_{2,m,R}(\eta',\eta_b')}{\eta_b'-\eta_b} d\eta_b' d\eta'\right] \tag{B.38}$$

where $m = 1$ or 2.

The first three integrals in equation B.36 are identical the infinite sphere integrals, and the singularity at $\eta = 0$ can be handled as described in Section B.1. The next two integrals have the following kernels:

$$K_{D,3}(\eta,\eta') = \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')} d\tau' = \ln\left(\frac{(1+\eta')(\eta'-\eta_b)}{\eta'(\eta'-\eta)}+\right)\frac{\eta_b}{\eta'} - \frac{1+\eta}{1+\eta'} \tag{B.39}$$

and

$$K_{E,3}(\eta,\eta') = \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')} d\tau' = \ln\left(\frac{(1+\eta')(\eta'-\eta_b)}{\eta'(\eta'+\eta)} + \right)\frac{\eta_b}{\eta'} - \frac{1-\eta}{1+\eta'}. \tag{B.40}$$

At $\eta = 0$, the kernels are:

$$K_{D,3}(0,\eta') = \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')} d\tau' = \ln\left(\frac{(1+\eta')(\eta'-\eta_b)}{\eta'^2} + \right)\frac{\eta_b}{\eta'} - \frac{1}{1+\eta'} \tag{B.41}$$

and

$$K_{E,3}(0,\eta') = \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\tau'}{(1-\tau')} d\tau' = \ln\left(\frac{(1+\eta')(\eta'-\eta_b)}{\eta'^2} + \right)\frac{\eta_b}{\eta'} - \frac{1}{1+\eta'}. \tag{B.42}$$

Additionally, at $\eta = 0$, the two Heaviside functions multiplying these integrals are equal. Subtracting the two integral from each other at $\eta = 0$ would result in $\frac{0}{0}$, allowing us to use L'Hospital's rule. Taking into account that the kernels are singular at $\eta' = \eta_b$ and that Subtraction of Singularity must be used on these integrals, we obtain the following at $\eta = 0$:

$$\begin{aligned}
&\int_{\frac{\eta_b}{1-\eta_b}}^{1} \left[F_{2,inf}(\eta') - F_{2,inf}(\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\left[K_{E,3}(\eta,\eta') - K_{D,3}(\eta,\eta')\right]}{\eta} d\eta' \\
&= \int_{\frac{\eta_b}{1-\eta_b}}^{1} \left[F_{2,inf}(\eta') - F_{2,inf}(\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[K_{E,3}(\eta,\eta') - K_{D,3}(\eta,\eta')\right]}{\frac{d}{d\eta}\eta} d\eta' \\
&= \int_{\frac{\eta_b}{1-\eta_b}}^{1} \left[F_{2,inf}(\eta') - F_{2,inf}(\eta_b)\right] \eta' \left(\frac{-2}{\eta'(1+\eta')}\right) d\eta'.
\end{aligned} \tag{B.43}$$

The integrals of the $K_{D,3}(\eta,\eta')$ and $K_{E,3}(\eta,\eta')$ at $\eta = 0$ are:

$$\lim_{\eta\to 0} \int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} K_{D,3}(\eta,\eta')\eta' d\eta' = \frac{-1}{2} + \eta_b + \ln(2) + (1-\eta_b^2)\ln(1-\eta_b) + \eta_b^2 \ln(\eta_b) \quad \text{(B.44)}$$

and

$$\lim_{\eta\to 0} \int\limits_{\frac{\eta_b}{1-\eta_b-\eta}}^{1} K_{E,3}(\eta,\eta')\eta' d\eta' = \frac{-1}{2} + \eta_b + \ln(2) + (1-\eta_b^2)\ln(1-\eta_b) + \eta_b^2 \ln(\eta_b). \quad \text{(B.45)}$$

Combining these two equations would give $\frac{0}{0}$, so we will use L'Hospital's rule.

$$\begin{aligned} F_{2,inf}(\eta_b) \lim_{\eta\to 0} \frac{1}{\eta} & \left[\int\limits_{\frac{\eta_b}{1-\eta_b-\eta}}^{1} K_{E,3}(\eta,\eta')\eta' d\eta' - \int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} K_{D,3}(\eta,\eta')\eta' d\eta' \right] \\ &= F_{2,inf}(\eta_b) \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[\int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1} K_{E,3}(\eta,\eta')\eta' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1} K_{D,3}(\eta,\eta')\eta' d\eta' \right]}{\frac{d}{d\eta}\eta} \\ &= -2F_{2,inf}(\eta_b). \end{aligned} \quad \text{(B.46)}$$

Turning now to equation B.37, we note that the first integral will be equal to zero when $\eta = 0$. The kernels of the second and third integrals are:

$$Kt3_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'-\eta)}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1-\eta'} \tag{B.47}$$

and

$$Kt5_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')(\eta'+\eta)}\right) - \frac{1-\eta}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}. \tag{B.48}$$

At $\eta = 0$, the kernels equal:

$$Kt3_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')\eta'}\right) - \frac{1}{1+\eta'} + \frac{2m\eta_b}{1-\eta'} \tag{B.49}$$

and

$$Kt5_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1-\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(1-2m\eta_b-\eta')(1+\eta')}{(1-\eta')\eta'}\right) - \frac{1}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}. \tag{B.50}$$

The Heaviside functions multiplying these two integrals are equal at $\eta = 0$. Subtracting the kernels will result in $\frac{0}{0}$, so we can use L'Hospital's rule:

$$
\begin{aligned}
&\int_{0}^{\frac{1-2m\eta_b}{1+2m\eta_b}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta,\eta_b)\right]\eta' \lim_{\eta\to 0} \frac{\left[Kt3_{n=3}(\eta,\eta') - Kt5_{n=3}(\eta,\eta')\right]}{\eta} d\eta' \\
&= \int_{0}^{\frac{1-2m\eta_b}{1+2m\eta_b}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta,\eta_b)\right]\eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[Kt3_{n=3}(\eta,\eta') - Kt5_{n=3}(\eta,\eta')\right]}{\frac{d}{d\eta}\eta} d\eta' \\
&= \int_{0}^{\frac{1-2m\eta_b}{1+2m\eta_b}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta,\eta_b)\right]\eta' \left(\frac{2}{\eta'(1+\eta')}\right) d\eta'.
\end{aligned}
\tag{B.51}
$$

Note the singularity at $\eta' = \eta$. The integrals of the kernels at $\eta = 0$ are:

$$
\begin{aligned}
\lim_{\eta\to 0} \int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt3_{n=3}(\eta,\eta')\eta' d\eta' = \frac{-1}{2} + m\eta_b + (1-2m\eta_b)\ln(2) - 2(m\eta_b)^2\ln(2m\eta_b) \\
- \frac{1-(2m\eta_b)^2}{2}\ln(1+2m\eta_b)
\end{aligned}
\tag{B.52}
$$

and

$$
\begin{aligned}
\lim_{\eta\to 0} \int_{0}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt5_{n=3}(\eta,\eta')\eta' d\eta' = \frac{-1}{2} + m\eta_b + (1-2m\eta_b)\ln(2) - 2(m\eta_b)^2\ln(2m\eta_b) \\
- \frac{1-(2m\eta_b)^2}{2}\ln(1+2m\eta_b).
\end{aligned}
\tag{B.53}
$$

