The Differential Geometry of Twisted Coil Windings

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ABSTRACT

The techniques of differential geometry have been applied to the problem of predicting the shape of thick twisted coil windings as successive turns and layers of turns are applied to a winding form. The explicit expressions for the required Christoffel symbols for parallel surfaces are derived in terms of the starting surface parameterization. Expressions for geodesic windings on a particular surface, called the rectifying developable, and the family of surfaces parallel to it are derived. The advantages of the rectifying developable from the point of view of coil fabrication are discussed.

INTRODUCTION

For magnet coils wound in a plane, such as the circular or D-shaped toroidal field coils of a tokamak, the winding cavity is well approximated by the envelope of rectangles of constant dimensions centered on the central filament of the winding pack and lying in the plane formed by the in-plane normal to the central filament and the normal to the winding plane. Various workers have (incorrectly) generalized the above idea to the case of twisted coil windings, taking the winding cavity to be, for example, the envelope of rectangles with sides parallel to the normal $\hat{N}$ to the winding surface along the central filament of the first layer and top and bottom parallel to the binormal vector $\hat{B} = \hat{T}\times\hat{N}$, where $\hat{T}$ is the tangent to the central filament. However, the above envelope does not represent the true shape taken by the windings as successive layers are wound. Knowledge of the true shape of the winding cavity envelope and direction of the current filaments is necessary for detailed design of twisted coils and accurate calculation of the magnetic fields produced by them, and requires use of differential geometry. In the present work, the mathematical tools required for such a description are described.

Two different approaches to the problem of defining twisted coil windings can be used. The first is to start with a defining surface (e.g., a toroid for modular stellarators) and define a closed curve on the defining surface in terms of a relation between two surface parameters. The closed curve represents either a first or central turn in a layer wound on the defining surface. The second approach is not to use the defining surface as a winding surface, but to start with a space curve, which may be defined by the defining surface, or by some other method, and construct a particular winding surface, called the rectifying developable, from it. The rectifying developable offers distinct advantages in coil fabrication, including the fact that the turns in a layer are geodesics on it.

In the following discussion, it is assumed that turns have constant transverse dimensions and that effects due to deformation of individual turns can be neglected; the latter approximation is good for coils with many turns and conversely for those with few turns.

WINDINGS ON AN ARBITRARY SURFACE

Given a surface with cartesian coordinates $\vec{R}(p,q)$, where p and q are surface parameters, and a starting curve representing the first turn in a layer, the problem is to find the family of geodesic parallels to the starting curve. The center filaments of successive turns in a layer are members of this family. In a plane, the solution is to find the normals to the starting curve at each point along it; the parallel curves are the loci of points equidistant from the starting curve along the normals. The analogous procedure on a curved surface is to find at each point along the starting curve the geodesic orthogonal to the starting curve. The geodesic parallels are the loci of points that are a constant arc length away from the starting curve along the orthogonal geodesics (see Fig. 1).

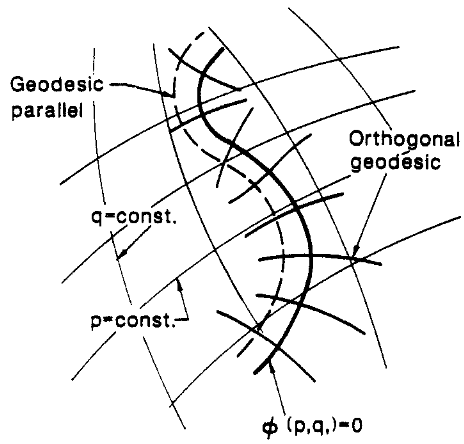

Fig. 1. Construction of geodesic parallels on a curved surface.

