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ABSTRACT

The synchrotron radiation loss from weakly relativistic electrons in typical Tokamak reactors have been calculated numerically. The line broadening due to relativistic Doppler shift and inhomogeneity of the toroidal magnetic field have been taken into account. The parametric dependence of the radiant energy has been investigated in detail and an emperical formula was obtained. The values calculated by these formulas agree with the numerical results within 10% for the temperature range 3 to 100 KeV.

I. Introduction

Rosenbluth has calculated the energy loss of electron synchrotron radiation for Tokamak fusion reactors at low temperatures (<9 keV).[1] Considering the line broadening due to the field inhomogenity of the toroidal magnetic field, he obtained for the radiant energy

$$P_c = .16 \frac{R_o^3 T_e}{NR_o} \sum_{i=1,2} \sum_{m=1}^{\infty} m^3 \frac{Z_m \Lambda_m^i}{1+Z_m \Lambda_m^i} \text{ keV/sec/electron,}$$

where the functions

$$Z_m \begin{pmatrix} \Lambda_m^1 \\ \Lambda_m^2 \end{pmatrix} = 1.25 \times 10^{-3} (\beta_e R_o B_o)(10^{-3} T_e)^{m-2} (me)^{m-2} \begin{pmatrix} 1 \\ \frac{1}{2m+1} \end{pmatrix}.$$

Hence $\beta_e = 4 \times 10^{-8} NT_e/B_o^2$, N is the electron density, T_e is the electron temperature in KeV, R_o is the major radius, B_o is the magnetic field on axis and m is the harmonic number. However, P_c given by this equation is not a converging function of m and thus is not valid for temperatures above 9 keV.

For the purpose of plasma energy balance studies, it is necessary to recalculate the energy loss for a Tokamak reactor for a larger range of temperatures and to find the scaling laws with respect to the plasma parameters and reactor size. To do this, the broadening due to the relativistic Doppler shift will also be considered in addition to the field inhomogeneity. We used the method of Hirschfield[(3)] et al. to calculate the average optical depth and black-body cut-off frequency ω^* from the single particle theory by Trubnikov.[(2)] This method is justified by the agreement of ω^* obtained here with those calculated on the basis of kinetic equation by Drummond et. al.[4] for plasma in a uniform field. The power loss is then calculated from the black body radiation neglecting the transparent loss. An emperical formula is obtained which estimates the loss within 10% from the calculated results.

II. The Energy Loss Calculation

The relativistic emmissivity of a single electron in a vacuum moving along a spiral path in a magnetic field is given by the formula[2]

$$I_\omega(\theta) = \frac{e^2\,\omega^2}{2\pi C\,\omega_o \sin^2\theta(1+P^2)^{1/2}} \sum_{m=1}^{\infty} \{[(1+P^2)^{1/2}\cos\theta - P_{||}]^2 J_m^{\,2}(x) + P_\perp^{\,2}\sin^2\theta J_m'^{\,2}(x)\}$$

$$\cdot\delta\{m - \frac{\omega}{\omega_b}(1+\frac{\rho}{R})[(1+P^2)^{1/2} - P_{11}\cos\theta]\}.\ (1)$$

Here $x = \frac{\omega}{\omega_o} P_\perp(P^2+1)^{-1/2}\sin\theta$, $\vec{P}$ is the momentum of the electron in units of $m_e c$, θ is the angle between the magnetic field $\vec{B}_o$ and the propagation direction of the radiation, and J_m and J_m' are Bessel functions of order m and their derivative. Shown in Fig. 1, is the toroidal dependence of the magnetic field and given by

$$\begin{aligned} B &= B_o \frac{1}{1 + \frac{r\cos\phi}{R_o}}\ , \\ &= B_o \frac{1}{1 + \frac{\rho}{R}}\ , \end{aligned} \qquad (2)$$

and

$$\omega_o = \omega_b \frac{1}{\sqrt{1+P^2}} \; \frac{1}{1 + \rho/R_o} \quad , \tag{3}$$

where $\omega_b = eB_o/m_eC$. We assume that the electron distribution is Maxwellian in a Tokamak and is given by

$$f(P) = \frac{N}{4\pi \; m_e^2 c^2 kT_e K_2(\mu)} \; e^{-\mu\sqrt{1 + P^2}} \quad .$$

