Thermalization of an Energetic Heavy Ion in Multispecies Plasmas

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These FDM's are preliminary and informal and as such may contain errors not yet eliminated.

Abstract

The thermalization of an energetic heavy ion in a multispecies plasma is studied assuming binary collisions. Several conclusions can be drawn from the results:

1) The slowing-down time is sensitive only to the electron density and temperature: $\tau_{SD} \sim T_e^{3/2}/N_e$. It is insensitive to the types of ions in the plasma (as long as charge neutrality is maintained) and insensitive to the temperatures of the ion species.

2) The total fraction of the test particle's energy going to the electrons, U_{Te}, is insensitive to the ion temperatures, the plasma density and the types of ions present in the plasma. It is sensitive only to the electron temperature.

3) If $U_{Ti} = 1 - U_{Te}$ represents the total fraction of the energy going to all ion species in the plasma, then a simple approximate expression for U_{Tj}, the fraction of the energy going to specie j, is given by:

$$U_{Tj} = \frac{Z_j N_j}{N_e} U_{Ti}$$

i.e., U_{Tj} is proportional to the charge density of specie j relative to the electron density.

I. Introduction

The results of Butler and Buckingham[1] are generalized to study the relaxation of a fast test ion in a uniform multispecies Maxwellian plasma. The plasma is assumed to be fully ionized and possess charge neutrality. Only binary coulomb collisions are assumed with cutoffs for large angle scattering and Debye shielding. No collective effects are considered and the effects of the magnetic field are neglected.

Shkarofsky, Johnston and Backynski[2] obtain the same results as Butler and Buckingham using the Fokker-Planck collision model. Conn[3] has applied their results in studying the thermalization of thermonuclear alpha particles in a D-T plasma (a lumped ion species of mass 2.5 and electrons), with densities and temperatures in the anticipated Tokamak reactor regime.

Sigmar and Joyce[4] use the Balescu-Lenard kinetic equation which includes Cerenkov radiation and Debye shielding automatically. They conclude that energy dependence and quantum effects for close collisions results in somewhat higher values of ln Λ. The effect of a magnetic field is also found to be very small. The results obtained here will be compared with those of Sigmar and Joyce where possible.

II. Energy Loss Equation

Using the assumption of binary collisions, the total rate of energy loss of the test particle may be found by summing over the species present in the plasma:

$$1)\quad \frac{d\bar{E}_T}{dt} = -8\pi^{1/2} \left(\frac{Ze^2}{4\pi\varepsilon_o}\right)^2 \sum_j \frac{Z_j^2\, N_j\, \ln\Lambda_j\, F(v/w_{tj})}{m_j w_{tj}}$$

$$2)\quad F(v/w_{tj}) = \frac{w_{tj}}{v} \int_o^{v/w_{tj}} e^{-y^2}\, dy - \left(1 + \frac{m_j}{M}\right) e^{-v^2/w_{tj}^2}$$

where $\overline{E}_T$ - mean test particle energy

M_j, Z_j, n_j - mass, charge and density of specie j

w_{tj} - thermal velocity of specie j = $\sqrt{2 kT_j/m_j}$

M,Z,v - mass, charge and velocity of test particle

$\Lambda_j = 1/\theta_{oj}$

θ_{oj} - cutoff angle for small angle scattering of test particle for specie j

The cutoff angle, θ_{oj}, is the larger of the classical and quantum mechanical determinations:

$$\theta_{oj} = \frac{b_{oj}(90°)}{\lambda d} \quad \text{classical}$$

$$= \frac{\lambda_j}{\lambda_d} \quad \text{quantum}$$

where: λd - Debye length

λ_j - center of mass wavelength in two body scattering

$b_{oj}(90°)$ - 90° impact parameter

$$\frac{1}{\lambda d} = \left[\frac{e^2}{2\varepsilon_o} \sum_j \frac{Z_j^{\,2} N_j}{kT_j} \right]^{1/2}$$

$$\lambda_j = \frac{\hbar (M + m_j)}{M m_j \left| \overline{v} - \overline{w}_j \right|}$$

$$b_{oj}(90°) = \frac{Z Z_j e^2 (M + m_j)}{4\pi\varepsilon o \, M m_j \left| \overline{v} - \overline{w}_j \right|^2}$$

To determine whether to use the classical or quantum mechanical cutoff, calculate $(bo/\lambda)^2$ for each specie. Let $M = A_T m_p$ where A_T is the mass no. of the test particle and m_p is the proton mass. Then, for plasma ion specie j :

$$\left(\frac{bo}{\lambda}\right)_j^2 = 24.7\,\frac{Z_j^2 Z^2 A_T}{E_T} \qquad E_T\,(\mathrm{kev})$$

and for the plasma electrons:

$$\left(\frac{bo}{\lambda}\right)_e^2 = 13.5 \times 10^{-3}\,\frac{Z^2}{T_e} \qquad T_e\,(\mathrm{kev})$$

where it has been assumed that $|\bar{v}-\bar{w}_e| \simeq w_{Te}$ for electrons and $|\bar{v}-\bar{w}_j| \simeq v$ for ion specie j.

For the electrons, the quantum determination should be used since for test particles of low Z and electron temperatures of several keV then $b_{oe} << \lambda_e$. For the jth ion specie use the quantum determination for test particle energies greater than $25 Z_j^2 Z^2 A_T$ keV and the classical determination for lower test particle energies:

3) $$\Lambda_e^2 = \frac{4\,me\varepsilon o\,kTe}{\hbar^2 e^2}\left[\frac{Ne}{kTe} + \sum_{ions}\frac{Z_j^2 N_j}{kT_j}\right]^{-1}$$

$$= 2.9 \times 10^{33}\,Te\left[\frac{Ne}{Te} + \sum_{ions}\frac{Z_j^2 N_j}{T_j}\right]^{-1}$$

4) $$\Lambda_j^2 = \frac{4 m_j^2\,Meo\,E_T}{\hbar^2 e^2 (M + m_j)^2}\left[\frac{Ne}{kTe} + \sum_{ions}\frac{Z_j^2 N_j}{kT_j}\right]^{-1}$$

$$= 5.34\times10^{36}\,\frac{E_T A_T A_j^2}{(A_T + A_j)^2}\left[\frac{Ne}{Te} + \sum_{ions}\frac{Z_j^2 N_j}{T_j}\right]^{-1} \qquad E_T > 25 Z_j^2 Z^2 A_T\,\mathrm{keV}$$

$$5)\qquad \Lambda_j^2 = \frac{8\pi^2 \varepsilon_o^3 m_j^2 E_T^2}{Z^2 Z_j^2 e^6 (M + m_j)^2} \left[\frac{Ne}{kTe} + \sum_{ions} \frac{Z_j^2 N_j}{kT_j} \right]^{-1}$$

$$= \frac{2.09\text{x}10^{35} A_j^2 E_T^2}{Z^2 Z_j^2 (A_j + A_T)^2} \left[\frac{Ne}{Te} + \sum_{ions} \frac{Z_j^2 N_j}{T_j} \right]^{-1} \qquad E_T < 25 Z^2 Z_j^2 A_T \text{ keV}$$

where the units are m^{-3} and keV.

Plots of ln Λ for deuterium, tritium, helium and argon are shown in Fig. 1 for alpha test particles as a function of the test particle energy and the plasma parameter $\gamma = [{}^{Ne}/Te + \sum_{ions} N_j Z_j^2/T_j]^{-1}$. Also included are plots of ln Λe vs. Te for various values of γ.