Finding the positions of successive turns in a layer therefore reduces to the problem of finding the geodesics orthogonal to an arbitrary curve in a surface. In general, constructing the orthogonal geodesics requires solution of the following pair of ordinary differential equations:[1,2]

$$\frac{d^2p}{ds^2} + \Gamma_1^{11}\left(\frac{dp}{ds}\right)^2 + 2\,\Gamma_1^{12}\frac{dp}{ds}\frac{dq}{ds} + \Gamma_2^{22}\left(\frac{dq}{ds}\right)^2 = 0 \qquad (1)$$

$$\frac{d^2q}{ds^2} + \Gamma_2^{11}\left(\frac{dp}{ds}\right)^2 + 2\,\Gamma_2^{12}\frac{dp}{ds}\frac{dq}{ds} + \Gamma_2^{22}\left(\frac{dq}{ds}\right)^2 = 0 \; . \qquad (2)$$

Here s is arc length along the orthogonal geodesic and the coefficients are Christoffel symbols, defined by the following equations:

$$\Gamma_1^{11} = (\vec{R}_{11}\cdot\vec{F})/V \qquad \Gamma_2^{11} = -\,(\vec{R}_{11}\cdot\vec{G})/V$$

$$\Gamma_1^{12} = (\vec{R}_{12}\cdot\vec{F})/V \qquad \Gamma_2^{12} = -\,(\vec{R}_{12}\cdot\vec{G})/V$$

$$\Gamma_1^{22} = (\vec{R}_{22}\cdot\vec{F})/V \qquad \Gamma_2^{22} = -\,(\vec{R}_{22}\cdot\vec{G})/V \qquad (3)$$

where the subscripts 1 and 2 on $\vec{R}$ signify partial differentiation of $\vec{R}$ by p and q, respectively and $V = |\vec{R}_1 \times \vec{R}_2|$. The vectors $\vec{F}$ and $\vec{G}$ are derived from the unit surface normal vector $\hat{N}$ and first derivatives of the coordinate vector $\vec{R}$ as follows:

$$\vec{F} = \vec{R}_2 \times \hat{N}$$

$$\vec{G} = \vec{R}_1 \times \hat{N} \; . \qquad (4)$$

Finally, the unit surface normal vector $\hat{N}$ is simply

$$\hat{N} = (\vec{R}_1 \times \vec{R}_2)/V \; . \qquad (5)$$

From the theory of differential equations, the solution to the above equations is uniquely determined for each point on the starting curve by the starting values and derivatives at each point. Their solution, in general, requires numerical techniques. The starting derivatives dp/ds and dq/ds can be easily found from the defining relation for the starting curve (e.g., $p = f(q)$ or $\phi(p,q) = 0$) and the expression for arc length:

$$ds^2 = (\vec{R}_1\,dp + \vec{R}_2\,dq)^2 \; . \qquad (6)$$

Then $\frac{dp}{ds} =$

$$\frac{dp}{(\vec{R}_1^{\,2}dp^2 + 2\,\vec{R}_1\cdot\vec{R}_2\,dpdq + \vec{R}_2^{\,2}dq^2)^{1/2}} \qquad (7)$$

and $\frac{dq}{ds} =$

$$\frac{dq}{(\vec{R}_1^{\,2}dp^2 + 2\,\vec{R}_1\cdot\vec{R}_2\,dpdq + \vec{R}_2^{\,2}dq^2)^{1/2}} \; . \qquad (8)$$

PARALLEL SURFACES

As successive layers are wound on a coil, the surfaces formed constitute a family of parallel surfaces, described by the parametric expression

$$\vec{R}^*(p,q,h) = \vec{R}(p,q) + \hat{N}(p,q)h \qquad (9)$$

where h is the winding depth and is fixed for a particular layer. In the above expression, p and q are parameters with some geometrical definition on the starting surface; their geometrical analogs for the parallel surface are in general given by complicated expressions. Parallel turns are found as before by constructing the family of geodesics orthogonal to the starting curve. The latter curve must be specified independently for each layer. A reasonable, but not unique, choice would be to take the normal projection of the center turn of the first layer: i.e., $\vec{r}_c(s_o,h) = \vec{r}_c(s_o) + \hat{N}(s_o)h$ for the center of each succeeding layer.

Explicit expressions for the Christoffel symbols of the parallel surface in terms of the parametric expression for the starting surface are found by substituting Eq. (9) in Eqs. (3), (4) and (5). The analogous quantities for the parallel surface in place of those of the starting surface are denoted by asterisks.