Here $\mu = m_e c^2/kT$, and $K_2(\mu)$ is the McDonald function or Hyperbolic Bessel function. The emissivity for the plasma averaged over the momentum space can be written as

$$\eta_\omega(\theta) = \sum_{m=1}^{\infty} \int \int \frac{Ne^2 \; \omega^2 \; (1+\rho/R_o)}{4\pi \; m_e^2 C^3 kT_e \omega_b \; \sin^2\theta (1+P^2)^{1/2} K_2(\mu)} \; P_\perp dP_\perp d \; P_{11}$$

$$\cdot$$

$$\{[1+P^2)^{1/2}\cos\theta - P_{||}]^2 J_m^2(x) + P_\perp{}^2 \sin^2\theta \; J_m'^{\,2} \; (x)\} \tag{4}$$

$$\cdot \delta\{m - \frac{\omega}{\omega_o}(1+\frac{\rho}{Ro})[(1 + P^2)^{1/2} - P_{11} \; \cos\theta]\} \; e^{-\mu\sqrt{1 + P^2}}$$

It is interesting to see how the inhomogeneity of the field contributes to the width of the harmonics in addition to the broadening due to Doppler shift. This can be studied by calculating the emissivity perpendicular to the magnetic field ($\theta = \frac{\pi}{2}$) where the ordinary and extraordinary waves are separated. For $\theta = \frac{\pi}{2}$, $(\omega/\omega_b)^2 << 1$ and $\mu >> 1$ the integration of equation (4) can be carried out according to Trubnikov and the result is

$$\eta_\omega^{(\parallel,\perp)}(y) = I_B \frac{\omega_P^2}{C\omega_b} \frac{\mu^{5/2} e^{2y}}{2^{3/2} y^{5/2}} \sum_{m \geq y}^{\infty} e^{-\mu\left(\frac{m}{y} - 1\right) + 2y\sqrt{\frac{m^2}{y^2} - 1}}$$

$$\cdot \left(\frac{m-y}{m+y}\right)^m \left\{ \frac{m^2-y^2}{2y3} \quad , \; 1 \right\} \quad , \qquad (5)$$

where $y = \frac{\omega}{\omega_b}\left(1 + \frac{\rho}{R_o}\right) = \nu\left(1 + \frac{\rho}{R_o}\right)$.

Here I_B is the black-body intensity, $\omega_P = \frac{4\pi Ne^2}{m_e}$, $\nu = \frac{\omega}{\omega_b}$, and $\eta_\omega^{\parallel}$ and $\eta_\omega^{\perp}$ denote ordinary and extraordinary waves. Referring to Figure 1, we integrate η_ω over the plasma diameter along ρ and obtain the average emissivity

$$\bar{\eta}_\omega = \frac{1}{\rho_2 - \rho_1} \int_{\rho_1}^{\rho_2} \eta_\omega(\rho)\, d\rho$$

With some simple transformations, we write

$$\bar{\eta}_\omega^{(\parallel,\perp)}(y) = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} \eta_\omega^{(\parallel,\perp)}(y)\, dy$$

where $y_1 = \nu\left(1 - \frac{1}{A}\right)$ and $y_2 = \nu\left(1 + \frac{1}{A}\right)$ and A is the aspect ratio $\frac{Ro}{n}$. The values of ν, y_1, and y_2 have to be $\leq$ m. The function

$$\phi_\omega^{(\parallel)} = \bar{\eta}_\omega^{(\parallel)} / \left(I_B\, \omega_p^2 / C\omega_b\right)$$

is computed numerically as functions of ν for m harmonics from 1 to 16 and at kT_e = 15 and 25 keV for both uniform and toroidal magnetic fields. The results are shown in Fig. 2 for 25 keV and A = 5. The function $\phi_\omega^{(\parallel)}$ for uniform field has been calculated by Hirschfield et. al[3] for different temperatures. As one can see, the line widths are indeed broader and the total absorption coefficients $\bar{\alpha}=\eta_{\omega}^{\parallel}/I_B$ are larger for toroidal field, however the change is not drastic.

The absorption spectra calculated from the equation

$$I_{\omega}(\tfrac{\pi}{2}) = I_B(\omega)(2 - e^{-\Lambda\phi^{\parallel}} - e^{-\Lambda\phi^{\perp}})$$

for $\Lambda = \omega_p^2 a/C\omega_b = 3000$ and for $T_e = 15$ and 25 kev are shown in Fig. 3. The dashed curves are for a plasma slab in a uniform field which have been calculated by Trubnikov. The spectra for the toroidal field are smoother and their amplitudes higher as well as peaked at higher frequencies.