III. Numerical Solution of the Slowing Down Problem

Evaluating the constants in equations 1) and 2) and letting $v^2 = 2\bar{E}_T/M$ gives:

$$6)\ \frac{d\bar{E}_T}{dt} = -6.41\text{x}10^{-18} Z^2 \sum_j Z_j^2 N_j \ln\Lambda_j \left(\frac{1}{A_j T_j}\right)^{1/2} \left\{ \left(\frac{A_T T_j}{A_j \bar{E}_T}\right)^{1/2} \frac{\sqrt{\pi}}{2} \text{erf} \left[\left(\frac{A_j \bar{E}_T}{A_T T_j}\right)^{1/2} \right] - \left(1 + \frac{A_j}{A_T}\right) e^{-A_j \bar{E}_T / A_T T_j} \right\}$$

Eqn. 6) can be solved numerically to give $\bar{E}_T(t)$. $\ln\Lambda_j$ for ions is dependent on the test particle energy but may be averaged over the slowing down range to simplify the calculations. Letting $x_j^2 = A_j \bar{E}_T / A_T T_j$ then:

$$7)\ \frac{d\bar{E}_T}{dt} = -6.41\text{x}10^{-18} Z^2 \sum_j Z_j^2 N_j \ln\Lambda_j \left(\frac{1}{A_j T_j}\right)^{1/2} \left\{ \frac{1}{xj} \frac{\sqrt{\pi}}{2} \text{erf}(x_j) - \left(1 + \frac{A_j}{A_T}\right) e^{-x_j^2} \right\}$$

Abramowitz and Stequn[5] give the following approximation to the error function, erf(x):

$$\mathrm{erf}(x) = 1 - (a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5)e^{-x^2} + \varepsilon(x)$$

$t = 1/(1+px)$

$p = .3275911$

$a_1 = .254829592$

$a_2 = -.284496736$

$a_3 = 1.421413741$

$a_4 = -1.453152027$

$a_5 = 1.061405429$

$$|\varepsilon(x)| \leq 1.5 \times 10^{-7}$$

$$F_j(\bar{E}_T) = 6.41\mathrm{x}10^{18}\, Z^2\, Z_j^{\,2}\, n_j \ln\Lambda j \left(\frac{1}{A_jT_j}\right)^{1/2} \left\{ \frac{1}{xj} \frac{\sqrt{\pi}}{2}\, \varepsilon\mathrm{rf}(x_j) - \left(1 + \frac{A_j}{A_T}\right) e^{-x_j^2} \right\}$$

and let $E_T^{\,i}$ denote the test particle energy after i time steps

$$\frac{d\bar{E}_T}{dt} = \frac{E_T^{\,i+1} - E_T^{\,i}}{\Delta t} = -\sum_j F_j(\bar{E}_T^{\,i})$$

8) $$\bar{E}_T^{\,i+1} = \bar{E}_T^{\,i} - \Delta t \sum_j F_j\,(\bar{E}_T^{\,i})$$

The fraction of the test particle energy being given to each specie, U_{Tj}, can be found from F_j:

$$U_{Tj}\,(t) = \frac{\int_o^t F_j dt'}{\sum_j \int_o^t F_j dt'}$$

$$\sum_j \int_o^t F_j dt' = -\int_o^t \frac{d\bar{E}_T}{dt'}\, dt' = -\bar{E}_T\,(t) + \bar{E}_{To} = \bar{E}_{To} - \bar{E}_T$$

$$\int_o^t F_j dt' = \sum_{i=o}^{N} F_j(\bar{E}_T^{\,i})\,\Delta t$$

9) $$U_{Tj}(\bar{E}_T^{\,N}) = \frac{\sum_{i=o}^{N} F_j(\bar{E}_T^{\,i})\,\Delta t}{\bar{E}_{To} - \bar{E}_T^{N}}$$

IV. Approximate Analytic Solution to the Slowing Down Problem.

Two approximations can be made for $F(v/w_{tj})$ given by eqn. 2). First, for $v/w_{tj} \geq 2$

$$\int_o^{v/w_{tj}} e^{-y^2} dy = \frac{\sqrt{\pi}}{2} \text{ erf } (v/w_{tj}) \cong \frac{\sqrt{\pi}}{2}$$

$$e^{-v^2/w_{tj}^2} \sim 0$$

10). $$F(v/w_{tj}) \sim \frac{w_{tj}}{v} \frac{\sqrt{\pi}}{2}$$

From Fig. 2 it can be seen that these approximations are valid to within a few percent. Furthermore $v/w_{tj} \geq 2$ will be assumed to be valid for all ion species, j.

In the second approximation, for $v/w_{tj} << 1$, the exponentials in $F(v/w_{tj})$ can be expanded in a power series. This approximation is valid for electrons:

$$F(v/w_{te}) \simeq \frac{w_{te}}{v} \left(\frac{v}{w_{te}} - \frac{v^3}{3w_{te}^3}\right) - \left(1 + \frac{me}{M}\right) \left(1 - \frac{v^2}{w_{te}^2} + \dots \right)$$

$$\simeq \frac{2v^2}{3w_{te}^2} - \frac{me}{M}$$

11). $$F(v/w_{te}) \simeq \frac{2}{3} \frac{v^2}{w_{te}^2}$$

Figure 2) shows that the accuracy of keeping only the first two terms of the expansions for erf (x) and e^{-x^2} is within a few percent for $v/w_{te} \leq .5$. The most restrictive approximation comes in dropping the me/M term which requires $E_T >> 3/2$ Te but dropping this term leads to a simple analytic solution of the slowing down process.

Using eqns 10) and 11) in eqn. 1) yields an analytic equation for a heavy energetic test particle slowing down in a multispecies plasma:

$$\frac{d\bar{E}_T}{dt} = -8\pi^{1/2} \left(\frac{Ze^2}{4\pi\varepsilon o}\right)^2 \left[\sum_{ions} \frac{Z_j^2 \; N_j \ln\Lambda_j}{m_j \; v} \frac{\sqrt{\pi}}{2} + \frac{N e \ln\Lambda e}{me} \frac{2}{3} \frac{v^2}{w_{te}^{\;3}} \right]$$

Using $\bar{E}_T = 1/2\ Mv^2$ and multiplying the above equation by v gives:

$$\frac{d}{dt} (\bar{E}_T^{\;3/2}) = -12 \left(\frac{M\pi}{2}\right)^{1/2} \left(\frac{Ze^2}{4\pi\varepsilon o}\right)^2 \left[\frac{\sqrt{\pi}}{2} \sum_{ions} \frac{Z_j^{\;2} N_j \ln \Lambda j}{m_j} + \frac{2}{3} \text{Ne} \ln\Lambda e \left(\frac{me}{M}\right)^{1/2} \frac{\bar{E}_T^{\;3/2}}{Te^{3/2}} \right]$$

12.) $$\frac{d}{dt} (\bar{E}_T^{\;3/2}) = -1.5\text{x}10^{-19} \frac{Z^2}{A_T} \left[57.\ A_T^{\;3/2} \sum_{ions} \frac{Z_j^{\;2} N_j \ln\Lambda_j}{A_j} + \text{Ne}\ln\Lambda e \frac{\bar{E}_T^{\;3/2}}{Te^{3/2}} \right]$$

$\bar{E}_T$, Te - (keV)

N - (m^{-3})

Z, A_T - charge and mass of test particle

Z_j, A_j - charge and mass of plasma ion specie j.