Introducing the definitions $L = \hat{N} \cdot \vec{R}_{11}$, $M = \hat{N} \cdot \vec{R}_{12}$, and $N = \hat{N} \cdot \vec{R}_{22}$, one can show that

$$\vec{F}^* = \vec{F} + \frac{h}{V}(M\vec{R}_2 - N\vec{R}_1) \text{ and} \tag{10}$$

$$\vec{G}^* = \vec{G} + \frac{h}{V}(L\vec{R}_2 - M\vec{R}_1) \ . \tag{11}$$

Finally, the three second partial derivatives $\vec{R}^*_{ij}$ for the parallel surface are given by the expressions:

$$\vec{R}^*_{11} = \vec{R}_{11} + \frac{h}{V}\Big[(2L\Gamma^{11}_1 + 2M\Gamma^{11}_2 - \hat{N}\cdot\vec{R}_{111})\vec{F} - (2L\Gamma^{12}_1 + 2M\Gamma^{12}_2 - \hat{N}\cdot\vec{R}_{112})\vec{G} + \frac{(L\vec{R}_2 - M\vec{R}_1)^2}{V}\hat{N}\Big] \tag{12}$$

$$\vec{R}^*_{12} = \vec{R}_{12} + \frac{h}{V}\Big[(M\Gamma^{11}_1 + N\Gamma^{11}_2 + M\Gamma^{12}_2 + L\Gamma^{12}_1 - \hat{N}\cdot\vec{R}_{112})\vec{F} - (L\Gamma^{22}_1 + M\Gamma^{22}_2 + N\Gamma^{12}_2 + M\Gamma^{12}_1 - \hat{N}\,\vec{R}_{221})\vec{G} + \frac{(\vec{L}R_2 - \vec{M}R_1)\cdot(M\vec{R}_2 - N\vec{R}_1)}{V}\hat{N}\Big] \tag{13}$$

$$\vec{R}^*_{22} = \vec{R}_{22} + \frac{h}{V}\Big[(2N\Gamma^{12}_2 + 2M\Gamma^{12}_1 - \hat{N}\cdot\vec{R}_{221})\vec{F} - (2N\Gamma^{22}_2 + 2M\Gamma^{22}_1 - \hat{N}\cdot\vec{R}_{222})\vec{G} - \frac{(M\vec{R}_2 - N\vec{R}_1)^2}{V}\hat{N}\Big] \ . \tag{14}$$

DEVELOPABLE SURFACES

A developable surface is a surface that can be developed (bent without stretching) to a plane. Developable surfaces are special cases of ruled surfaces. A ruled surface has the parametric form

$$\vec{R}(s,l) = \vec{r}_o(s) + l\hat{p}(s) \tag{15}$$

where s is arc length along the space curve $\vec{r}_o(s)$ and $\hat{p}(s)$ is a unit vector defined at every point of the space curve. $\vec{R}(s_o,l)$ for a fixed value of s_o is a straight line called a generator of the surface. A ruled surface is developable if and only if there exists a space curve to which all of the generators are tangent; this curve is called the edge of regression. An equivalent analytical definition is that a surface is developable if and only if the total or Gaussian curvature K_{tot} is zero, that is

$$K_{tot} = \frac{LN - M^2}{V^2} = 0 \ . \tag{16}$$

For any ruled surface of the form of Eq. (10),

$$\frac{\partial^2 \vec{R}}{\partial l^2} = 0,$$

and Eq. (16) reduces to M = 0 or

$$\hat{p}' \cdot (\vec{r}_o{}' \times \hat{p}) = 0 \ . \tag{17}$$

In practice, Eq. (17) is more useful than the previous geometrical definition of a developable for determining whether or not a particular parametric form represents a developable surface.

The angle ϕ between the generators of a ruled surface and a curve $\vec{r}$ lying on the surface is determined by the relation

$$\cos\phi = \vec{r}'\cdot\hat{p} \ . \tag{18}$$

In the above and following discussions, the prime symbol indicates differentiation with respect to arc length s. According to Eq. (17), if the ruled surface is developable, $\hat{p}'$ must lie in the $\vec{r}' - \hat{p}$ plane and one has the further requirement that

$$\sin\phi = \vec{r}'\cdot\hat{q} \tag{19}$$

where $\hat{q}$ is a unit vector in the $\vec{r}' - \hat{p}$ plane perpendicular to $\hat{p}$. Finally, differentiation of Eq. (18) yields