To calculate the net radiation energy loss, the angular dependence of the emissivity has to be taken into account. When the summation over m in Eq. (4) is replaced by integration using the integral representation of the δ-function and again following Trubnikov, the emmissivity is written as

$$\eta_{\omega}(\theta) = \frac{\omega^2 e^2 N}{4\pi^2 \omega_b} \frac{(1 + \rho/R_o)}{K_2(\mu)\sin\theta} \int_{-\infty}^{\infty} d\xi \Big[\cos^2\theta \frac{\partial^2}{\partial s^2} - 2i\cos\theta \frac{\partial^2}{\partial s \partial r_z} - \frac{\partial^2}{\partial r_z^2}$$

$$+ \sin^2\theta \cos^2\frac{\xi}{2}\left(\frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial r_z^2} - 1\right) + \sin^2\theta \frac{\partial^2}{\partial r_z^2}\Big]\frac{K_1(X)}{X} .$$

Here $X = (s^2 + r_y^2 + r_z^2)^{1/2} = \mu[1 - 2i\chi\xi + 4\chi^2\sin^2\theta(\sin^2\frac{\xi}{2} - \frac{\xi^2}{2})^{1/2} = \mu Z$

and $\chi = y/\mu$.

By carrying out the differentiation of the McDonald function $K_1(x)$ we have

$$\eta_\omega(\theta) = \frac{\omega^2 e^2 N}{4\pi^2\omega_b} \frac{\chi^2\sin^2\theta}{K_2(\mu)} \quad (1 + \frac{\rho}{R_o})$$

$$\cdot\int_{-\infty}^{\infty} d\xi\{[(\cos\xi - 1)^2 + \frac{\cos\theta}{\chi^2\sin^4\theta}] \frac{K_3(\mu z)}{z^3} + [1 + \cot^2\theta + (1+\cot^2\theta)\cos\xi] \frac{K_2(\mu z)}{\mu\chi^2 z^2}\}. \quad (5)$$

Trubnikov used the first term to obtain an approximate formula for a plasma slab in uniform field. In this calculation both terms were used as explained below. By using the leading term of the asymtotic expression for large argument for the McDonald functions K_2, K_3 and evaluating the integral by using the saddle-point method, we have

$$\eta_\omega(\theta) = \frac{Ne^2\omega^2}{(2\pi)^{3/2}c\omega_b} \frac{\sin^2\theta\, e^{-\mu(Z_o - 1)}}{\sqrt{\mu}\,\chi\, Z_o} \quad (1 + \frac{\rho}{R_o})$$

$$\left\{ \frac{1 + 2Z_o^{-2}\cot^2\theta}{\sqrt{\sin^2\theta \cosh t - \sin^2\theta}} + \frac{Z_o^{-1}(\cosh t + 1)}{\mu\sqrt{\cosh t - 1}\,\sin\theta} \right\}, \quad (6)$$

where $t = \sinh t - \frac{1}{\chi \sin^2\theta}$ and

$$Z_o = [\chi^2 \sin^2\theta (\cosh t - 1)^2 - \cot^2\theta]^{1/2} .$$

The angular distribution of $\eta_\omega(\theta)$ for $\mu = 10$ and $m = 5$ is shown in Fig. 4. The solid curve is the sum calculated from both terms while the dashed curve was calculated from the first term only. The difference of the two curves is not negligible and is larger at 90°. Therefore, both terms should be used in the calculation of energy loss.

The approximate optical depth τ which is the average over all directions and over the plasma volume, but normalized to the black-body intensity for two polarizations $2\, I_B(\omega) = k\, T_e \omega^2/4\pi^3 c^2$, can be calculated from the following equation,

$$\tau = \Lambda \left\{ \pi\mu \frac{A}{\sqrt{2\pi\mu}} \int_0^{\pi/2} \int_{1-\frac{1}{A}}^{1+\frac{1}{A}} \frac{e^{-\mu(Z_o-1)}}{\chi Z_o} \rho' \sin\theta \, d\theta \, d\rho' \right.$$

$$\cdot \left\{ \frac{1 + 2Z_o^{-2}\cot^2\theta}{\sqrt{\sin^2\theta \cosh t - \sin^2\theta}} + \frac{Z_o^{-1}(\cosh t + 1)}{\mu\sqrt{\cosh t - 1}\, \sin\theta} \right\} \qquad (7)$$

where $\rho' = 1 + \frac{\rho}{R_o}$.

The cut-off frequency $\omega^* = m^*\omega_b$ is identified by the point of the frequency spectrum at which $\tau = 1$ for a given Λ.