The test particle energy at which the rate of energy loss to the electrons is equal to the rate of energy loss to all ion species will be defined as E_{crit}.

$$57\, A_T^{3/2} \sum_{ions} \frac{Z_j^{2} N_j \ln\Lambda_j}{A_j} = Ne \ln\Lambda e \frac{E_{crit}^{3/2}}{Te^{3/2}}$$

13). $$E_{crit} = 14.8\ Te\ A_T \left[\frac{1}{Ne\ln\Lambda e} \sum_{ions} \frac{Z_j^{2} N_j \ln\Lambda_j}{A_j}\right]^{2/3}$$

Note that E_{crit} is a weak function of the test particle energy through its dependence on $\ln\Lambda_j$ and can be assumed constant by choosing values of $\ln\Lambda_j$ at some average test particle energy. The slowing down equation can then be written as:

14). $$\frac{d(\bar{E}_T^{3/2})}{dt} = -1.5\text{x}10^{-19} \frac{Z^2}{A_T} \frac{Ne\ln\Lambda e}{Te^{3/2}} \left[E_{crit}^{3/2} + \bar{E}_T^{3/2} \right]$$

In eqn. 13) define β_j and β such that the contributions from each ion specie may be evaluated:

15). $$\beta_j = \frac{57.A_T^{3/2}}{Ne \ln \Lambda e} \frac{Z_j^{2} N_j \ln \Lambda_j}{A_j}.$$

16). $$\beta = \sum_j \beta_j$$

17). $$E_{crit} = \beta^{2/3}\ Te$$

18). $$\frac{d(\bar{E}_T^{3/2})}{dt} = +1.5\text{x}10^{-19} \frac{Z^2}{A_T} \frac{Ne\ln\Lambda e}{Te^{3/2}} \left[Te^{3/2} \sum_j \beta_j + \bar{E}_T^{3/2} \right]$$

β_j/β then represents the ratio of the test particle energy going to specie j to the test particle energy going to all ion species and is independent of the test particle energy.

Now define a parameter τ as a characteristic relaxation time for the energetic test particles.

19). $$\frac{1}{\tau} \equiv 1.5 \times 10^{-19} \frac{Z^2}{A_T} \frac{Ne\ln\Lambda e}{Te^{3/2}}$$

Eqn. 14) then has the form:

20). $$\frac{d(\bar{E}_T^{\ 3/2})}{dt} = -\frac{1}{\tau}\left(E_{crit}^{\ 3/2} + \bar{E}_T^{\ 3/2}\right)$$

with the solution:

21). $$\bar{E}_T^{\ 3/2}(t) = E_{To}^{\ 3/2}\, e^{-t/\tau} - E_{crit}^{\ 3/2}\left(1 - e^{-t/\tau}\right)$$

Equation 21) is the same result obtained by Butler and Buckingham except that τ and E_{crit} have been generalized to the multispecies case.

To find the slowing-down time, solve eqn. 21) for t and define $t = \tau_{SD}$ at $\bar{E}_T = Te$.

$$\tau_{SD} \simeq \tau \ln\left(\frac{E_{To}^{\ 3/2} + E_{crit}^{3/2}}{Te^{3/2} + E_{crit}^{3/2}}\right)$$

Since $E_{crit}^{\ 3/2} >> Te^{3/2}$, τ_{SD} can be simplified

22). $$\tau_{SD} \simeq \tau \ln\left(1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2}\right)$$

Where E_{To} is the initial energy of the test ion.

To find the fraction of the test particle energy going to each ion specie start with equation 20).

$$\frac{d(E_T^{\ 3/2})}{dt} = -\frac{1}{\tau}\left(E_{crit}^{\ 3/2} + \bar{E}_T^{\ 3/2}\right)$$

$$\frac{d\bar{E}_T}{dt} = -\frac{2}{3\tau}\left(\frac{E_{crit}^{\ 3/2}}{\bar{E}_T^{\ 1/2}} + \bar{E}_T\right)$$

23). $$\frac{d\bar{E}_T}{dt} = -\frac{2}{3\tau}\left(\sum_j \frac{\beta_j \; Te^{3/2}}{\bar{E}_T^{\;1/2}} + \bar{E}_T\right)$$

Define the contribution from each ion specie j as $d\bar{E}_T/dt)_j$ and the contribution from all ion species as $d\bar{E}_T/dt)_{ions}$ then:

24). $$\left.\frac{d\bar{E}_T}{dt}\right)_j = -\frac{2}{3\tau}\beta_j \frac{Te^{3/2}}{\bar{E}_T^{\;1/2}}$$

25). $$\left.\frac{d\bar{E}_T}{dt}\right)_{ions} = -\frac{2}{3\tau}\beta \frac{Te^{3/2}}{\bar{E}_T^{\;1/2}}$$

and similarly for electrons:

26). $$\left.\frac{d\bar{E}_T}{dt}\right)_e = -\frac{2}{3\tau}\bar{E}_T$$

The fraction of test particle energy given to specie j after time t, U_{Tj}. is then:

27). $$U_{Tj}(t) = \frac{\int_o^t d\bar{E}_T/dt')_j dt'}{\int_o^t (d\bar{E}_T/dt')\, dt'}$$

28). $$U_{Tions}(t) = \sum_{ions} U_{Tj}(t)$$

29). $$U_{Te}(t) = 1 - U_{Tions}(t)$$

Where $U_{Tions}(t)$ represents the fraction going to all ion species after a time t. Using eqn. 21) to relate t and $\bar{E}_T$, $U_{Tions}(\bar{E}_T)$ and $U_{Tj}(\bar{E}_T)$ can be found:

$$U_{Tions}(\bar{E}_T) = \frac{1}{\bar{E}_{To} - \bar{E}_T}\int_{\bar{E}_T}^{\bar{E}_{To}} \frac{E_{crit}^{\;3/2}}{\bar{E}_T'^{\;3/2} + E_{crit}^{\;3/2}}\, d\bar{E}_T'$$

30). $$U_{Tions}(\bar{E}_T) = \frac{\beta\ Te^{3/2}}{E_{To} - \bar{E}_T} \int_{E_T}^{E_{To}} \frac{d\ \bar{E}_T}{\bar{E}_T'^{\,3/2} + E_{crit}^{\,3/2}}$$

$$U_{Tj}(\bar{E}_T) = \frac{1}{E_{To} - \bar{E}_T} \int_{E_T}^{E_{To}} \frac{\beta_j\ Te^{3/2}\ d\bar{E}_T'}{\bar{E}_T'^{\,3/2} + E_{crit}^{3/2}}$$

31). $$U_{Tj}(\bar{E}_T) = \frac{\beta_j}{\beta} U_{Ti}(\bar{E}_T) \quad .$$

Eqn. 30) has the solution :

