A Brief Note on Deriving the Trapped Ion Mode Dispersion Relation Using a Fluid Model

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1. Introduction:

The purpose of this brief note is to give a derivation of the dissipative trapped ion mode (TI Mode) dispersion relation from a fluid theory model. Kadomtsev and Pogutse [1] have done just that but it appears that there has been some trouble in arriving at their result using their model! The relation one wishes to obtain appears below as

$$\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{T_i}+\frac{1}{T_e}\right)=\frac{1}{T_i}\,\frac{\omega-\omega_{*i}}{\omega+i\nu_i/\varepsilon}+\frac{1}{T_e}\,\frac{\omega+\omega_{*e}}{\omega+i\nu_e/\varepsilon}\ , \qquad (1)$$

where all the terms will be defined carefully during the course of the derivation.

I shall include, mostly for pedagogical purposes, a species dependent "gravitational" force in order to model ∇B and curvature drift effects, i.e.

$$g_j \simeq T_j/m_j R\ .$$

This does not complicate the derivation.

First, all terms to be used will be defined and the geometry in which the problem is to be solved will be examined. Then the appropriately generalized fluid equations for both the circulating particle "fluid" and the trapped particle "fluid" for each species, assuming fully ionized singly charged ions will be given. By systematically ignoring "inconsequential" terms in the fluid equations, one degeneralizes to a sufficient degree that equation (1) can be obtained.

2. Definitions and Geometry (See Figure #1)

$\vec{B}_o$ = uniform, time independent B field.

$\vec{g}_j = |g_j|\hat{x}$ = constant "gravity" for j^{th} species

$\nabla n_o = \frac{dn_o}{dx}\hat{x} = -\left|\frac{dn_o}{dx}\right|\hat{x}$ = density gradient

$\vec{k} = k_y\hat{y}$, $k_y > 0$ = propagation vector for electrostatic wave in plasma

$\vec{E} = -\nabla\hat{\phi} = ik_y\, \tilde{\phi}\hat{y} - \frac{\partial\tilde{\phi}}{\partial z}\,\hat{z}$ = electric field due to perturbation (wave) in plasma.

$\vec{V}_{g_j} = \frac{\vec{g}_j \times \vec{B}}{\Omega_j B} = \frac{-g_j}{\Omega_j}\,\hat{y}$ = gravitational drift velocity for j^{th} species

$\omega_{g_j} = \frac{-k_y g_j}{\Omega_j}$ = drift frequency due to gravity for j^{th} species (i.e. $\vec{k}_y \cdot \vec{V}_{g_j}$)

$\Omega_j = e_j B/m_j$

$\vec{V}_{dj} = \frac{\hat{b} \times \nabla n_o}{n_o}\,\frac{T_j}{m_j\Omega_j}$ = diamagnetic drift velocity of j^{th} species

$= \frac{-T_j}{m_j\Omega_j}\,\frac{1}{n_o}\left|\frac{dn_o}{dx}\right|\hat{y}$ (assuming $\nabla T = 0$)

$$\omega_j^* = k_y\frac{T_j}{e_jB}\,\frac{1}{n_o}\,\frac{dn_o}{dx} = \vec{k}_y \cdot \vec{V}_d = -k_y\frac{T_j}{e_jB}\,\frac{1}{n_o}\left|\frac{dn_o}{dx}\right|$$

3. The Equations

In this derivation the trapped particles, both ions and electrons, will be treated as separate "fluids" distinct from the circulating ions and electrons. Thus any species subscripts should be considered as running from 1 to 4. As there are mechanisms in the plasma (notably collisions in this case [2]) which can cause trapped particles to become detrapped and circulating particles to become trapped, we must take this into account in a 4 fluid picture. This is done by adding a type of source or sink term in the fluid continuity equations. We shall thus write

$$\frac{\partial n_j}{\partial t} + \nabla \cdot (n_j\,\vec{V}_j) = S_j \qquad (2)$$

where the form of S_j will be given later. Remembering that what we have set out to do is find the wave electric field $(-\nabla\phi)$ in the plasma and agreeing that we will settle for knowing whether or not it will grow in time given some "reasonable" initial perturbation, one can analyze this electrostatic wave using Poisson's equation

$$\nabla^2\phi = -\frac{1}{\varepsilon_o}\sum_{j=1}^{4} e_j n_j \quad . \tag{3}$$