Table I gives a comparison of m* calculated for a plasma slab in a uniform field with results given by Drummond et. at. and Trubnikov taken from reference 4 by considering the dielectric property of the plasma. The agreement is remarkable and is better than Trubnikov's results using the first term only. This suggests that the single particle model is a good approximation.

III. Parametric Dependence of the Radiant Energy Loss

The total power loss is given by

$$W_c = m^{*3}\omega_b^{\,3} kT_e/12\,\pi^2 C^2$$

with $\Lambda=\omega_p^{\,2}a/\omega_b$ or by

$$W_C = \frac{m^{*3}\ \omega_b^{\,3} kT_e}{1\,2\ \pi^2 C^2}(1-\gamma)$$

with $\Lambda=\omega_p^{\,2}a\ \omega_b c(1-\gamma)$, when the wall reflectivity γ is included. W_c implicitly depends on the parameters T_e, A, Λ, and B_o via the cut-off frequency m* ω_b. In order to see the variations of W_c with respect to those parameters, families of curves m* vs. A, Λ, or T_e are presented in Figs. 5, 6 and 7. From those figures, it is observed that m* is a linear function of A at constant T_e and Λ with an average slope of about 0.17, and varying as $\sqrt{\Lambda}$. The variations of m^{*3} with respect to T_e is slightly higher than given by first order. If we choose A = 5 as the reference point, the following equation would give the best overall reproduction of the calculated values of W_c

$$W_c = 10^{-7}[5+.18(5-A)]^3 \ (1-\gamma) \ \Lambda^{1/2} \ B_o^{\ 3} \ T_e^{\ 2.1} \ \mathrm{kw/m^2} \tag{8}$$

where B_o is in Tesla and T_e in keV. The parameter Λ can be written as $\Lambda = 6.05\times10^{-17} Na/(1-\gamma)B_o$ if N is in $1/m^3$ and a in meters. Fig. 8 plots a family of W_c vs. T_e for A=4 and 5. The open circles in Fig. 9 indicate the values calculated by emperical formulas of Eq.(8) . These formulas tend to overestimate the values at low temperatures and under-estimate them at high temperatures, but the differences are within about 10% for the temperature range 3 to 100 keV. For T_e <8 keV, W_c approaches Rosenbluth's low temperature limit. Recently Trubnikov has published a different emperical formula[4], where the power loss varies as $T_e^{5/2}$ and therefore gives a larger estimate at higher temperatures. For comparison purposes, his results are also calculated for Λ = 1,000, 3,000 and 10,000 and A = 5 and are shown as the dashed curves in Fig. 9.

For a typical Tokamak reactor[(5)] of A = 5, a = 2.5m, B_o = 5 Tesla T_e = 15 keV and $N = 10^{20}$ m^{-3} the power loss is 19.4 kw/m^3. for zero reflectivity. This power loss is only about the value of bremsstrahlung. Synchrotron radiation can be important, however, at higher temperature and, because of its temperature and magnetic field dependence, may be helpful in obtaining a thermally stable equilibrium.

REFERENCES

1. M. N. Rosenbluth, N. F. 10, 340 (1970).

2. B. A. Trubnikov, Phys. Fluids 4, 195 (1961) and ref. Cited therein.

3. Jay L. Hirshfield et. al. Phys. Fluids 4, 198 (1961).

4. W. E. Drummond and M. N. Rosenbluth, Phys. Fluids 6, 276 (1963).

5. University of Wisconsin Fusion Feasibility Group, proceedings of Texas Symposium on the Technology of Controlled Thermonuclear Fusion Experiments and the Engineering Aspects of Fusion Reactors. (To be Published)

Table I

Te(keV)	βeBL*	Λ	m_{DR}	m*	m^*_{TR}
40	8.5×10^6	6.38×10^4	10.9	10.9	11.5
50	9.5×10^6	5.7×10^4	11.7	12.0	13.
60	1.9×10^7	9.5×10^4	15.2	14.8	16
100	5.15×10^8	15.5×10^5	34.8	34.0	37

FIGURE CAPTIONS

Figure 1 - Tokamak geometry

Figure 2 - Line broadenings of the synchrotron radiations (a) for plasma in uniform field (after Hirschfield et. al.), (b) for plasma in toroidal field.

Figure 3 - Absorption spectra in the direction perpendicular to the magnetic field, dashed curves for plasma slab in uniform field (after Trubnikov), solid curves for plasma in toroidal field.