Combining these two equations would give $\frac{0}{0}$, so we will use L'Hospital's rule:

$$
\begin{aligned}
&F_{2,m,L}(\eta,\eta_b)\lim_{\eta\to 0}\frac{1}{\eta}\left[\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt3_{n=3}(\eta,\eta')\eta' d\eta' - \int_{0}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt5_{n=3}(\eta,\eta')\eta' d\eta'\right] \\
&= F_{2,m,L}(\eta,\eta_b)\lim_{\eta\to 0}\frac{\frac{d}{d\eta}\left[\int_{\eta}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt3_{n=3}(\eta,\eta')\eta' d\eta' - \int_{0}^{\frac{1+\eta-2m\eta_b}{1+\eta+2m\eta_b}} Kt5_{n=3}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta} \\
&= 2F_{2,m,L}(\eta,\eta_b)\ln\left(\frac{2}{1+2m\eta_b}\right).
\end{aligned}
\tag{B.54}
$$

We turn now to the fourth and seventh integrals in equation B.37. The η_b' limits of integration are identical to the integrals we just examined, so we know we can use L'Hospital's rule. These integrals do not contain any singularities, so at $\eta = 0$, the fourth and seventh integrals combined become:

$$
\begin{aligned}
&\int_{\frac{1}{2m+1}}^{\frac{1-2m\eta_b}{1+2m\eta_b}} F_{2,m,L}(\eta',\eta_b)\eta' \lim_{\eta\to 0}\frac{[Kt5_{n=3}(\eta,\eta') - Kt3_{n=3}(\eta,\eta')]}{\eta} d\eta' \\
&= \int_{\frac{1}{2m+1}}^{\frac{1-2m\eta_b}{1+2m\eta_b}} F_{2,m,L}(\eta',\eta_b)\eta' \lim_{\eta\to 0}\frac{\frac{d}{d\eta}[Kt5_{n=3}(\eta,\eta') - Kt3_{n=3}(\eta,\eta')]}{\frac{d}{d\eta}\eta} d\eta' \\
&= \int_{\frac{1}{2m+1}}^{\frac{1-2m\eta_b}{1+2m\eta_b}} F_{2,m,L}(\eta',\eta_b)\eta' \left(\frac{-2}{\eta'(1+\eta')}\right) d\eta'.
\end{aligned}
\tag{B.55}
$$

We turn now to the final two integrals in equation B.37, the fifth and sixth integrals.

$$Kt8_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(\eta' - \eta_b)(1+\eta')}{(\eta' + \eta)\eta'}\right) - \frac{1+\eta}{1+\eta'} + \frac{\eta_b}{\eta'} \tag{B.56}$$

and

$$Kt7_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(\eta' - \eta_b)(1+\eta')}{(\eta' - \eta)\eta'}\right) - \frac{1+\eta}{1+\eta'} + \frac{\eta_b}{\eta'}. \tag{B.57}$$

At $\eta = 0$, the kernels equal:

$$Kt8_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(\eta' - \eta_b)(1+\eta')}{(\eta' - \eta)\eta'}\right) - \frac{1+\eta}{1+\eta'} + \frac{\eta_b}{\eta'} \tag{B.58}$$

and

$$Kt7_{n=3} = \int_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\eta'} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{1}{\eta_b' - \eta_b} d\eta_b' \\ = \ln\left(\frac{(\eta' - \eta_b)(1+\eta')}{\eta'^2}\right) - \frac{1}{1+\eta'} + \frac{2m\eta_b}{1-\eta'}. \tag{B.59}$$

The Heaviside functions multiplying these two integrals are equal at $\eta = 0$. Subtracting the kernels will result in $\frac{0}{0}$, so we can use L'Hospital's rule:

$$\int_{\frac{\eta_b}{1-\eta_b}}^{\frac{1}{2m+1}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta_b,\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\left[Kt8_{n=3}(\eta,\eta') - Kt7_{n=3}(\eta,\eta')\right]}{\eta} d\eta'$$

$$= \int_{\frac{\eta_b}{1-\eta_b}}^{\frac{1}{2m+1}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta_b,\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[Kt8_{n=3}(\eta,\eta') - Kt7_{n=3}(\eta,\eta')\right]}{\frac{d}{d\eta}\eta} d\eta'$$

$$= \int_{\frac{\eta_b}{1-\eta_b}}^{\frac{1}{2m+1}} \left[F_{2,m,L}(\eta',\eta_b) - F_{2,m,L}(\eta_b,\eta_b)\right] \eta' \left(\frac{-2}{\eta'(1+\eta')}\right) d\eta'. \tag{B.60}$$

Note the singularity at $\eta' = \eta_b$. The integrals of the kernels at $\eta = 0$ are:

$$\lim_{\eta\to 0} \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m+1}} Kt7_{n=3}(\eta,\eta')\eta' d\eta' = -\frac{1-2m\eta_b}{2(2m-1)} + \frac{1}{2}\ln(2m) + \frac{(1+\eta)^2}{2}\ln\left(\frac{1-\eta_b}{(2m-1)}\right)$$
$$+ \frac{1}{2(2m-1)^2}\ln\left(2m(1+\eta_b-2m\eta_b)\right) + \frac{\eta_b^2}{2}\ln\left(\frac{\eta_b^2(2m-1)}{(1+\eta_b-2m\eta_b)(1-\eta_b)}\right) \tag{B.61}$$

and

$$\lim_{\eta\to 0} \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}} Kt8_{n=3}(\eta,\eta')\eta' d\eta' = -\frac{1-2m\eta_b}{2(2m-1)} - \frac{1}{2}\ln(2m) + \frac{1}{2}\ln\left(\frac{1-\eta_b}{(2m-1)}\right)$$
$$+ \frac{1}{2(2m-1)^2}\ln\left(2m(1+\eta_b-2m\eta_b)\right) + \frac{\eta_b^2}{2}\ln\left(\frac{\eta_b^2(2m-1)}{(1+\eta_b-2m\eta_b)(1-\eta_b)}\right). \tag{B.62}$$

Combining these two equations would give $\frac{0}{0}$, so we will use L'Hospital's rule:

$$
\begin{aligned}
& F_{2,m,L}(\eta_b,\eta_b)\lim_{\eta\to 0}\frac{1}{\eta}\left[\int\limits_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}} Kt8_{n=3}(\eta,\eta')\eta' d\eta' - \int\limits_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m+1}} Kt7_{n=3}(\eta,\eta')\eta' d\eta'\right] \\
&= F_{2,m,L}(\eta_b,\eta_b)\lim_{\eta\to 0}\frac{\frac{d}{d\eta}\left[\int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{2m+1}} Kt8_{n=3}(\eta,\eta')\eta' d\eta' - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{2m+1}} Kt7_{n=3}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta} \\
&= -2F_{2,m,L}(\eta_b,\eta_b)\ln\left(\frac{2m(1-\eta_b)}{2m-1}\right).
\end{aligned}
\tag{B.63}
$$