We will actually use the quasi-neutrality condition [3] $\sum_j e_j n_j = 0$ because we will be concerned with long wavelength $\lambda_\perp > \lambda_{Debye}$ modes. We also assume that ϕ does not vary in the direction of the density gradient i.e. $d\phi/dx = 0$ which is called the "local approximation". Both conditions can be reintroduced at the expense of a great deal of simplicity. Since K & P did not consider these effects in [1] and we wish to get their result, I will also ignore them. The non-local condition however is not negligible in terms of deciding the effect of the mode on plasma behavior.

To get back to the derivation we see from [2] that to find n_j (given S_j) one must find $\vec{V}_j$. One can do this using the fluid equations of motion. The best reference is Lehnert[4] and taking his result one has for the $\perp$ fluid motion of the j^{th} species the general equation

$$n_j\vec{V}_{\perp j} = n_j\frac{\vec{E}\times\vec{B}_o}{B^2} - \frac{(\nabla\cdot\overleftrightarrow{P}_j)\times\vec{B}_o}{e_jB^2} + n_j\frac{m_j}{e_j}\frac{\vec{g}_j\times\vec{B}_o}{B^2} \tag{4}$$

$$- n_j\frac{m_j}{e_j}\left(\frac{\partial}{\partial t}\vec{V}_j + \vec{V}_j\cdot\nabla\vec{V}_j\right)_\perp \times\frac{\vec{B}_o}{B^2} + \text{collision terms}$$

The parallel motion comes from

$$m_jn_j\left(\frac{\partial}{\partial t}\vec{V}_j + \vec{V}_j\cdot\nabla\vec{V}_j\right)_\parallel = n_je_jE_\parallel - (\nabla\cdot\overleftrightarrow{P}_j)_\parallel + \text{collosion terms} \tag{5}$$

where $\hat{b} = \vec{B}_o/B$ and

$$\nabla\cdot\overleftrightarrow{P}_j = \underbrace{[\nabla_\parallel P_\parallel - (P_\parallel - P_\perp)\nabla B/B]}_{\text{component}} + \underbrace{[\nabla_\perp P_\perp + (P_\parallel - P_\perp)(\hat{b}\cdot\nabla)\hat{b}]}_{\text{component}} \quad . \tag{6}$$

To proceed we make the following simplifying assumptions:

1) isotropic pressure $P_\perp = P_\parallel = P$

2) ignore inertial effects and collisions in equations of motion

3) $\vec{B}$ field is considered constant and any "real" ∇B effects are to be modeled into $\vec{g}_j$

4) circulating particles sample each ψ surface so quickly (and completely) compared to the period of the wave we will be considering that we shall assume these circulating particles come into a "quasi-static" equilibrium with the wave electric field $(-\nabla\phi)$ along the $\vec{B}$ field only.

With these assumptions one has for circulating particles, using equation (5),

$$n_{ej} = (1 - \sqrt{\varepsilon})\, n_o(x) \exp\left(-\frac{e_j\phi}{Tj}\right) \tag{8}$$

where $\varepsilon^{1/2}$ is the fraction of particles trapped if one had an isotropic equilibrium distribution for species j. The steps to show this are in Appendix A. Since we now have an expression for n_j for circulating particles one does not need to use the circulating particle continuity equations. This may seem like a cheat but when we imposed our assumptions (i.e. assumptions (4) and (2) in particular), we in essence legislated this result. The form given in (8) is often called (somewhat inappropriately) the "adiabatic" response of the j^{th} species to a perturbation ϕ. The plasma is in thermal equilibrium (at least along s) in the presence of potential ϕ which to the circulating particles must be very slowly changing in time for this approximation to be valid, i.e., $(\omega/k_\parallel)$ of the wave must be less than the characteristic thermal velocity of the species along the field line. See Appendix A for further discussion.