32). $$U_{Tions}(\bar{E}_T) = \frac{1}{3} \frac{E_{crit}}{(E_{To} - \bar{E}_T)} \left[\ln \left(\frac{E_{crit} - (E_{crit}\ E_{To})^{1/2} + E_{To}}{E_{crit} - (E_{crit}\ \bar{E}_T)^{1/2} + \bar{E}_T} \quad \frac{(E_{crit}^{1/2} + \bar{E}_T^{\,1/2})^2}{(E_{crit}^{1/2} + E_{To}^{1/2})^2} \right) \right.$$

$$\left. + 2\sqrt{3}\ \tan^{-1} \left(\frac{2\ E_{To}^{1/2} - E_{crit}^{1/2}}{\sqrt{3}\ E_{crit}^{1/2}} \right) - 2\sqrt{3}\ \tan^{-1} \left(\frac{2\ \bar{E}_T^{\,1/2} - E_{crit}^{1/2}}{\sqrt{3}\ E_{crit}^{1/2}} \right) \right] \quad .$$

When $\bar{E}_T = Te \ll E_{crit}$; $E_{To} \gg Te$, then:

33). $$U_{Tions} = \frac{1}{3} \frac{E_{crit}}{E_{To}} \left[\ln \left(\frac{E_{crit} - E_{crit}^{1/2}\ E_{To}^{1/2} + E_{To}}{E_{crit} + 2\ E_{crit}^{1/2}\ E_{To}^{1/2} + E_{To}} \right) \right.$$

$$\left. + 2\sqrt{3}\ \tan^{-1} \left(\frac{2\ E_{To}^{1/2} - E_{crit}^{1/2}}{\sqrt{3}\ E_{crit}^{1/2}} \right) + \frac{\sqrt{3}\pi}{3} \right] \quad .$$

Note that the expression for E_{crit} depends on all species in the plasma. However, this dependence is weak since $Z_j/A_j \sim 1/2$, $\ln \Lambda_j \sim 20$ and $\ln \Lambda e \sim 17$ for thermonuclear conditions. For charge neutrality $ne = \sum_j Z_j n_j$ so that

$$\bar{Z} \equiv \sum_j \frac{Z_j^2 N_j \ln \Lambda j}{A_j \, Ne \ln \Lambda e} \sim .6$$

For a 50-50 DT plasma:

34). $\bar{Z} \simeq .49$

$$E_{crit} \simeq 9.2 \, A_T \, Te$$

The slowing down time, τ_{SD}, given by eqn. 22) is then very insensitive to the ion species in the plasma since E_{crit} enters through a logarithmic expression. U_{Tions} given by eqn. 33) is independent of both the plasma ion species and the plasma density.

Assuming $Z_j \sim A_j/2$ and $\ln\Lambda_j = 20$, $\ln\Lambda e = 17$, a simplified expression for β_j is found:

35). $$\beta_j \simeq 33.5 \, A_T^{3/2} \, \frac{Z_j N_j}{Ne}$$

$$\beta = \sum_j \beta_j \simeq 33.5 \, A_T^{3/2}$$

$$\frac{\beta_j}{\beta} \simeq \frac{Z_j N_j}{Ne}$$

36). $$U_{Tj} \simeq \frac{Z_j N_j}{Ne} \, U_{Tions}$$

Eqn. 36) shows that the fraction of the test particle energy going to ion specie j is proportional to the charge density of specie j relative to the electron density of the plasma.

It can also be noted that for low enough electron temperatures and high enough initial energies for the test particle such that $E_{To} >> E_{crit}$

eqn. 33) for a primarily D-T plasma reduces To:

$$U_{Tions} \simeq \frac{5\sqrt{3}}{9} \quad \frac{E_{crit}}{E_{To}}$$

$$\simeq 8.75 \quad \frac{A_T \, T_e}{E_{To}}$$

$$\frac{E_{To}}{A_T} \sim \frac{1}{vo^2}$$

$$U_{Tions} \sim \frac{Te}{vo^2}$$

i.e., higher electron temperatures and lower initial test particle velocities increase the fraction of energy given to the ions. For the same energy, higher mass particles give more of their energy to the ions.

V. Examples of Thermonuclear Alpha Particle Thermalization

Figure 3 contains plots of the mean alpha energy as a function of time for $E_{\alpha o}$ = 3.5 MeV in a D-T plasma of density $10^{20}\ m^{-3}$. For Te = 15 keV both the approximate analytic solution, eqn. 21), and the more precise numerical solution, eqn. 8), are shown. The differences between the two results is very small. Also shown is a Te = 30 KeV case using the approximate analytic solution.

Figure 4 gives plots of $U_{\alpha e}$, $U_{\alpha D}$, $U_{\alpha T}$ and $U_{\alpha i} = U_{\alpha D} + U_{\alpha T}$ for Ne = $10^{20}\ m^{-3}$ using eqn. 32). The plasma is a 50-50 mixture of deuterium and tritium with no impurities. The results of Sigmar and Joyce[4)] are also indicated for comparison. The agreement with the more exact treatment of Sigmar and Joyce is very good for T_e greater than 30 KeV (within 5%), while for Te less than 30/keV the agreement is reasonable.

Figure 5 indicates the effects of impurities (argon and helium) in the plasma. For $Ne = 10^{20}\ m^{-3}$, $N_D = N_T$, $N_{Ar} = .01\ (N_D+N_T)$ and $N_{He} = .4\ (N_D+N_T)$, the fraction of energy given to the impurities is significant although the densities are low. This shows the charge dependence of the slowing down. The total fraction of the alpha particles energy going to all ion species, $U_{\alpha ions}$ is almost unchanged when the impurities are added.

It can also be noted that the ion-ion relaxation time is much shorter than τ_{ei} even for very heavy impurities. Hence the energy absorbed by the non-hydrogen isotopes will quickly go to heat the deuterium and tritium. All the ions can then be treated as being in equilibrium at the same temperature.

VI. The Average Energy of an Energetic Test Particle Over the Slowing Down Damage

The average energy of a test particle white slowing down can be found by time averaging the energy. Using the results of section IV a relatively simple expression can be found.

37.) $$\langle \bar{E}_T \rangle = \frac{1}{\tau_{SD}} \int_o^{\tau_{SD}} \bar{E}_T(t)\, dt$$

From eqn. 21) the energy and time can be related and the integral converted to an integration over energy.

$$\bar{E}_T^{3/2}(t) = E_{To}^{3/2}\, e^{-t/\tau} - E_{crit}^{3/2}\,(1 - e^{-t/\tau})$$

38.) $$t = \tau \ln\left(\frac{E_{To}^{3/2} + E_{crit}^{3/2}}{\bar{E}_T^{3/2} + E_{crit}^{3/2}} \right)$$

39.) $$dt = \frac{\tau}{\bar{E}_T^{3/2} + E_{crit}^{3/2}} \frac{3}{2} \bar{E}_T^{1/2} d\bar{E}_T$$

40.) $$\langle \bar{E}_T \rangle = -\frac{3}{2} \frac{\tau}{\tau_{SD}} \int_{E_{To}}^{\bar{E}_f} \frac{\bar{E}_T^{3/2} \, d\bar{E}_T}{\bar{E}_T^{3/2} + E_{crit}^{3/2}}$$

where $\bar{E}_f$ represents the final energy of the test particle and τ_{SD} is given by eqn. 22).

41.) $$\langle \bar{E}_T \rangle = \frac{3}{2} \frac{1}{\ln \left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2} \right]} \int_{\bar{E}_f}^{E_{To}} \frac{\bar{E}_T^{3/2} \, d\bar{E}_T}{\bar{E}_T^{3/2} + E_{crit}^{3/2}}$$

For a constant source of particles at energy E_{To} the distribution function for their mean energies, $\bar{E}_T$, is found from eqn. 41).