We can now turn to the integrals in equation B.38. Note that the third integral in equation B.38 will be equal to zero at $\eta = 0$. Now we can turn to the first and fourth integrals in equation B.38. The kernels of these integrals are:

$$
\begin{aligned}
Kt4_{n=3} &= \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b'-\eta_b} \\
&= \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'-\eta}\right) - \frac{1+\eta}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}
\end{aligned}
\tag{B.64}
$$

and

$$
\begin{aligned}
Kt6_{n=3} &= \int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b'-\eta_b} \\
&= \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'+\eta}\right) - \frac{1-\eta}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}.
\end{aligned}
\tag{B.65}
$$

At $\eta = 0$, the kernels are equal to:

$$
\begin{aligned}
Kt4_{n=3} &= \int\limits_{\frac{\eta_b(1+\eta')}{1+\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b'-\eta_b} \\
&= \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'}\right) - \frac{1}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}
\end{aligned}
\tag{B.66}
$$

and

$$
\begin{aligned}
Kt6_{n=3} &= \int\limits_{\frac{\eta_b(1+\eta')}{1-\eta}}^{\frac{1+\eta'}{2m}} \left(\frac{\eta_b}{\eta_b'}\right)^2 \frac{d\eta_b'}{\eta_b' - \eta_b} \\
&= \ln\left(\frac{1-2m\eta_b+\eta'}{\eta'}\right) - \frac{1}{1+\eta'} + \frac{2m\eta_b}{1+\eta'}.
\end{aligned}
\tag{B.67}
$$

The Heaviside functions multiplying these two integrals are equal at $\eta = 0$. Subtracting the kernels will result in $\frac{0}{0}$, so we can use L'Hospital's rule:

$$
\begin{aligned}
&\int\limits_0^{\frac{1}{2m-1}} \left[F_{2,m,R}(\eta',\eta_b) - F_{2,m,R}(\eta,\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\left[Kt6_{n=3}(\eta,\eta') - Kt4_{n=3}(\eta,\eta')\right]}{\eta} d\eta' \\
&= \int\limits_0^{\frac{1}{2m-1}} \left[F_{2,m,R}(\eta',\eta_b) - F_{2,m,R}(\eta,\eta_b)\right] \eta' \lim_{\eta\to 0} \frac{\frac{d}{d\eta}\left[Kt6_{n=3}(\eta,\eta') - Kt4_{n=3}(\eta,\eta')\right]}{\frac{d}{d\eta}\eta} d\eta' \\
&= \int\limits_0^{\frac{1}{2m-1}} \left[F_{2,m,R}(\eta',\eta_b) - F_{2,m,R}(\eta,\eta_b)\right] \eta' \left(\frac{-2}{\eta'(1+\eta')}\right) d\eta'.
\end{aligned}
\tag{B.68}
$$

Note the singularity at $\eta' = \eta$. The integrals of the kernels at $\eta = 0$ are:

$$
\begin{aligned}
\lim_{\eta\to 0} \int\limits_0^{\frac{1}{2m-1}} Kt6_{n=3}(\eta,\eta')\eta' d\eta' &= -\frac{1-2m\eta_b}{2(2m-1)} + \frac{1}{2}\ln(2m) + 2(m\eta_b)^2 \ln\left(\frac{2m-1}{2m}\right) \\
&- \frac{1}{2}\ln(2m-1) + \frac{1}{2(2m-1)^2}\ln\left(2m(1+\eta_b-2m\eta_b)\right) \\
&+ \frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b}{1+\eta_b-2m\eta_b}\right)
\end{aligned}
\tag{B.69}
$$

and

$$\lim_{\eta\to 0}\int_{\eta}^{\frac{1}{2m-1}} Kt4_{n=3}(\eta,\eta')\eta' d\eta' = -\frac{1-2m\eta_b}{2(2m-1)} + \frac{1}{2}\ln(2m) + 2(m\eta_b)^2\ln\left(\frac{2m-1}{2m}\right)$$
$$-\frac{1}{2}\ln(2m-1) + \frac{1}{2(2m-1)^2}\ln\left(2m(1+\eta_b-2m\eta_b)\right)$$
$$+\frac{(1-2m\eta_b)^2}{2}\ln\left(\frac{1-2m\eta_b}{1+\eta_b-2m\eta_b}\right). \tag{B.70}$$

Combining these two equations would give $\frac{0}{0}$, so we will use L'Hospital's rule:

$$F_{2,m,R}(\eta,\eta_b)\lim_{\eta\to 0}\frac{1}{\eta}\left[\int_{0}^{\frac{1}{2m-1}} Kt6_{n=3}(\eta,\eta')\eta' d\eta' - \int_{\eta}^{\frac{1}{2m-1}} Kt4_{n=3}(\eta,\eta')\eta' d\eta'\right]$$
$$= F_{2,m,R}(\eta,\eta_b)\lim_{\eta\to 0}\frac{\frac{d}{d\eta}\left[\int_0^{\frac{1}{2m-1}} Kt6_{n=3}(\eta,\eta')\eta' d\eta' - \int_\eta^{\frac{1}{2m-1}} Kt4_{n=3}(\eta,\eta')\eta' d\eta'\right]}{\frac{d}{d\eta}\eta}$$
$$= -2F_{2,m,R}(\eta,\eta_b)\ln\left(\frac{2m}{2m-1}\right). \tag{B.71}$$

The final two integrals to consider are second and fifth. These two integrals have the same η_b' limits of integration as the fifth and sixth integrals in equation B.37, and were already considered above.

Appendix C

Analytic Expression for $n = 2$ Finite Sphere Shape Factor

This appendix reproduces the $n = 2$ shape factor for the finite sphere. The analytic expression for the $n = 2$ shape factor is used as the initial source for the numerical implementation of finite sphere neutron transport method described in Sec. 4.2.