The rest of the derivation will settle about two points. First, what is the trapped particle response to a perturbation $\phi = \tilde{\phi}(x,s)\, e^{ik_y y - i\omega t}$, remembering that x will be considered merely a label indicating where in the plasma we are considering the analysis. Second, what is the form of S_j, the source term in the continuity equation for trapped particles only? Implicitly we have already assumed that a source or sink term in the trapped particle continuity equations is important but its effect back upon the response and/or number of circulating particles shall be ignored in this approach. This is probably not too bad as long as $\varepsilon^{1/2}$ is small. What is small?

4. <u>The derivation</u>

First I shall break the trapped particle density into two parts:

$$n_{jt} = \sqrt{\varepsilon}\, n_{oj} + \tilde{n}_{jt} \tag{9}$$

where $\varepsilon^{1/2} n_{oj}$ is the fraction of particles which would be trapped in an isotropic distribution in equilibrium with a potential ϕ. Remember that ϕ is determined self consistently from the number densities of both circulating and trapped particles where the nature of each species response is taken into account; it is <u>not</u> an input parameter. Using $n_{oj} = n_o \exp(-\, e_j\phi/T_j)$ one finds that by adding circulating and trapped particles one has (take ions for example) for $\phi = 0$,

$$\underbrace{(1 - \sqrt{\varepsilon}\,)\, n_o(x)}_{\text{circulating}} + \underbrace{\sqrt{\varepsilon}\, n_o(x)}_{\text{trapped}} = n_o(x)$$

We shall for simplicity assume there is <u>no</u> equilibrium electric field present so that $n_o(x)$ = density of ions = density of electrons, locally i.e., about x.

In the presence of the potential ϕ one now must consider the response of the plasma. Both the circulating particles and the trapped particles will try to respond <u>along</u> the field lines, as best they can. So for circulating particles where one has already assumed that they will respond by setting up a quasi-static equilibrium in the presence ϕ , one has equation (8). For trapped particles one finds their response from solving

$$\frac{\partial n_{jt}}{\partial t} + \nabla \cdot n_{jt}\vec{V}_{jt} = S_{jt} \qquad (12)$$

where by $\frac{\partial}{\partial t} n_{jt}$ one means

$$\frac{\partial n_{jt}}{\partial t} = \sqrt{\varepsilon}\ \frac{\partial}{\partial t}\ n_{oj} + \frac{\partial \tilde{n}_{jt}}{\partial t} \quad .$$

Some discussion is in order here. As I have written it, equation (12) is fine provided that the term $\tilde{n}_{jt}$ includes in it two effects. First, it must give the perpendicular response of the trapped particles to the perturbation ($k\ \phi$), and second, one would think that it must also (somehow) contain the information that trapped particles can not completely respond "adiabatically" along the field lines (as the $\sqrt{\varepsilon}\ n_{oj}$ term would tend to imply) because they have a $\mu_j \partial B/\partial s$ force preventing them from doing so. To have the circulating particles treat the wave in a quasi-static manner we have implicitly assumed that $\omega/k_{\parallel}$ must be $\ll (T_j/m_j)^{1/2}$. This wave (known from more rigorous kinetic theory) has a frequency $\omega \ll \omega_{bj}$ where $\omega_{bj} \simeq \sqrt{\varepsilon}\ (T_j/m_j)^{1/2}/gR$. Therefore, in order for the trapped particles to respond "adiabatically" along $\vec{B}$ the parallel wavelength of the wave must be such that $\omega/k_{\parallel} \ll \sqrt{\varepsilon}\ (T_j/m_j)^{1/2}$ or $k_{\parallel} \gtrsim \frac{1}{qR_o}$ which implies it must have a parallel wavelength less than or equal to the "trapping" distance of the particles in the field gradient (i.e. $\lambda_{\parallel} < qR_o$). As $k_{\parallel} \to 0$ we get away from these "drift" waves and must begin to consider flute instabilites which depend on curvature more than density gradient effects. i.e., If $k_{\parallel}$ were $\ll 1/qR_o$ then the plasma would respond more hydrodynamically than adiabatically. i.e. wave in cold plasma case. In this limit one should not have a $\sqrt{\varepsilon}\ n_o \exp(-\ e_j\phi/T_j)$ term for the response of the trapped particles to $\partial\phi/\partial s$. In actuality, the more self consistent approach[s] takes this into account and as we already know $\phi(x,t)$ turns out to be peaked on the outside of the torus ($\vec{E} = -\nabla\phi = 0$ at outside however!).