42.) $$f(\bar{E}_T) = \frac{3}{2} \frac{S}{\ln \left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2} \right]} \frac{\bar{E}_T^{1/2}}{\bar{E}_T^{3/2} + E_{crit}^{3/2}}$$

where S represents the source strength.

43.) $$\langle \bar{E}_T \rangle = \frac{1}{S} \int_{\bar{E}_f}^{E_{To}} \bar{E}_T \, f(\bar{E}_T) \, d\bar{E}_T$$

Eqn. 41) can be broken into two integrals by letting $\bar{E}_T^{3/2} \rightarrow \bar{E}_T^{3/2} + E_{crit}^{3/2} - E_{crit}^{3/2}$

$$\langle\bar{E}_T\rangle = \frac{3}{2}\,\frac{1}{\ln\left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2}\right]}\left[\int_{\bar{E}_f}^{E_{To}} d\bar{E}_T - E_{crit}^{3/2}\int_{\bar{E}_f}^{E_{To}} \frac{d\bar{E}_T}{\bar{E}_T^{3/2} + E_{crit}^{3/2}}\right]$$

The second integral was evaluated in section IV. From eqn. 30) it can be written as

$$E_{crit}^{3/2}\int_{\bar{E}_f}^{E_{To}} \frac{d\bar{E}_T}{\bar{E}_T^{3/2} + E_{crit}^{3/2}} = (E_{To} - \bar{E}_f)\, U_{Tions}$$

$$\langle\bar{E}_T\rangle = \frac{3}{2}\,\frac{(E_{To} - \bar{E}_f)}{\ln\left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2}\right]}\,(1 - U_{Tions})$$

44.) $$\langle\bar{E}_T\rangle = \frac{3}{2}\,\frac{(E_{To} - \bar{E}_f)\, U_{Te}}{\ln\left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2}\right]}$$

Define $$F(\bar{E}_T/E_{crit}) \equiv \frac{(\bar{E}_T/E_{crit})^{1/2}}{1 + (\bar{E}_T/E_{crit})^{3/2}}$$

then $$f(\bar{E}_T) = F(\bar{E}_T/E_{crit})\,\frac{3}{2}\,\frac{S}{E_{crit}}\,\frac{1}{\ln\left[1 + \left(\frac{E_{To}}{E_{crit}}\right)^{3/2}\right]}$$

F gives the shape of the distribution function for mean energies and is plotted in Figures 6 and 7. The maximum occurs at $\bar{E}_T = .63\, E_{crit}$.

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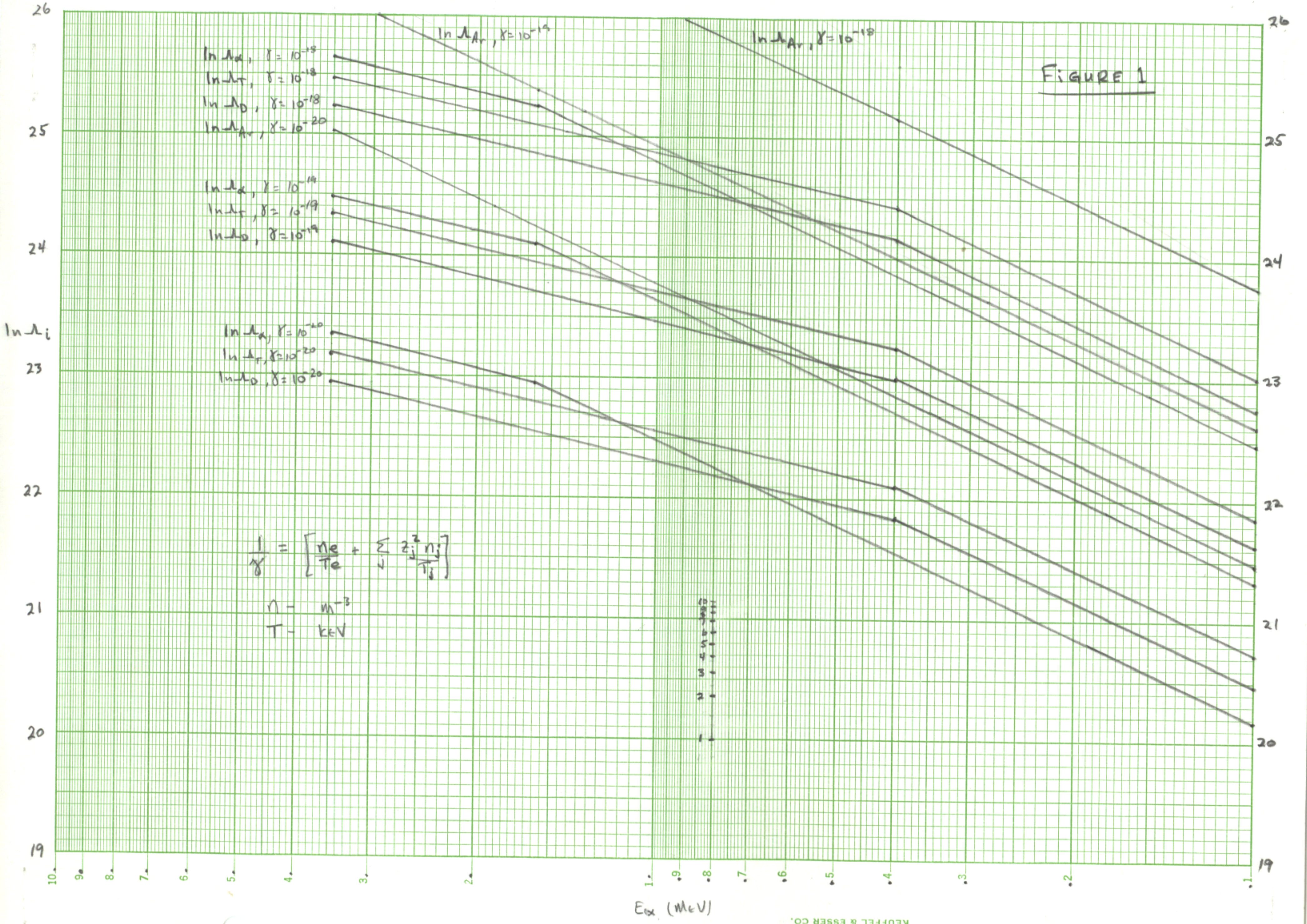
FIGURE 1
$\ln \Lambda_{Ar}$, $\gamma = 10^{-18}$
$\ln \Lambda_{Ar}$, $\gamma = 10^{-19}$
$\ln \Lambda_\alpha$, $\gamma = 10^{-18}$
$\ln \Lambda_T$, $\gamma = 10^{-18}$
$\ln \Lambda_D$, $\gamma = 10^{-18}$
$\ln \Lambda_{Ar}$, $\gamma = 10^{-20}$
$\ln \Lambda_\alpha$, $\gamma = 10^{-19}$
$\ln \Lambda_T$, $\gamma = 10^{-19}$
$\ln \Lambda_D$, $\gamma = 10^{-19}$
$\ln \Lambda_\alpha$, $\gamma = 10^{-20}$
$\ln \Lambda_T$, $\gamma = 10^{-20}$
$\ln \Lambda_D$, $\gamma = 10^{-20}$
$\frac{1}{\gamma} = \left[\frac{n_e}{T_e} + \sum_j \frac{z_j^2 n_j}{T_j} \right]$
n – m^{-3}
T – keV
$\ln \Lambda_i$
E_α (MeV)
26
25
24
23
22
21
20
19
10.
9.
8.
7.
6.
5.
4.
3.
2.
1.
.9
.8
.7
.6
.5
.4
.3
.2
.1
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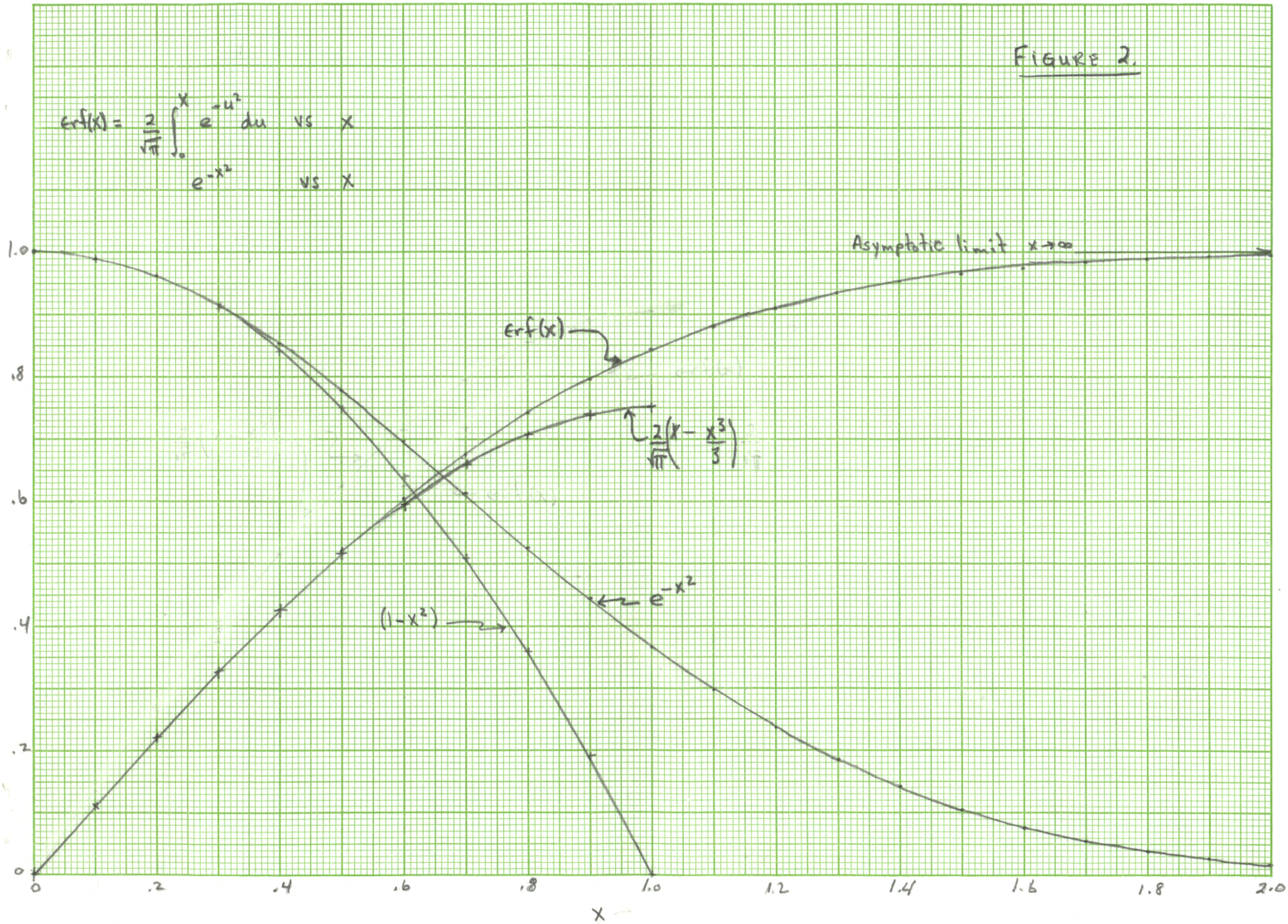
FIGURE 2.
$Erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$ vs X
e^{-x^2} vs X
Asymptotic limit $x \to \infty$
Erf(x)
$\frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3}\right)$
e^{-x^2}
$(1-x^2)$
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
1.2
1.4
1.6
1.8
2.0
X