Figure 4 - The angular distribution of the emissivity for $\mu = 10$ and $m = 5$.

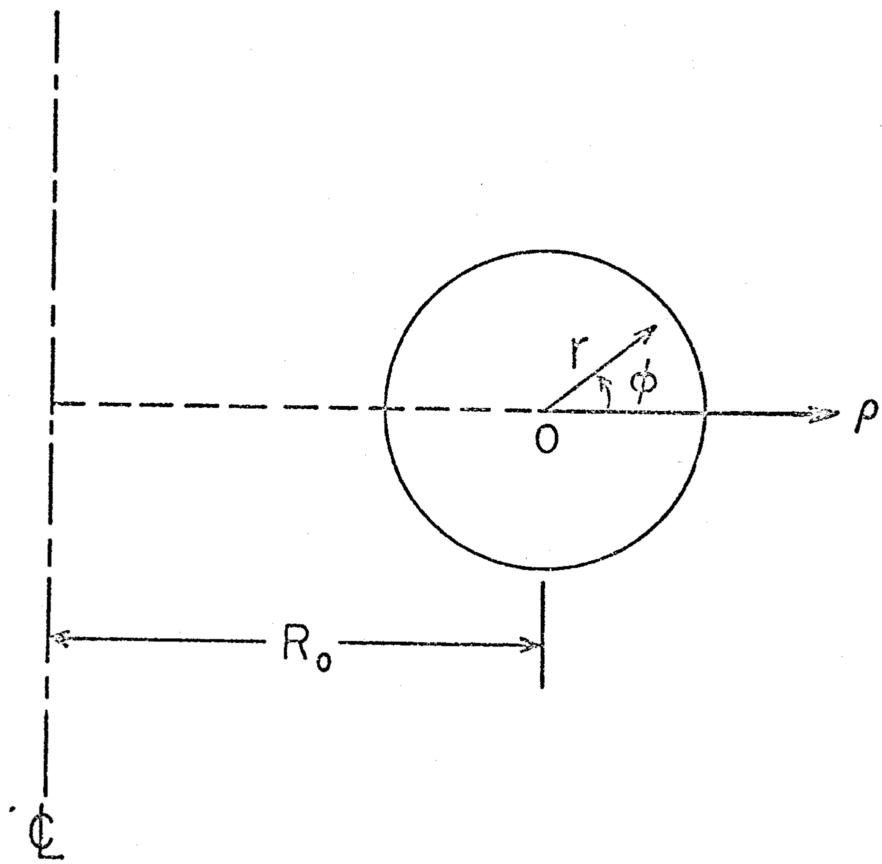
Figure 5 - The black-body cut-off frequency m^* in unit of ω_b as a function of Λ for constant T_e and A.

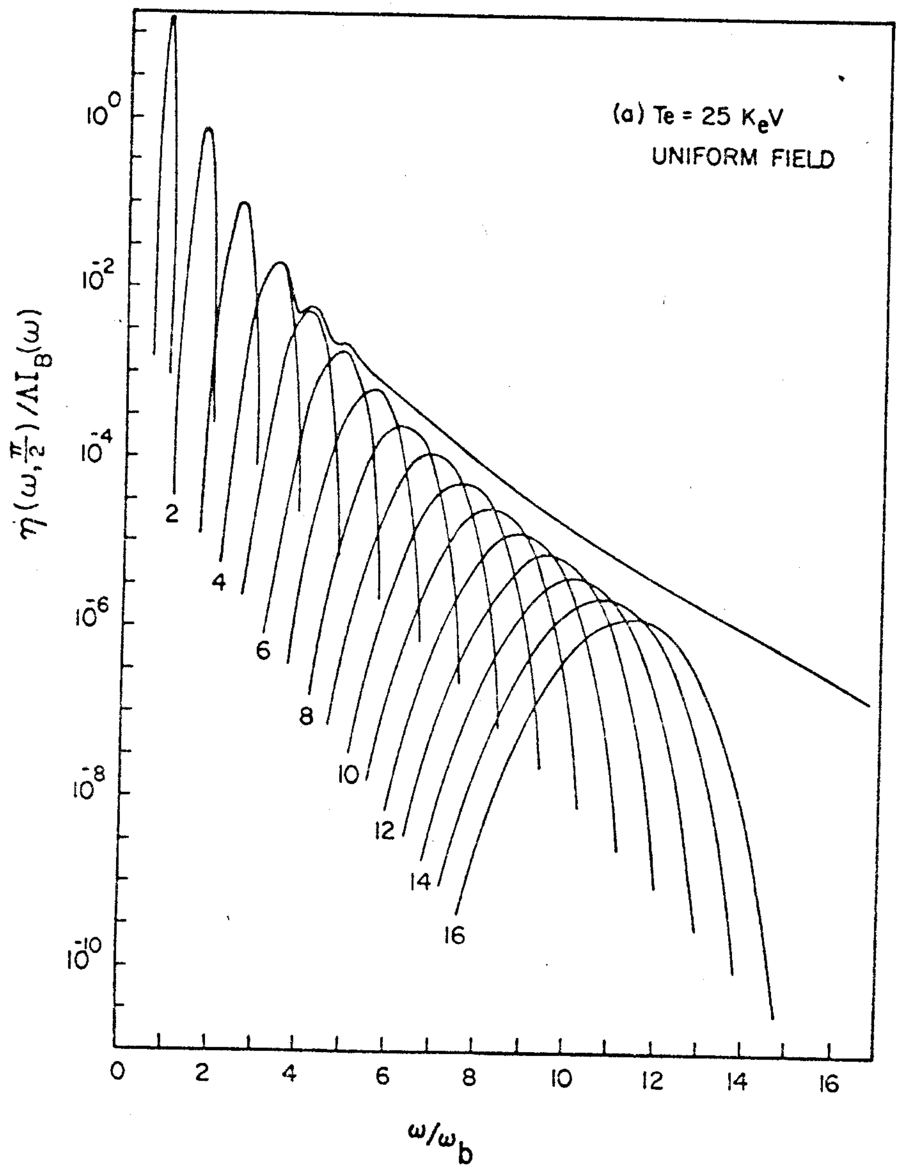
Figure 6 - The black-body cut-off frequency m^{*3} in unit of ω_b as a function of Λ for constant T_e and A.