In concluding this remark then one must confess that one can not happily resolve this problem within the context of a linear approach in a fluid theory and with an assumed isotropic pressure. The effects of a $\mu\nabla B$ force show up in a fluid equation context only in two places: 1) in the acceleration term $\vec{V}\cdot\nabla\vec{V}$ which is non-linear (i.e. needed for "trapping" of fluid) and, 2) as an anisotropic pressure where one can see from equation (6) that $\frac{P_\perp}{B}\frac{\partial B}{\partial s}$ is the $\mu\frac{\partial B}{\partial s}$ force. In Appendix A, I show that one obtains the Boltzmann factor equilibrium term (see equation (8)) by taking the plasma to be isothermal along the field lines. One could equally well take an anisotropic pressure and the double adiabatic equations of state $\frac{d}{dt}\left(\frac{P_\parallel B^2}{n^3}\right) = 0$ and $\frac{d}{dt}\left(\frac{P_\perp}{nB}\right) = 0$ and try to arrive at some other equation relating n_j to ϕ, B, $T_\perp$, and $T_\parallel$. Good Luck! In short, the extra work of assuming anisotropy, while physically more appealing, is mathematically unattractive and from the more rigorous kinetic theory results [5] one finds that the simple model where one assumes the trapped particles to have an adiabatic response term $\sqrt{\varepsilon}\, n_o \exp(-e_j\phi/T_j)$ and a response term $(\tilde{n}_{jt})$ due to $\vec{E}\times\vec{B}$ and $\vec{g}_j\times\vec{B}$ convection will be entirely adequate. This is the power of 20-20 hindsight and should <u>not</u> be considered immediately obvious. What do you want from a fluid theory -- perfection? We all will settle for simplicity!

With this realization we note that $\frac{\partial \tilde{n}_{jt}}{\partial t}$ can be expressed as

$$\sqrt{\varepsilon}\, n_o(x)\, \exp\left(-\frac{e_j\phi}{T_j}\right)\frac{\partial}{\partial t}\left(-\frac{e_j\phi}{T_j}\right) + \frac{\partial n_{jt}}{\partial t}$$

Taking $\tilde{n}_{jt}, \phi$ as $\sim\exp(ik_y y - i\omega t)$ one has

$$\frac{\partial n_{jt}}{\partial t} = i\omega\sqrt{\varepsilon}\, n_o \exp\left(-\frac{e_j\phi}{T_j}\right) - i\omega\tilde{n}_{jt} \qquad (13)$$

The cross field convective velocity taken from (4) and with reference to Figure #1 is

$$\vec{V}_j = -i\frac{k_y\phi}{B}\hat{x} - \frac{g_j}{\Omega_j}\hat{y} \qquad (14)$$

where one notes that for B = constant $\nabla\cdot\vec{V}_j = 0$, i.e., incompressible motion.