C.1 Infinite Medium Source

The $n = 2$ infinite sphere shape factor is:

$$\begin{aligned} F_{2,inf}(\eta) =& \frac{1}{2\pi\eta}\left[\pi^2\eta + \frac{3}{2}(1-\eta)\ln\left(\frac{1-\eta}{2}\right)^2 - \frac{3}{2}(1+\eta)\ln\left(\frac{1+\eta}{2}\right)^2\right. \\ &\left. + 3(1-\eta)\,\mathrm{Li}_2\left(\frac{1-\eta}{2}\right) - 3(1+\eta)\,\mathrm{Li}_2\left(\frac{1+\eta}{2}\right)\right]. \end{aligned} \tag{C.1}$$

C.2 $m = 1$ Depletion Waves

The $n = 2, m = 1$ right depletion, corresponding to the $H(1-2\eta_b+\eta)$ Heaviside function, where m is the reflection number, is:

$$
\begin{aligned}
F_{2,m1,R}(\eta,\eta_b) = H(1-2\eta_b+\eta)\Bigg\{ & \frac{-\pi(5-6\eta_b+10\eta)}{48\eta} + \frac{7-\eta_b+4\eta}{8\pi\eta}\ln(2)^2 \\
& -\frac{1-\eta_b}{4\pi\eta}\ln(2)\ln(1-\eta_b) + \frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)^2 + \frac{1}{2\pi\eta}\ln(2)\ln(\eta_b) \\
& -\frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b) - \frac{\eta_b}{8\pi\eta}\ln(\eta_b)^2 - \frac{1+\eta_b}{4\pi\eta}\ln(\eta_b)\ln(1+\eta_b) \\
& +\frac{1+\eta_b}{8\pi\eta}\ln(1+\eta_b)^2 - \frac{1}{2\pi}\ln(2)\ln(1-\eta) - \frac{-1+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-\eta) \\
& -\frac{1-\eta}{4\pi\eta}\ln(\eta_b)\ln(1-\eta) - \frac{1-\eta}{2\pi\eta}\ln(1-\eta)^2 - \frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b-\eta) \\
& -\frac{\eta_b-\eta}{4\pi\eta}\ln(\eta_b)\ln(\eta_b-\eta) - \frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(\eta_b-\eta) \\
& +\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(\eta_b-\eta) - \frac{1-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1+2\eta_b-\eta) \\
& -\frac{3}{2\pi\eta}\ln(2)\ln(1+\eta) + \frac{2-\eta}{2\pi\eta}\ln(1-\eta)\ln(1+\eta) + \frac{1+\eta}{4\pi\eta}\ln(\eta_b-\eta)\ln(1+\eta) \\
& -\frac{1+\eta}{4\pi\eta}\ln(1+2\eta_b-\eta)\ln(1+\eta) + \frac{1+\eta}{4\pi\eta}\ln(1+\eta)^2 - \frac{1+\eta}{2\pi\eta}\ln(2)\ln(1-2\eta_b+\eta) \\
& -\frac{1+\eta}{4\pi\eta}\ln(\eta_b)\ln(1-2\eta_b+\eta) + \frac{1+\eta}{4\pi\eta}\ln(1+2\eta_b-\eta)\ln(1-2\eta_b+\eta) \\
& +\frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(1-2\eta_b+\eta) + \frac{1}{2\pi\eta}\ln(2)\ln(1-\eta_b+\eta) \\
& +\frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-\eta_b+\eta) - \frac{1-\eta_b+\eta}{4\pi\eta}\ln(\eta_b)\ln(1-\eta_b+\eta) \\
& +\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(1-\eta_b+\eta) - \frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-\eta_b+\eta) \\
& -\frac{1-\eta_b+2\eta}{4\pi\eta}\ln(\eta_b-\eta)\ln(1-\eta_b+\eta) - \frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(1-\eta_b+\eta) \\
& +\frac{2-\eta_b+2\eta}{8\pi\eta}\ln(1-\eta_b+\eta)^2 - \frac{3+\eta_b}{4\pi\eta}\mathrm{Li}_2(\eta_b) - \frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left(\frac{1+\eta_b}{2}\right) \\
& -\frac{1}{4\pi\eta}\mathrm{Li}_2\left(\frac{-\eta_b}{1-\eta_b+\eta}\right) + \frac{1-2\eta_b+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(\eta_b-\eta)}{(1-\eta_b)(1-\eta_b+\eta)}\right] \\
& +\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left[\frac{-\eta_b(\eta_b-\eta)}{(1+\eta_b)(1-\eta_b+\eta)}\right] + \frac{1}{2\pi}\mathrm{Li}_2\left(\frac{1+2\eta_b-\eta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{1-\eta}{(1-\eta_b)(1+2\eta_b-\eta)}\right]-\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1-2\eta_b+\eta)(1-\eta)}{(1+2\eta_b-\eta)(1+\eta)}\right]\\
&+\frac{3+\eta}{2\pi\eta}\mathrm{Li}_2\left(\frac{1+\eta}{2}\right)-\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(\eta_b-\eta)(1+\eta)}{(1-\eta_b+\eta)(1-\eta)}\right]+\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{1-\eta}{1+2\eta_b-\eta}\right)\\
&-\frac{1-\eta_b+2\eta}{4\pi\eta}\mathrm{Li}_2(1-\eta_b+\eta)+\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{1+\eta}{2-2\eta_b+2\eta}\right)\Bigg\}.
\end{aligned}
\tag{C.2}
$$

The $n=2, m=1$ left depletion, corresponding to the $H(1-2\eta_b-\eta)$ Heaviside function, where m is the reflection number, is:

$$
\begin{aligned}
F_{2,m1,L}(\eta,\eta_b) = H(1-2\eta_b-\eta)\Bigg\{&-\frac{\pi(1-6\eta_b-10\eta)}{48\eta}-\frac{7-\eta_b-4\eta}{8\pi\eta}\ln(2)^2\\
&+\frac{1-\eta_b}{4\pi\eta}\ln(2)\ln(1-\eta_b)-\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)^2-\frac{1}{2\pi\eta}\ln(2)\ln(\eta_b)\\
&+\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b)+\frac{\eta_b}{8\pi\eta}\ln(\eta_b)^2+\frac{1+\eta_b}{4\pi\eta}\ln(\eta_b)\ln(1+\eta_b)\\
&-\frac{1+\eta_b}{8\pi\eta}\ln(1+\eta_b)^2+\frac{3}{2\pi\eta}\ln(2)\ln(1-\eta)-\frac{1-\eta}{4\pi\eta}\ln(1-\eta)^2\\
&+\frac{1-\eta}{2\pi\eta}\ln(2)\ln(1-2\eta_b-\eta)+\frac{1-\eta}{4\pi\eta}\ln(\eta_b)\ln(1-2\eta_b-\eta)\\
&-\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-2\eta_b-\eta)-\frac{1}{2\pi\eta}\ln(2)\ln(1-\eta_b-\eta)\\
&-\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-\eta_b-\eta)+\frac{1-\eta_b-\eta}{4\pi\eta}\ln(\eta_b)\ln(1-\eta_b-\eta)\\
&-\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(1-\eta_b-\eta)+\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-\eta_b-\eta)\\
&-\frac{2-\eta_b-2\eta}{8\pi\eta}\ln(1-\eta_b-\eta)^2+\frac{1-\eta}{2\pi\eta}\ln(2)\ln(1+\eta)-\frac{1+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1+\eta)\\
&+\frac{3+\eta}{4\pi\eta}\ln(\eta_b)\ln(1+\eta)-\frac{2+\eta}{2\pi\eta}\ln(1-\eta)\ln(1+\eta)\\
&+\frac{1+\eta}{4\pi\eta}\ln(1-\eta_b-\eta)\ln(1+\eta)+\frac{1+\eta}{2\pi\eta}\ln(1+\eta)^2\\
&+\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b+\eta)+\frac{\eta_b+\eta}{4\pi\eta}\ln(\eta_b)\ln(\eta_b+\eta)\\
&+\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(\eta_b+\eta)-\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(\eta_b+\eta)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1-\eta_b-2\eta}{4\pi\eta}\ln(1-\eta_b-\eta)\ln(\eta_b+\eta)-\frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(\eta_b+\eta)\\
&-\frac{1}{2\pi\eta}\ln(2)\ln(1+2\eta_b+\eta)+\frac{1+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1+2\eta_b+\eta)\\
&-\frac{1}{2\pi\eta}\ln(\eta_b)\ln(1+2\eta_b+\eta)+\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1+2\eta_b+\eta)\\
&-\frac{1-\eta}{4\pi\eta}\ln(1-2\eta_b-\eta)\ln(1+2\eta_b+\eta)-\frac{1}{2\pi\eta}\ln(1+\eta)\ln(1+2\eta_b+\eta)\\
&+\frac{1}{2\pi\eta}\ln(1+2\eta_b+\eta)^2+\frac{3+\eta_b}{4\pi\eta}\mathrm{Li}_2(\eta_b)+\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left(\frac{1+\eta_b}{2}\right)-\frac{3-\eta}{2\pi\eta}\mathrm{Li}_2\left(\frac{1-\eta}{2}\right)\\
&+\frac{1-\eta_b-2\eta}{4\pi\eta}\mathrm{Li}_2(1-\eta_b-\eta)+\frac{1}{4\pi\eta}\mathrm{Li}_2\left(\frac{-\eta_b}{1-\eta_b-\eta}\right)-\frac{1}{2\pi\eta}\mathrm{Li}_2\left[\frac{1-\eta}{2(1-\eta_b-\eta)}\right]\\
&-\frac{1-2\eta_b-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(\eta_b+\eta)}{(1-\eta_b)(1-\eta_b-\eta)}\right]-\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left[\frac{-\eta_b(\eta_b+\eta)}{(1+\eta_b)(1-\eta_b-\eta)}\right]\\
&+\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1-\eta)(\eta_b+\eta)}{(1+\eta)(1-\eta_b-\eta)}\right]+\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{2\eta_b}{1+2\eta_b+\eta}\right)\\
&+\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{1+\eta}{(1-\eta_b)(1+2\eta_b+\eta)}\right]+\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1+\eta)(1-2\eta_b-\eta)}{(1-\eta)(1+2\eta_b+\eta)}\right]\\
&+\frac{1}{2\pi}\mathrm{Li}_2\left(\frac{1+2\eta_b+\eta}{2}\right)\Bigg\}.
\end{aligned}
\tag{C.3}
$$

C.3 $m = 2$ Depletion Waves

The $n = 2, m = 2$ right depletion, corresponding to the $H(1-4\eta_b+\eta)$ Heaviside function, where m is the reflection number, is:

$$
\begin{aligned}
F_{2,m2,R}(\eta,\eta_b) = H(1-4\eta_b+\eta)\Bigg\{ & \frac{(1-\eta_b)\pi}{24\eta} + \frac{3+2\eta_b-\eta}{4\pi\eta}\ln(2)^2 - \frac{1+2\eta_b}{8\pi\eta}\ln(3)^2 \\
& -\frac{\eta_b}{2\pi\eta}\ln(2)\ln(1-3\eta_b) + \frac{\eta_b}{2\pi\eta}\ln(3)\ln(1-3\eta_b) - \frac{\eta_b}{2\pi\eta}\ln(3)\ln(1-\eta_b) \\
& +\frac{\eta_b}{2\pi\eta}\ln(1-3\eta_b)\ln(1-\eta_b) + \frac{1}{2\pi\eta}\ln(2)\ln(\eta_b) - \frac{\eta_b}{2\pi\eta}\ln(1-\eta_b)\ln(\eta_b) \\
& +\frac{\eta_b}{8\pi\eta}\ln(\eta_b)^2 + \frac{1+\eta_b}{4\pi\eta}\ln(\eta_b)\ln(1+\eta_b) - \frac{1+\eta_b}{8\pi\eta}\ln(1+\eta_b)^2 \\
& -\frac{1-\eta}{2\pi\eta}\ln(2)\ln(1-2\eta_b-\eta) - \frac{1+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-2\eta_b-\eta) \\
& -\frac{1-\eta}{4\pi\eta}\ln(\eta_b)\ln(1-2\eta_b-\eta) + \frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-2\eta_b-\eta) \\
& +\frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b-\eta) + \frac{\eta_b-\eta}{4\pi\eta}\ln(\eta_b)\ln(\eta_b-\eta) \\
& +\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(\eta_b-\eta) - \frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(\eta_b-\eta) - \frac{1}{2\pi\eta}\ln(2)\ln(1+\eta) \\
& -\frac{1}{2\pi\eta}\ln(\eta_b)\ln(1+\eta) + \frac{1+\eta}{4\pi\eta}\ln(1-2\eta_b-\eta)\ln(1+\eta) \\
& -\frac{1+\eta}{4\pi\eta}\ln(\eta_b-\eta)\ln(1+\eta) - \frac{1}{2\pi\eta}\ln(2)\ln(1-\eta_b+\eta) \\
& -\frac{1-2\eta_b+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-\eta_b+\eta) + \frac{1-\eta_b+\eta}{4\pi\eta}\ln(\eta_b)\ln(1-\eta_b+\eta) \\
& -\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(1-\eta_b+\eta) + \frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-\eta_b+\eta) \\
& +\frac{1-\eta_b+2\eta}{4\pi\eta}\ln(\eta_b-\eta)\ln(1-\eta_b+\eta) + \frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(1-\eta_b+\eta) \\
& -\frac{2-\eta_b+2\eta}{8\pi\eta}\ln(1-\eta_b+\eta)^2 + \frac{1}{2\pi\eta}\ln(2)\ln(1+2\eta_b+\eta) \\
& +\frac{1+\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1+2\eta_b+\eta) + \frac{1-\eta}{4\pi\eta}\ln(\eta_b)\ln(1+2\eta_b+\eta) \\
& -\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1+2\eta_b+\eta) - \frac{1}{2\pi}\ln(1-2\eta_b-\eta)\ln(1+2\eta_b+\eta) \\
& +\frac{1-\eta}{4\pi\eta}\ln(1+\eta)\ln(1+2\eta_b+\eta) - \frac{1-\eta}{4\pi\eta}\ln(1+2\eta_b+\eta)^2 - \frac{1}{4\pi\eta}\mathrm{Li}_2\left(-\frac{1}{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\eta_b}{2\pi\eta}\mathrm{Li}_2\left(\frac{1-3\eta_b}{3-3\eta_b}\right)-\frac{1-\eta_b}{4\pi\eta}\mathrm{Li}_2(3\eta_b)+\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left(\frac{3(1+\eta_b)}{4}\right)\\
&+\frac{1}{4\pi\eta}\mathrm{Li}_2\left(\frac{-\eta_b}{1-\eta_b+\eta}\right)-\frac{1-2\eta_b+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(\eta_b-\eta)}{(1-\eta_b)(1-\eta_b+\eta)}\right]\\
&-\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left[\frac{-\eta_b(\eta_b-\eta)}{(1+\eta_b)(1-\eta_b+\eta)}\right]+\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(\eta_b-\eta)(1+\eta)}{(1-\eta_b+\eta)(1-\eta)}\right]\\
&+\frac{1-\eta_b+2\eta}{4\pi\eta}\mathrm{Li}_2(1-\eta_b+\eta)-\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{2\eta_b}{1+2\eta_b+\eta}\right)\\
&+\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(1-2\eta_b-\eta}{(1-\eta_b)(1+2\eta_b+\eta)}\right]-\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1+\eta)(1-2\eta_b-\eta)}{(1-\eta)(1+2\eta_b+\eta)}\right]\\
&-\frac{1}{2\pi}\mathrm{Li}_2\left(\frac{1+2\eta_b+\eta}{2}\right)-\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{1+\eta}{2-2\eta_b+2\eta}\right)\Bigg\}.
\end{aligned}
\tag{C.4}
$$