$$- \sin\phi \ \phi' = \vec{r}''\cdot\hat{p} + \vec{r}'\cdot\hat{p}' \ .$$

From the Frenet-Serret formulas[3] for space curves, one has the relation $\vec{r}'' = \kappa\hat{n}$ where κ is the curvature and $\hat{n}$ is the principal normal to $\vec{r}$. If $\vec{r}$ is a geodesic, by definition $\hat{n}$ must be parallel to the surface normal along $\vec{r}$ and $\vec{r}'' \cdot \hat{p} = 0$. We therefore have for geodesics on a developable surface

$$\phi' = - \hat{p}'\cdot\hat{q} = \pm \ |\hat{p}'| \ . \qquad (20)$$

Formal integration of Eq. (20) shows that if $\vec{r}$ is a geodesic on a developable surface, then any other geodesic cuts the generators at angles that differ at most by a constant angle from the angles at which $\vec{r}$ cuts the generators. This follows geometrically from the fact that geodesics are straight lines when the surface is developed to a plane. Geodesics parallel to $\vec{r}$ must therefore cut the generators at the same angles as $\vec{r}$.

THE RECTIFYING DEVELOPABLE

A particular developable surface, called the rectifying developable, can be generated from any smoothly varying space curve $\vec{r}_0$. The advantages of using the rectifying developable for a winding surface have been pointed out by various authors.[4,5] Among them is the fact that the $\vec{r}$ and its geodesic parallels are geodesics on the surface and therefore have no tendency to slip sideways under tension; continuous clamping of turns is therefore not required. Another advantage is the fact that the coil bobbin can be formed by rolling of plate stock since it is a developable surface. Finally, flat ribbon-like conductors can be used in order to minimize bending strains since bending in the "hard" direction is not required.

The rectifying developable has the parametric form of Eq. (15) with $\hat{p}$ having the form

$$\hat{p} = \frac{\frac{\tau}{\kappa}\hat{t} + \hat{b}}{[1 + (\frac{\tau}{\kappa})^2]^{1/2}} \qquad (21)$$

where τ and κ are the torsion and normal curvature, respectively, and $\hat{t}$ and $\hat{b}$ the tangent and binormal unit vectors of the space curve $\vec{r}_0$. Since the surface is developable (a fact that is easily shown by use of Eq. (17) and the Frenet-Serret equations for a space curve), geodesics become straight lines when the surface is developed to a plane. In particular, the geodesic parallels to $\vec{r}_0$ (including $\vec{r}_0$ itself) are parallel straight lines on the developed surface (hence the word rectifying). If the curve $\vec{r}_0$ is a closed curve in space (as in coil windings), the geodesic parallels have equal length. The latter statement can be verified analytically as follows. For the rectifying developable, Eqs. (18) and (19) yield the result

$$\cot\phi_0 = \sigma \qquad (22)$$

where the subscript indicates that the angle refers to $\vec{r}_0$ and where the symbol $\sigma = \tau/\kappa$ has been introduced. A geodesic parallel to $\vec{r}_0(s_0)$ can be written in the form

$$\vec{r}(s_0) = \vec{r}_0(s_0) + u(s_0)\ \hat{p}(s_0) \qquad (23)$$

where the function $u(s_0)$ is the length along the generator between $\vec{r}_0(s_0)$ and $\vec{r}(s_0)$ (see Fig. 2). An explicit expression for u can then be obtained by substituting Eq. (23) in Eqs. (18) and (19) and using Eq. (22). The result is

$$u(s_0) = u_0 \frac{\sin\phi_0(o)}{\sin\phi_0(s_0)} = \frac{w}{\sin\phi_0(s_0)} \qquad (24)$$

where w is the (constant) perpendicular distance between $\vec{r}_0$ and $\vec{r}$ in the developed surface. Substituting Eq. (24) in Eq. (23), the differential of arc length ds along $\vec{r}(s_0)$ is found to be

$$ds = |\vec{r}'| \ ds_0 = (1 + w\sigma') \ ds_0 \ . \qquad (25)$$

Since w is constant, $w\sigma' ds_0$ is an exact differential and when integrated around a closed curve must yield zero. This completes the analytic proof that geodesic parallels on the rectifying developable have equal length when $\vec{r}_0(s_0)$ is a closed curve in space.