A quick word should be given here concerning my apparent neglect of the $\frac{\nabla p \times B}{e_j B^2}$ term of equation (4), and also why I did not consider the effect of collisions on the equations of motion of the trapped particles. The answer to the first part comes from considering the source of the $\nabla p \times \vec{B}$ term. We know that only true guiding center drifts (such as ∇B, curvature, etc.) can produce change separation and for $\frac{\partial}{\partial t}\rho e$ to be $\neq 0$ (ρe = charge density $\sum_j n_j \cdot e_j$) one requires $\nabla\cdot\vec{J}\neq 0$. $\vec{J}$ has two components [to order (Larmor radius/scale length)^2] and they are [6]

$$\vec{J} = \vec{J}_{g.c.} + \nabla \times \vec{M} \quad . \tag{15}$$

$\vec{J}_{g.c.}$ is produced in the model from $\vec{g}_j \times \vec{B}$ drift (since $\vec{B}$ = const.). $\nabla \times \vec{M}$ produces the $\nabla p \times \vec{B}$ term and since $\nabla\cdot(\nabla \times \vec{M}) = 0$ and can therefore not produce charge separation one knows that $\nabla p \times \vec{B}$ term need not be included in stability analysis. It is of course still needed to have equilibrium. If ∇B and curvature drift were present then $\nabla\cdot\left(\frac{\nabla p \times B}{B^2}\right)$ would give terms such as

$$\nabla\cdot\left(n\mu \frac{\vec{B} \times \nabla B}{B^2} + nmv_\parallel^2 \frac{\vec{B} \times (\hat{b}\cdot\nabla)\hat{b}}{B^2}\right) + \nabla\cdot(\nabla \times \vec{M}) \tag{16}$$

which is just the divergence of the guiding center drifts. I simply decided to model these effects into $\vec{g}_j$ and while $\nabla\cdot\vec{V}$ would not be zero if $\nabla B\neq 0$ this compressional effect is small in the case we consider here. The answer to the second point, i.e. neglecting collisions in equations of motion, can only really be answered by going to the kinetic theory, ordering the equations, and solving order by order. If one does so, one sees that the collisional effects are so "slow" ($\nu_{coll} < \omega_{bounce\ ion}$) that the particles pretty much undergo their motion unimpeded and the primary effect of collisions is to really populate or depopulate the trapped part of phase space. This effect indicates that ν_{ej} should enter into

S_j the source term in the continuity equations. Before finding S_j though let us go back and find $\nabla\cdot n_j \vec{V}_{jt}$ of each trapped species.

The divergence of the particle flux is found to be:

$$\nabla\cdot(n_j \vec{V}_j) = (-i \frac{k_y\phi}{B}\hat{x} + V_{gj}\hat{y}) \cdot \nabla(\sqrt{\varepsilon}\, n_{oj} + \tilde{n}_{jt})$$

where $\nabla\cdot\vec{V}_j = 0$ and $\vec{V}_{gj} = -g_j/\Omega_j\hat{y}$. Now

$$\nabla(\sqrt{\varepsilon}\, n_{oj}) = \sqrt{\varepsilon}\, n_{oj} \{\frac{1}{n_o} \frac{dn_o}{dx}\hat{x} - ik_y \frac{e_j\phi}{T_j}\hat{y}\}$$

and one has

$$\nabla\cdot(n_j \vec{V}_j) = -i\sqrt{\varepsilon}\, n_{oj}\cdot(\omega_{*j} + \omega_{gj}) \frac{e_j\phi}{T_j} + i\omega_{gj}\tilde{n}_{jt} \quad . \tag{17}$$

The continuity equation (linearited) for the j^{th} species of trapped particles becomes, using (2), (13), and (17)

$$-i(\omega-\omega_{gj})\tilde{n}_{jt} + i\sqrt{\varepsilon}\, n_o(\omega-\omega_{gj}-\omega_{*j}) \frac{e_j\phi}{T_j} = S_j \tag{18}$$

and we have only to find an appropriate form for S_j. K & P[1] ask us to consider the following argument: "In equilibrium the number of trapped particles must be the $\sqrt{\varepsilon}$ -th fraction of the total number of particles: $n_{jt} = \sqrt{\varepsilon}\, n_o$. If the number of trapped particles exceeds this value, they will be ejected from the confinement cone by the collisions. If their number is smaller, the opposite effect will take place and the transit particles will fill up the cone."