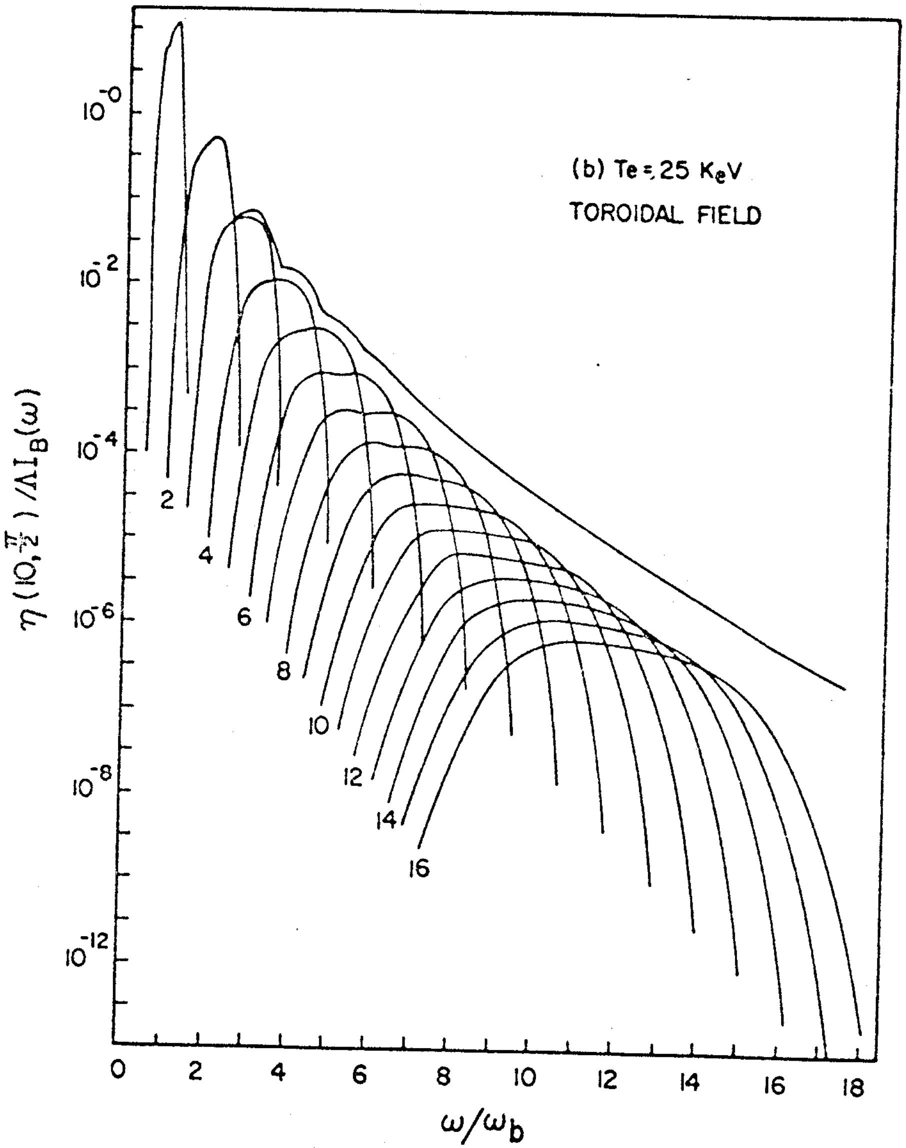
Figure 7 - The black-body cut-off frequency m^{*3} in unit of ω_b as a function of T_e for constant A and Λ.