GEODESICS ON THE FAMILY OF SURFACES PARALLEL TO THE RECTIFYING DEVELOPABLE

An explicit form for the family of surfaces parallel to the rectifying developable can be found using Eqs. (9), (15) and (21). The result is

$$\vec{R} = \vec{r}_0(s_0) - \hat{n}_0(s_0)h + l\hat{p}_0(s_0) \qquad (26)$$

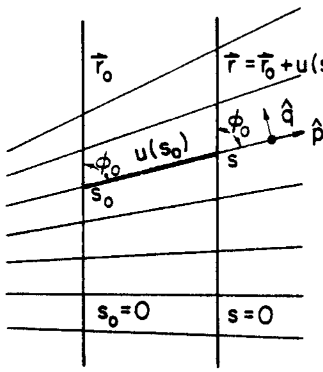

Fig. 2. Parallel geodesics on the rectifying developable. The surface has been developed to a plane.

where $\hat{n}_0$, the principal normal to the curve $\vec{n}_0$ is parallel to the surface normal $\hat{N}$ along $\vec{r}_0$. This is a consequence of the fact $\vec{r}_0$ is a geodesic on the rectifying developable. Equation (26) has the same form as Eq. (15), with $\vec{r}_0 - \hat{n}_0 h$ substituting for $\vec{r}_0(s_0)$. Using Eq. (17), it can be shown that the surface given by Eq. (26) is also developable. From Eq. (20), it then follows that for a given value of s_0, the angle between a geodesic on the parallel surface and the generator differs at most by a constant from the angle between $\vec{r}_0$ and the corresponding generator.

If this constant angle is chosen to be zero, then $\phi = \phi_0 = \cot^{-1} \sigma(s_0)$. Proceeding as before, one writes an expression analogous to Eq. (23) for a geodesic on the parallel surface. (See Fig. 3)

$$\vec{r}(s_0,h) = \vec{r}_0 - \hat{n}_0 h + u(s_0)\, \hat{p}(s_0) \; . \qquad (27)$$

Use of Eqs. (18-21) yields a first order differential equation for $u(s_0)$:

$$u'\sin\phi_0 + u\cos\phi_0\phi_0' = d(u\sin\phi_0) = h\tau. \quad (28)$$

Equation (28) is integrated to yield

$$u(s_0) = \frac{h}{\sin\phi_0} \int_0^{s_0} \tau(s)ds + \frac{w}{\sin\phi_0} \; . \qquad (29)$$

The above result (Eq. 29) is not physically correct, however, since in general the integral

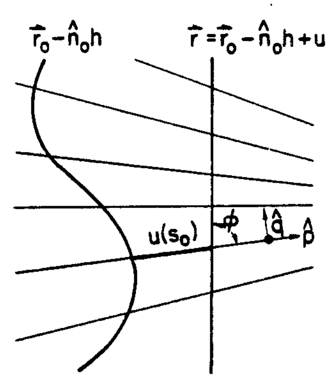

Fig. 3. Construction of a geodesic on a surface parallel to the rectifying developable. The surface has been developed to a plane.

of the torsion τ around a closed space curve is not zero; there will be an offset after the turn has returned to its starting position. This must be compensated by winding at an angle ϕ that differs by a constant α from ϕ_0; by Eq. (20) the curve can still be a geodesic. The resultant expression for u is

$$u = \frac{1}{\sin(\phi_0 - \alpha)} \left[\sin\alpha \int_0^{s_0} (1 + \kappa h)ds + \cos\alpha \int_0^{s_0} h\tau ds + w\right] .$$

α is determined by the condition $u(o) = u(L_0)$ where L_0 is the length of the curve $\vec{r}_0$ and is given by the expression:

$$\tan\alpha = \frac{- h \int_0^{L_0} \tau(s)ds}{L_0 + h \int_0^{L_0} \kappa(s)ds} \; .$$

CONCLUSIONS

The mathematical tools for determining the growth of twisted coil windings as they are wound have been described. The resultant description of the precise shape of the winding pack is expected to be especially useful in the detailed design of actual machines, where the exact shape of the winding cavity must be known, and in precise calculation of magnetic fields. A particular winding surface, the rectifying developable, has desirable properties for coil fabrication.

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