Since the fraction of velocity space through which trapped particles must scatter to become untrapped is $\mathcal{O}(\sqrt{\varepsilon})$, the effective collision frequency for these trapped particles is

$$\left(\frac{\pi/2}{\sqrt{\varepsilon}}\right)^2 (\nu_j)_{90°} \approx \frac{2.5}{\varepsilon} (\nu_j)_{90°}$$

which is taken to be essentially ν_j/ε .

And the "excess" of trapped particles above $\sqrt{\varepsilon}\, n_o$ is $n_{jt}-\sqrt{\varepsilon}\, n_o$ so that equation (18) becomes

$$-i(\omega-\omega_{gj})\tilde{n}_{jt} + i\sqrt{\varepsilon}\, n_o(\omega-\omega_{gj}-\omega_{*j}) \frac{e_j\phi}{T_j} = S_j \tag{19}$$

and now one defines S_j to be

$$\begin{aligned} S_j &= -\frac{\nu_j}{\varepsilon}(n_{jt} - \sqrt{\varepsilon}\, n_o) \\ &= -\frac{\nu_j}{\varepsilon}[\sqrt{\varepsilon}\, n_o \exp(-\frac{e_j\phi}{T_j}) + \tilde{n}_{jt} - \sqrt{\varepsilon}\, n_o] \\ &\simeq -\frac{\nu_j}{\varepsilon}(\tilde{n}_{jt} - \sqrt{\varepsilon}\, n_o \frac{e_j\phi}{T_j}) \quad . \end{aligned} \tag{20}$$

Solving for $\tilde{n}_{jt}$ one has

$$[-i(\omega-\omega_{gj}) + \frac{\nu_j}{\varepsilon}]\tilde{n}_{jt} = \sqrt{\varepsilon}\frac{e_j\phi}{T_j}[\frac{\nu_j}{\varepsilon} n_o - n_o i(\omega-\omega_{gj}-\omega_{*j})] \quad .$$

This becomes

$$\tilde{n}_{jt} = \sqrt{\varepsilon}\, n_o \frac{e_j\phi}{T_j} \frac{[(\omega-\omega_{gj}-\omega_{*j}) + i\nu_j/\varepsilon]}{\omega-\omega_{gj} + i\nu_j/\varepsilon} \tag{21}$$

Using the quasi-neutrality condition $\sum_j n_j e_j = 0$ one has

$$n_{ci} + n_{ti} = n_{ce} + n_{te}$$

or

$$(1-\sqrt{\varepsilon})n_o \exp(\frac{e\phi}{T_e}) + \sqrt{\varepsilon}\, n_o \exp(\frac{e\phi}{T_e}) + \tilde{n}_{et} = (1-\sqrt{\varepsilon})n_o \exp(-\frac{e\phi}{T_i}) + \sqrt{\varepsilon}\, n_o \exp(-\frac{e\phi}{T_i}) + \tilde{n}_{it}.$$

Expanding the exponential one has

$$n_o e\phi(\frac{1}{T_e} + \frac{1}{T_i}) = \tilde{n}_{it} - \tilde{n}_{et}$$

and using (21) one has

$$\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{T_i}+\frac{1}{T_e}\right)=\frac{1}{T_i}\left(\frac{\omega-\omega_{gi}-\omega_{*i}+i\nu_i/\varepsilon}{\omega-\omega_{gi}+i\nu_i/\varepsilon}\right)+\frac{1}{T_e}\left(\frac{\omega-\omega_{ge}-\omega_{*e}+i\nu_e/\varepsilon}{\omega-\omega_{ge}+i\nu_e/\varepsilon}\right) \tag{22}$$