The $n=2, m=2$ left depletion, corresponding to the $H(1-4\eta_b-\eta)$ Heaviside function, where m is the reflection number, is:

$$
\begin{aligned}
F_{2,m2,L}(\eta,\eta_b) = H(1-4\eta_b-\eta)\Bigg\{&-\frac{(1-\eta_b)\pi}{24\eta}-\frac{(3+2\eta_b+\eta)}{4\pi\eta}\ln(2)^2+\frac{(1+2\eta_b)}{8\pi\eta}\ln(3)^2\\
&+\frac{\eta_b}{2\pi\eta}\ln(2)\ln(1-3\eta_b)-\frac{\eta_b}{2\pi\eta}\ln(3)\ln(1-3\eta_b)+\frac{\eta_b}{2\pi\eta}\ln(3)\ln(1-\eta_b)\\
&-\frac{\eta_b}{2\pi\eta}\ln(1-3\eta_b)\ln(1-\eta_b)-\frac{1}{2\pi\eta}\ln(2)\ln(\eta_b)+\frac{\eta_b}{2\pi\eta}\ln(1-\eta_b)\ln(\eta_b)\\
&-\frac{\eta_b}{8\pi\eta}\ln(\eta_b)^2-\frac{1+\eta_b}{4\pi\eta}\ln(\eta_b)\ln(1+\eta_b)+\frac{1+\eta_b}{8\pi\eta}\ln(1+\eta_b)^2+\frac{1}{2\pi\eta}\ln(2)\ln(1-\eta)\\
&+\frac{1}{2\pi\eta}\ln(\eta_b)\ln(1-\eta)+\frac{1}{2\pi\eta}\ln(2)\ln(1-\eta_b-\eta)\\
&+\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-\eta_b-\eta)-\frac{1-\eta_b-\eta}{4\pi\eta}\ln(\eta_b)\ln(1-\eta_b-\eta)\\
&+\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(1-\eta_b-\eta)-\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-\eta_b-\eta)\\
&+\frac{2-\eta_b-2\eta}{8\pi\eta}\ln(1-\eta_b-\eta)^2-\frac{1}{2\pi\eta}\ln(2)\ln(1+2\eta_b-\eta)\\
&-\frac{1-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1+2\eta_b-\eta)-\frac{1+\eta}{4\pi\eta}\ln(\eta_b)\ln(1+2\eta_b-\eta)\\
&-\frac{1+\eta}{4\pi\eta}\ln(1-\eta)\ln(1+2\eta_b-\eta)+\frac{1+\eta}{4\pi\eta}\ln(1+2\eta_b-\eta)^2
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1+\eta}{4\pi\eta}\ln(1-\eta_b-\eta)\ln(1+\eta)+\frac{1+\eta}{4\pi\eta}\ln(1+2\eta_b-\eta)\ln(1+\eta)\\
&+\frac{1+\eta}{2\pi\eta}\ln(2)\ln(1-2\eta_b+\eta)+\frac{1-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(1-2\eta_b+\eta)\\
&+\frac{1+\eta}{4\pi\eta}\ln(\eta_b)\ln(1-2\eta_b+\eta)-\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(1-2\eta_b+\eta)\\
&-\frac{1}{2\pi}\ln(1+2\eta_b-\eta)\ln(1-2\eta_b+\eta)-\frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(1-2\eta_b+\eta)\\
&-\frac{1-2\eta_b-\eta}{4\pi\eta}\ln(1-\eta_b)\ln(\eta_b+\eta)-\frac{\eta_b+\eta}{4\pi\eta}\ln(\eta_b)\ln(\eta_b+\eta)\\
&-\frac{1+\eta_b}{4\pi\eta}\ln(1+\eta_b)\ln(\eta_b+\eta)+\frac{1-\eta}{4\pi\eta}\ln(1-\eta)\ln(\eta_b+\eta)\\
&-\frac{1-\eta_b-2\eta}{4\pi\eta}\ln(1-\eta_b-\eta)\ln(\eta_b+\eta)+\frac{1+\eta}{4\pi\eta}\ln(1+\eta)\ln(\eta_b+\eta)+\frac{1}{4\pi\eta}\mathrm{Li}_2\left(-\frac{1}{3}\right)\\
&+\frac{\eta_b}{2\pi\eta}\mathrm{Li}_2\left(\frac{1-3\eta_b}{3-3\eta_b}\right)+\frac{1-\eta_b}{4\pi\eta}\mathrm{Li}_2(3\eta_b)-\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left[\frac{3(1+\eta_b)}{4}\right]\\
&-\frac{1-\eta_b-2\eta}{4\pi\eta}\mathrm{Li}_2(1-\eta_b-\eta)+\frac{1}{2\pi\eta}\mathrm{Li}_2\left(\frac{2\eta_b}{1+2\eta_b-\eta}\right)-\frac{1}{2\pi}\mathrm{Li}_2\left(\frac{1+2\eta_b-\eta}{2}\right)\\
&+\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1-2\eta_b+\eta)(1-\eta)}{(1+2\eta_b-\eta)(1+\eta)}\right]-\frac{1-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(1-2\eta_b+\eta)}{(1-\eta_b)(1+2\eta_b-\eta)}\right]\\
&-\frac{1}{4\pi\eta}\mathrm{Li}_2\left(\frac{-\eta_b}{1-\eta_b-\eta}\right)+\frac{1}{2\pi\eta}\mathrm{Li}_2\left[\frac{1-\eta}{2(1-\eta_b-\eta)}\right]\\
&+\frac{1-2\eta_b-\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{\eta_b(\eta_b+\eta)}{(1-\eta_b)(1-\eta_b-\eta)}\right]+\frac{1+\eta_b}{4\pi\eta}\mathrm{Li}_2\left[\frac{-\eta_b(\eta_b+\eta)}{(1+\eta_b)(1-\eta_b-\eta)}\right]\\
&-\frac{1+\eta}{4\pi\eta}\mathrm{Li}_2\left[\frac{(1-\eta)(\eta_b+\eta)}{(1+\eta)(1-\eta_b-\eta)}\right]\Bigg\}.
\end{aligned}
\tag{C.5}
$$