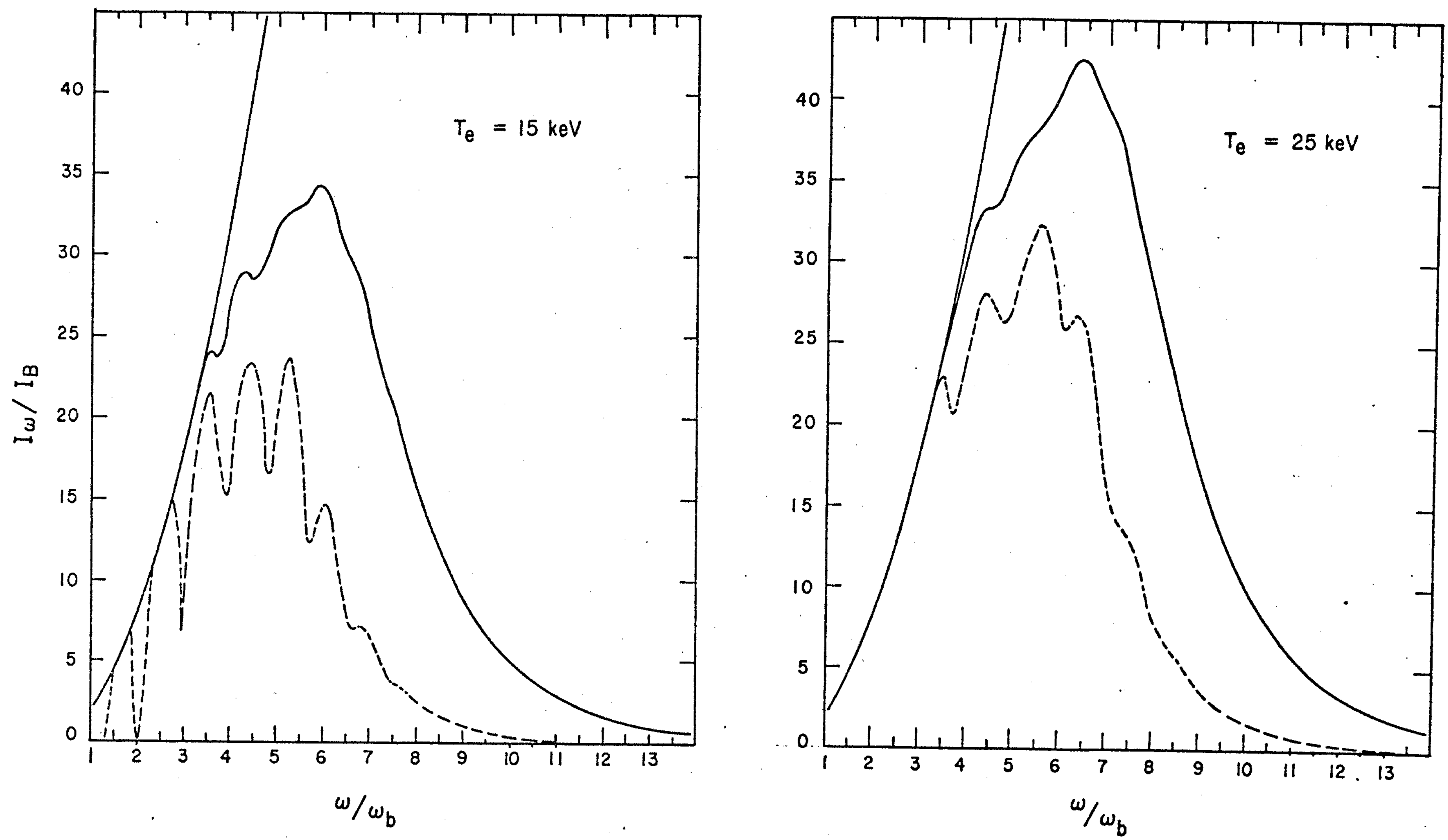
Figure 8 - The energy loss W_c vs. T_e for (a) $A = 4$, (b) $A = 5$.