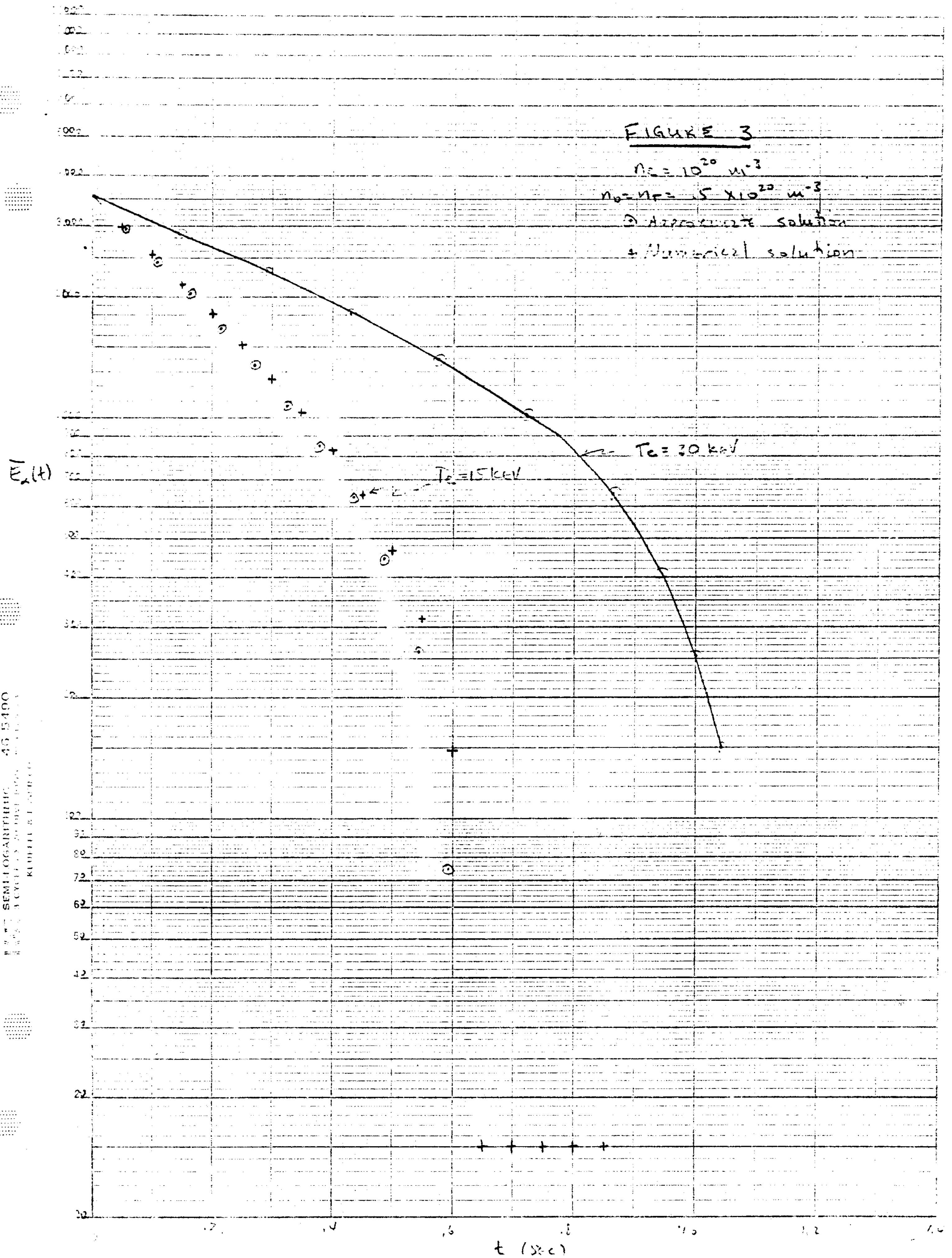
FIGURE 3
$n_e = 10^{20}$ m^{-3}
$n_D = n_T = .5 \times 10^{20}$ m^{-3}
⊙ Approximate solution
+ Numerical solution
T_e = 30 keV
T_e = 15 keV
$\bar{E}_\alpha(t)$
t (sec)