Appendix D

Finite Sphere Checks

One of the checks necessary to show internal consisitency in the finite sphere method was to match the limits of integration of various integrals evaluated to obtain the nth shape factor. The shape factors must go to zero after all the depletion waves have crossed the medium, and matching the limits of integration to show that the total is zero is a convienient way to check that the shape factor will go to zero without needing to calculate the shape factor. This appendix shows this process for the $n = 2$ shape factor for the reflective boundary formulation. This same process was followed for the $n = 2$ shape factor for the formulation not utilizing the reflective boundary condition, giving the same result.

$$
\begin{aligned}
F_2(\eta,\tau) = &\int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' - \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' \\
&+ \int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&+ \int_0^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_0^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&+ \int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_0^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&- \int_{\frac{1}{3}}^{\frac{1+\eta-2\eta_b}{1+\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&+ \int_{\frac{1}{3}}^{\frac{1-\eta-2\eta_b}{1-\eta+2\eta_b}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'
\end{aligned}
$$

$$
\begin{aligned}
F_2(\eta,\tau) = &\int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' - \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' \\
&+ \int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,inf}(\eta')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&+ \int_0^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_0^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,R}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&+ \int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\eta}^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' - \int_0^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' \\
&- \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{F_{1,L}(\eta',\tau')}{1-\tau'}d\tau' d\eta'
\end{aligned}
$$

where

$$
F_{1,inf}(\eta) = \frac{1}{2\pi\eta}\ln\left(\frac{1+\eta}{1-\eta}\right) \tag{D.1}
$$

$$
F_{1,L}(\eta,\tau) = -H(1-2\eta_b-\eta)\frac{1}{4\pi\eta}\left[\ln\left(\frac{1+\eta}{1-\eta}\right) - \ln\left(1-\frac{\eta_b}{\tau}\right) + \ln\left(\frac{\eta_b}{\tau}\right)\right] \tag{D.2}
$$

and

$$
F_{1,R}(\eta,\tau) = -H(1-2\eta_b+\eta)\frac{1}{4\pi\eta}\left[\ln\left(\frac{1+\eta}{1-\eta}\right) + \ln\left(1-\frac{\eta_b}{\tau}\right) - \ln\left(\frac{\eta_b}{\tau}\right)\right] \tag{D.3}
$$

Note that each of the three sources contain the term $\ln\left(\frac{1+\eta}{1-\eta}\right)$. We will begin by matching the integrals for only this integral argument.

$$
\begin{aligned}
&= \int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' - \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' \\
&- \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&+ \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' + \int_0^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&- \int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_{\eta}^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&+ \int_0^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' + \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' - \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{2\pi\eta'}d\tau' d\eta' \\
&- \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b-\eta}}^{1}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{\eta}\int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&- \int_0^{\eta}\int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{1}\int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&- \int_{\eta}^{1}\int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_{\eta}^{1}\int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' - \int_0^{\eta}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&- \int_{\eta}^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' + \int_0^{\frac{1}{3}}\int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta' \\
&- \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}}\int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta}\frac{1}{1-\tau'}\frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'}d\tau' d\eta'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\eta}^{1} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_0^{1} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' \\
&- \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{3}} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}} \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' \\
&+ \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_0^{\eta} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\eta}^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' \\
&- \int_0^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{\eta_b}{1+\eta-\eta_b}}^{\frac{1}{3}} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}} \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'
\end{aligned}
$$

$$= \int_0^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\eta}^{1} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_0^{1} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'$$

$$- \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_0^{\eta} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'$$

$$- \int_0^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\eta}^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\eta}^{\frac{1}{3}} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'$$

$$+ \int_0^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' + \int_0^{\frac{1}{3}} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'$$

$$= \int_{\frac{1}{3}}^{1} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{\frac{1-\eta}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta'$$

$$= \int_{\frac{1}{3}}^{1} \int_{\frac{1-\eta}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{\frac{1-\eta}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\ln\left(\frac{1+\eta'}{1-\eta'}\right)}{4\pi\eta'} d\tau' d\eta' = 0$$

Now we can match limits of integration for the remaining integrals,

$$
\begin{aligned}
&= \int\limits_{\frac{\eta_b}{1-\eta-\eta_b}}^{1} \int\limits_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int\limits_{\frac{\eta_b}{1+\eta-\eta_b}}^{1} \int\limits_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int\limits_{0}^{\eta} \int\limits_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int\limits_{0}^{1} \int\limits_{\frac{2\eta_b}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int\limits_{\eta}^{1} \int\limits_{\frac{2\eta_b}{1+\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int\limits_{0}^{\eta} \int\limits_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int\limits_{\eta}^{\frac{1}{3}} \int\limits_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)-\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int\limits_{0}^{\frac{1}{3}} \int\limits_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int\limits_{\frac{\eta_b}{1-\eta_b+\eta}}^{\frac{1}{3}} \int\limits_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int\limits_{\frac{\eta_b}{1-\eta-\eta_b}}^{\frac{1}{3}} \int\limits_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{0}^{\eta} \int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' + \int_{0}^{\eta} \int_{0}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{\frac{\eta_b}{\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int_{0}^{1} \int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{0}^{1} \int_{0}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int_{\eta}^{1} \int_{\frac{2\eta_b}{1+\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' + \int_{\eta}^{1} \int_{0}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int_{0}^{\eta} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)-\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{\eta}^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int_{0}^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta'
\end{aligned}
$$

$$
\begin{aligned}
&= \int_0^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' + \int_{\frac{1}{3}}^{1} \int_{\frac{1+\eta}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int_0^{1} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' + \int_{\eta}^{1} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int_0^{\eta} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_0^{\eta} \int_0^{\frac{1-\eta}{1-\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&- \int_{\eta}^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)-\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{\eta}^{\frac{1}{3}} \int_0^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' \\
&+ \int_0^{\frac{1}{3}} \int_{\frac{2\eta_b}{1-\eta'}}^{0} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' + \int_0^{\frac{1}{3}} \int_0^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta'
\end{aligned}
$$

$$= \int_{\frac{1}{3}}^{1} \int_{\frac{1+\eta}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{0}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta'$$

$$+ \int_{\frac{1}{3}}^{1} \int_{0}^{\frac{1+\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta'$$

$$= \int_{\frac{1}{3}}^{1} \int_{\frac{1+\eta}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' - \int_{\frac{1}{3}}^{1} \int_{\frac{1+\eta}{1+\eta'}}^{\frac{1-\eta}{1+\eta'}} \frac{\eta'}{\eta} \frac{1}{1-\tau'} \frac{\left[-\ln\left(1-\frac{\eta_b}{\tau'}\right)+\ln\left(\frac{\eta_b}{\tau'}\right)\right]}{4\pi\eta'} d\tau' d\eta' = 0$$