If one took the time to solve the above equation* for ω you would find that $|\omega_{gi}|<<\omega$ and can therefore be neglected. The ν_j/ε in the <u>numerator</u> does not affect much the solution for ω so it may also be dropped (in the numerator only!). With these deletions we finally obtain

$$\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{T_e}+\frac{1}{T_i}\right)=\frac{1}{T_i}\,\frac{\omega-\omega_{*i}}{\omega+i\nu_i/\varepsilon}+\frac{1}{T_e}\,\frac{\omega-\omega_{*e}}{\omega+i\nu_e/\varepsilon}\quad . \tag{23}$$

To arrive at K & P result one must compare <u>my</u> definition of ω_* to their definition [1] $(\omega_{*j})_{K\&P}=k_y\frac{T_j}{|e_j|B}\,\frac{1}{n_o}\,\frac{dn_o}{dx}$. Doing this one finds that

$$(\omega_{*i})_{me}=(\omega_{*i})_{K\&P} \quad \text{and} \quad (\omega_{*e})_{me}=-(\omega_{*e})_{K\&P}$$

so (22) becomes using K & P definitions

$$\frac{1}{\sqrt{\varepsilon}}\left(\frac{1}{T_e}+\frac{1}{T_i}\right)=\frac{1}{T_i}\,\frac{\omega-(\omega_{*i})_{K\&P}}{\omega+i\nu_i/\varepsilon}+\frac{1}{T_e}\,\frac{\omega+(\omega_{*e})_{K\&P}}{\omega+i\nu_e/\varepsilon} \tag{24}$$

which is their equation (5.4). QED.

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*An excercise for the student.

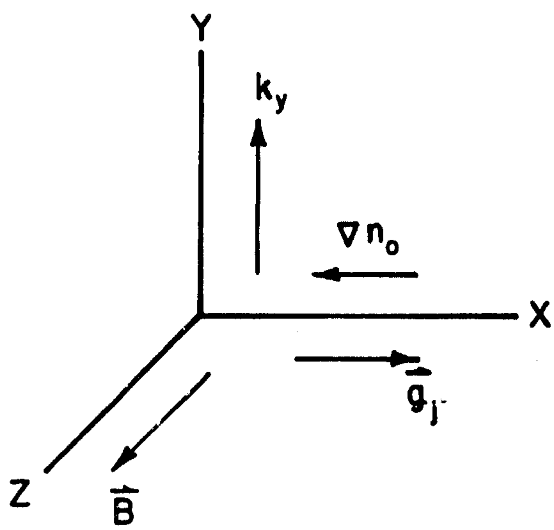

Fig. #i Geometry for TI mode

Appendix A

Take eqn. (5), assuming isotropic pressure, and ignoring inertial terms, and collisions one has

$$0 = -e_k \frac{\partial \phi}{\partial s} - \frac{1}{n_k} \frac{\partial}{\partial s} T_k n_k \tag{A1}$$

where s = distance along field line from an arbitrary reference point.

Now if one considers the plasma to be isothermal along a field line (which is a fairly reasonable expectation for fully ionized regions of the plasma) then one can take as an equation of state

$$0 = \frac{\partial}{\partial s}\left(\frac{n_k T_k}{n_k^{\gamma}}\right) = \frac{\partial}{\partial s} \frac{n_k T_k}{n_k} = \frac{\partial T_k}{\partial s} \tag{A2}$$

since isothermal $\Rightarrow \gamma = 1$. This ignoring of the inertial effects is valid provided $(\omega/k_{\parallel})$ of the wave is less than the characteristic thermal velocity of the particles. One has from the above eqns. (A1) and (A2)