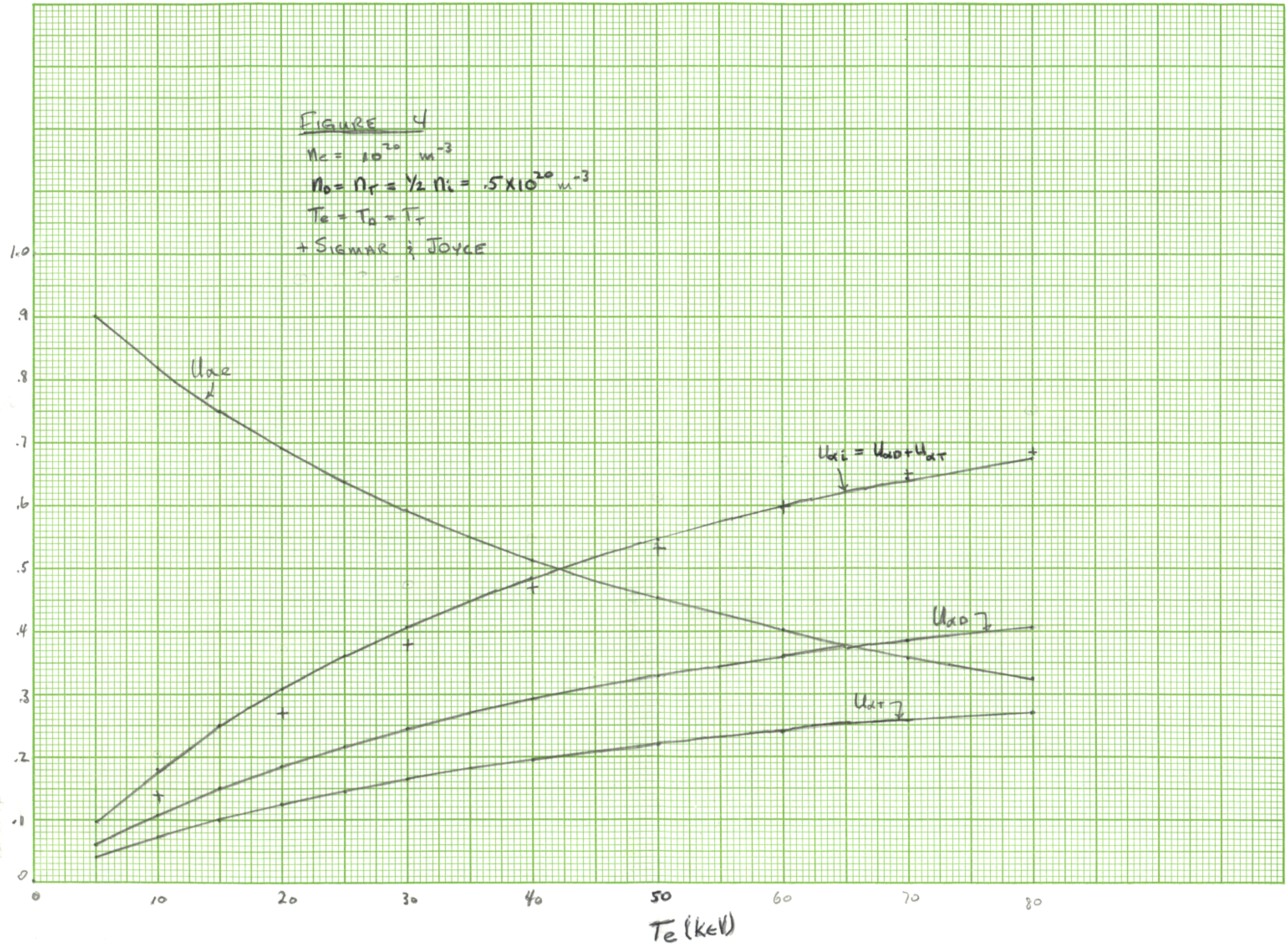
FIGURE 4
$n_e = 10^{20}\ m^{-3}$
$n_D = n_T = \frac{1}{2} n_i = .5 \times 10^{20}\ m^{-3}$
$T_e = T_D = T_T$
+ SIGMAR & JOYCE
$U_{\alpha e}$
$U_{\alpha i} = U_{\alpha D} + U_{\alpha T}$
$U_{\alpha D}$
$U_{\alpha T}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
10
20
30
40
50
60
70
80
Te (keV)