Appendix E

Deriving Heterogeneous Kernel

This appendix shows the derivation of the time-dependent, spherical shell heterogeneous medium integration kernel, using the Laplace transform method [21]. Consider the time dependent, one-speed neutron transport equation with isotropic scattering, for a heterogeneous medium:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \hat{\Omega}\cdot\nabla + \Sigma(\boldsymbol{r})\right)\psi\left(\boldsymbol{r},\hat{\Omega},t\right) = \frac{Q(\boldsymbol{r})}{4\pi}. \tag{E.1}$$

Taking the Laplace transform of equation E.1, the following equation is obtained:

$$\left(\hat{\Omega}\cdot\nabla + \tilde{\Sigma}(\boldsymbol{r})\right)\psi\left(\boldsymbol{r},\hat{\Omega},s\right) = \frac{\tilde{Q}\left(\boldsymbol{r}\right)}{4\pi} \tag{E.2}$$

where the steady-state macroscopic cross section has been replaced with:

$$\tilde{\Sigma}(\boldsymbol{r}) = \Sigma(\boldsymbol{r}) + \frac{s}{v}. \tag{E.3}$$

The corresponding integral equation to equation E.2 is

$$\tilde{\phi}(\boldsymbol{r}, s) = \int_{V'} \tilde{K}(\boldsymbol{r}, \boldsymbol{r}'; s)\tilde{Q}(\boldsymbol{r}', s)d\boldsymbol{r}'. \tag{E.4}$$

Taking the inverse Laplace transform of the both sides of equation E.4 gives

$$\phi(\boldsymbol{r}, t) = \int_{V'} \mathcal{L}^{-1}\left[\tilde{K}(\boldsymbol{r}, \boldsymbol{r}'; s)\tilde{Q}(\boldsymbol{r}', s)\right] d\boldsymbol{r}'. \tag{E.5}$$

Finally, the convolution theorem can be used on the right side of equation E.5 to obtain

$$\phi(\boldsymbol{r}, t) = \int_{V'} \int_{0}^{t} K(\boldsymbol{r}, \boldsymbol{r}'; t - t')Q(\boldsymbol{r}', t')dt'd\boldsymbol{r}'. \tag{E.6}$$

Henderson and Maynard [21] note that time-dependent integration kernels can be obtained from steady-state kernels. Therefore, the heterogeneous time-dependent spherical shell integration kernel can be found from the steady-state spherical shell kernel. The steady-state heterogeneous medium spherical shell kernel is:

$$K_{ss}(r, r') = \frac{1}{8\pi r r'} \left\{\mathrm{E}_1\left(\tau(r, r')\right) - \mathrm{E}_1\left(\tau(r, -r')\right)\right\} \tag{E.7}$$

where

$$\tau(r, r') = \int_{0}^{|r-r'|} \Sigma\left(r - w\frac{r - r'}{|r - r'|}\right) dw. \tag{E.8}$$

Making the substitution described above, equation E.3, in equation E.7, and then taking the Laplace transform of the resulting expression gives the following:

$$K_{ss}(r,r') = \frac{1}{8\pi r r'} \left\{ \mathrm{E}_1 \left(\int_0^{|r-r'|} \left[\Sigma \left(r - w \frac{(r-r')}{|r-r'|} \right) + \frac{s}{v} \right] dw \right) \right.$$
$$\left. - \mathrm{E}_1 \left(\int_0^{|r+r'|} \left[\Sigma \left(r - w \frac{(r+r')}{|r+r'|} \right) + \frac{s}{v} \right] dw \right) \right\}. \tag{E.9}$$

To find the time-dependent, heterogeneous spherical shell kernel is found by taking the Laplace transform of the above expression. The rest of the derivation proceeds as follows:

$$
\begin{aligned}
K_{ss}(r,r';t) = \frac{1}{8\pi r r'}\mathcal{L}^{-1}\Bigg\{ & \mathrm{E}_1\left(\int_0^{|r-r'|}\left[\Sigma\left(r - w\frac{(r-r')}{|r-r'|}\right) + \frac{s}{v}\right]dw\right) \\
& - \mathrm{E}_1\left(\int_0^{|r+r'|}\left[\Sigma\left(r-w\right) + \frac{s}{v}\right]dw\right)\Bigg\} \\
= \frac{1}{8\pi r r'}\Bigg\{ & \mathcal{L}^{-1}\left[\mathrm{E}_1\left(\tau(r,r') + \frac{s|r-r'|}{v}\right)\right] \\
& - \mathcal{L}^{-1}\left[\mathrm{E}_1\left(\tau(r,-r') + \frac{s(r+r')}{v}\right)\right]dw\Bigg\} \\
= \frac{1}{8\pi r r'}\Bigg\{ & \int_1^\infty \frac{e^{-u\tau(r,r')}}{u}\mathcal{L}^{-1}\left[e^{-\frac{s|r-r'|}{v}}\right]du \\
& - \int_1^\infty \frac{e^{-u\tau(r,-r')}}{u}\mathcal{L}^{-1}\left[e^{-\frac{s(r+r')}{v}}\right]du\Bigg\} \\
= \frac{1}{8\pi r r'}\Bigg\{ & \int_1^\infty \frac{e^{-u\tau(r,r')}}{u}\delta\left(t - \frac{u|r-r'|}{v}\right)du \\
& - \int_1^\infty \frac{e^{-u\tau(r,-r')}}{u}\delta\left(t - \frac{u(r+r')}{v}\right)du\Bigg\} \\
= \frac{1}{8\pi r r' t}\Bigg\{ & e^{-\frac{vt\tau(r,r')}{|r-r'|}}H\left(t - \frac{|r-r'|}{v}\right) - e^{-\frac{vt\tau(r,-r')}{|r+r'|}}H\left(t - \frac{|r+r'|}{v}\right)\Bigg\}
\end{aligned}
\tag{E.10}
$$

To obtain the final form of the time-dependent, steady-state, spherical shell kernel, replace t with $t - t'$.

$$
\begin{aligned}
K_{ss}(r,r';t-t') = & \frac{1}{8\pi r r'(t-t')}\Bigg\{e^{-\frac{v(t-t')\tau(r,r')}{|r-r'|}}H\left(t-t' - \frac{|r-r'|}{v}\right) \\
& - e^{-\frac{v(t-t')\tau(r,-r')}{|r+r'|}}H\left(t-t' - \frac{|r+r'|}{v}\right)\Bigg\}
\end{aligned}
\tag{E.11}
$$

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