$$0 = -\frac{e_k}{T_k} \frac{\partial \phi}{\partial s} - \frac{\partial}{\partial s} \ln(n_k)$$

or

$$n_k = n_{ok} \exp\left(-\frac{e_k \phi}{T_k}\right) \tag{A3}$$

but for circulating particles

$$n_{ok} = (1-\sqrt{\Sigma})n_o$$

Note in the opposite limit where $(\omega/k_{\parallel})$ of the wave is higher than the characteristic thermal velocity of the particles we can treat the plasma in the "hydrodynamic limit" (i.e., wave motion in cold plasma) and thus may ignore thermal effects. This gives

Appendix A (con't.)

$$V_{\parallel k} = \frac{e_k}{m_k} \frac{\phi}{\omega/k_{\parallel}} \quad .$$

It is usually the intermediate case where $\frac{\omega}{k_{\parallel}} \sim Vth$ that we must take both thermal effects and inertial terms into account. Landau damping effects thus must be considered for $\omega/k_{\parallel} \sim Vth$.

ACKNOWLEDGEMENT

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Bibliography

[1] Trapped Particles in Toroidal Magnetic Systems, by B. B. Kadomtsev, O. P. Pogutse, Nuc. Fus. 11, (1971) pg. 83-84.

[2] It should be noted that any waves present in the plasma of frequency $\omega > \Omega_j$ or of wavelength $\lambda_j < (\rho_{Larmor})_j$ can break the adiabaticity of μ_j, the magnetic moment of the particle. These typically high frequency instabilities (such as the loss cone instabilities) are not of direct interest to us in this calculation. They may be present however and should in "reality" be considered as an enhanced (or anamalous) scattering term for the trapped particles.

[3] The reader might be perplexed that besides the definition $\Sigma e_j n_j = 0$ as being the quasi-neutrality condition, one occasionally stumbles across another definition and that is $\nabla \cdot \vec{J} = 0$. They are essentially equivalent as can be seen below

$$\rho_e = \sum_j e_j n_j$$

and from the continuity of charge equation

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \vec{J} = 0$$

one sees that $\nabla \cdot \vec{J} = 0$ implies that

$$\frac{\partial}{\partial t} \sum_j e_j n_j = 0$$

Obviously, if $\Sigma e_j n_j = 0$ then $\nabla \cdot \vec{J} = 0$ is satisfied. What using $\nabla \cdot \vec{J} = 0$ does is allow for one to have an equilibrium (time independent) electric field in the plasma. (See Plasma Turbulence by B. B. Kadomtsev, pg. 8 for an example using $\nabla \cdot \vec{J} = 0$) where strictly speaking $\rho_e = 0$ does not allow this freedom. Another point also should be made here and that concerns the use of $\rho_e = 0$ or $\nabla \cdot \vec{J} = 0$. The use of either of these expressions does not mean that they are rigorously zero. One must have some charge separation or divergence of $\vec{J}$ in order for there to be a wave. The use of quasi-neutrality is thus only a good approximation because ρ_e is so small that $\nabla^2 \phi$ is negligible. One notes however that a current such as the magnetication current in a plasma $(\nabla \times \vec{M})$ gives $\nabla \cdot (\nabla \times M) = 0$ exactly so that $\nabla \times \vec{M}$ can not contribute to the charge separation which is responsible for the instability; thus it need not be included in calculating $\vec{J}$ to use in $\nabla \cdot \vec{J} = 0$.

[4] Dynamics of Charged Particles by B. Lehnert, North Holland Publishing Co., 1964, p. 118.

Bibliography (con't.)

[5] "Low Frequency Instabilities in Plasma with Magnetic Shear", N. T. Gladd, Ph.D. Thesis, University of Texas, Austin (1974).

[6] The Adiabatic Motion of Charged Particles by T. G. Northrop, Interscience Publishers, 1963, See Chapter 4 for outline of proof.