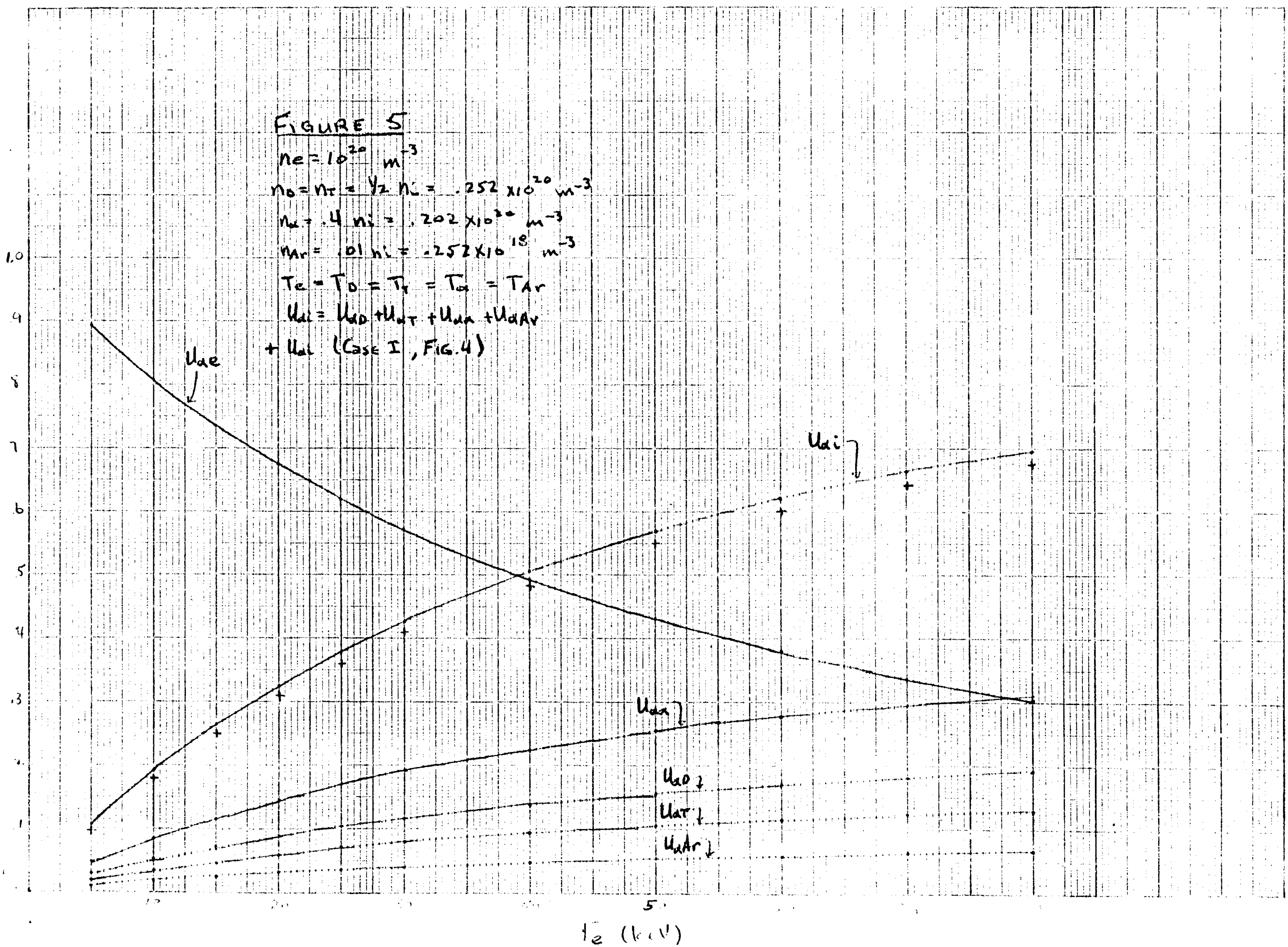
FIGURE 5
$n_e = 10^{20}\ m^{-3}$
$n_D = n_T = \frac{1}{2} n_i = .252 \times 10^{20}\ m^{-3}$
$n_\alpha = .4\ n_i = .202 \times 10^{20}\ m^{-3}$
$n_{Ar} = .01\ n_i = .252 \times 10^{18}\ m^{-3}$
$T_e = T_D = T_T = T_\alpha = T_{Ar}$
$U_{di} = U_{dD} + U_{dT} + U_{d\alpha} + U_{dAr}$
+ U_{di} (Case I, FIG. 4)
U_{de}
U_{di}
$U_{d\alpha}$
U_{dD}
U_{dT}
U_{dAr}
1.0
.9
.8
.7
.6
.5
.4
.3
.1
5
T_e (keV)

FIGURE 6

$$F(\bar{E}_T/E_{crit}) = \frac{(\bar{E}_T/E_{crit})^{1/2}}{1+(\bar{E}_T/E_{crit})^{3/2}}$$

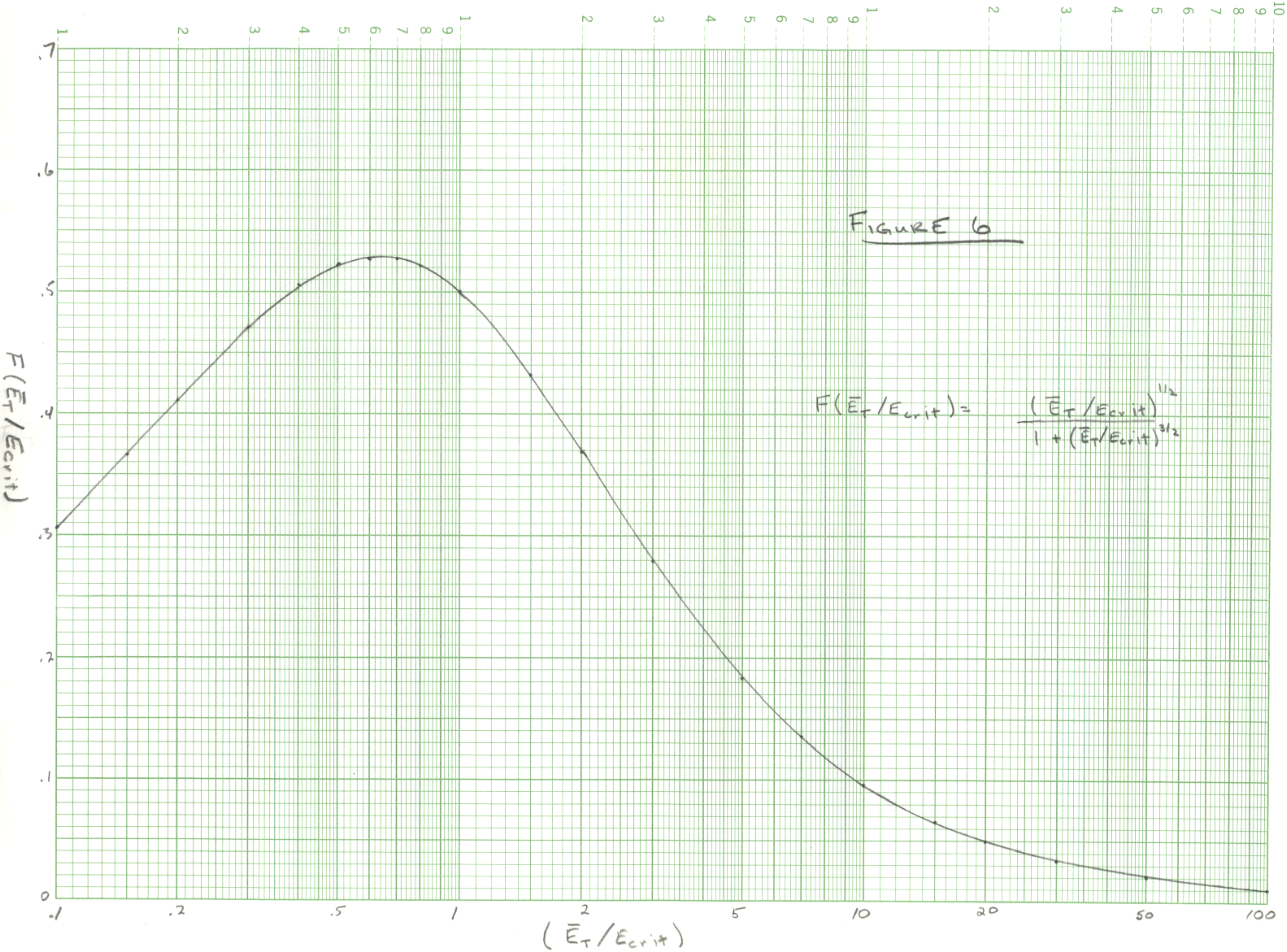


FIGURE 7
$F(\bar{E}_T/E_{crit}) = \frac{(\bar{E}_T/E_{crit})^{1/2}}{1+(\bar{E}_T/E_{crit})^{3/2}}$
$F(\bar{E}_T/E_{crit})$
.7
.6
.5
.4
.3
.2
.1
0
0
1
2
3
4
5