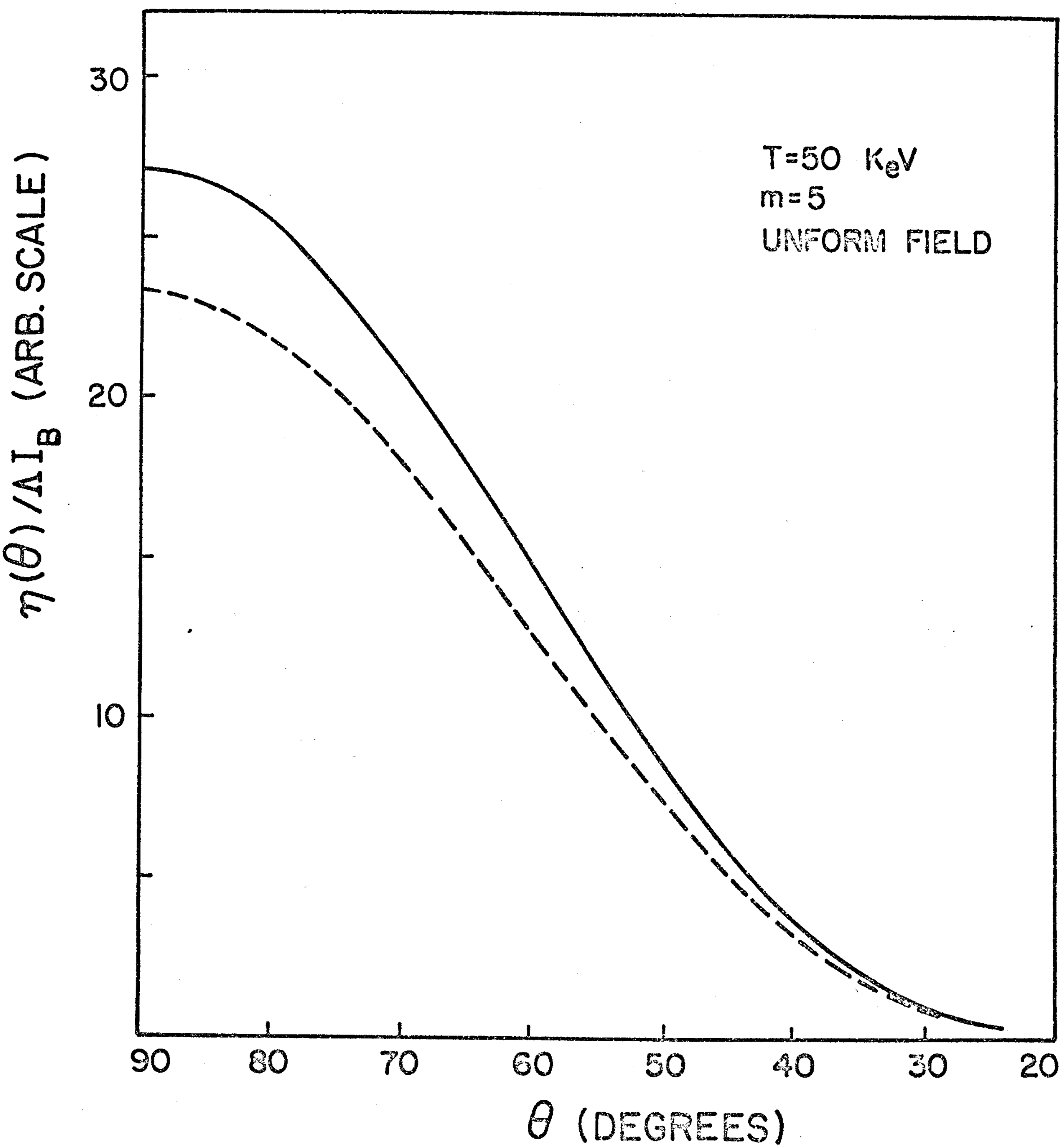
Figure 9 - The comparison of the energy loss W_c vs. T_e for $A = 5$ calculated with the emperical results. The dashed curves calculated from Trubnikov's formulas.

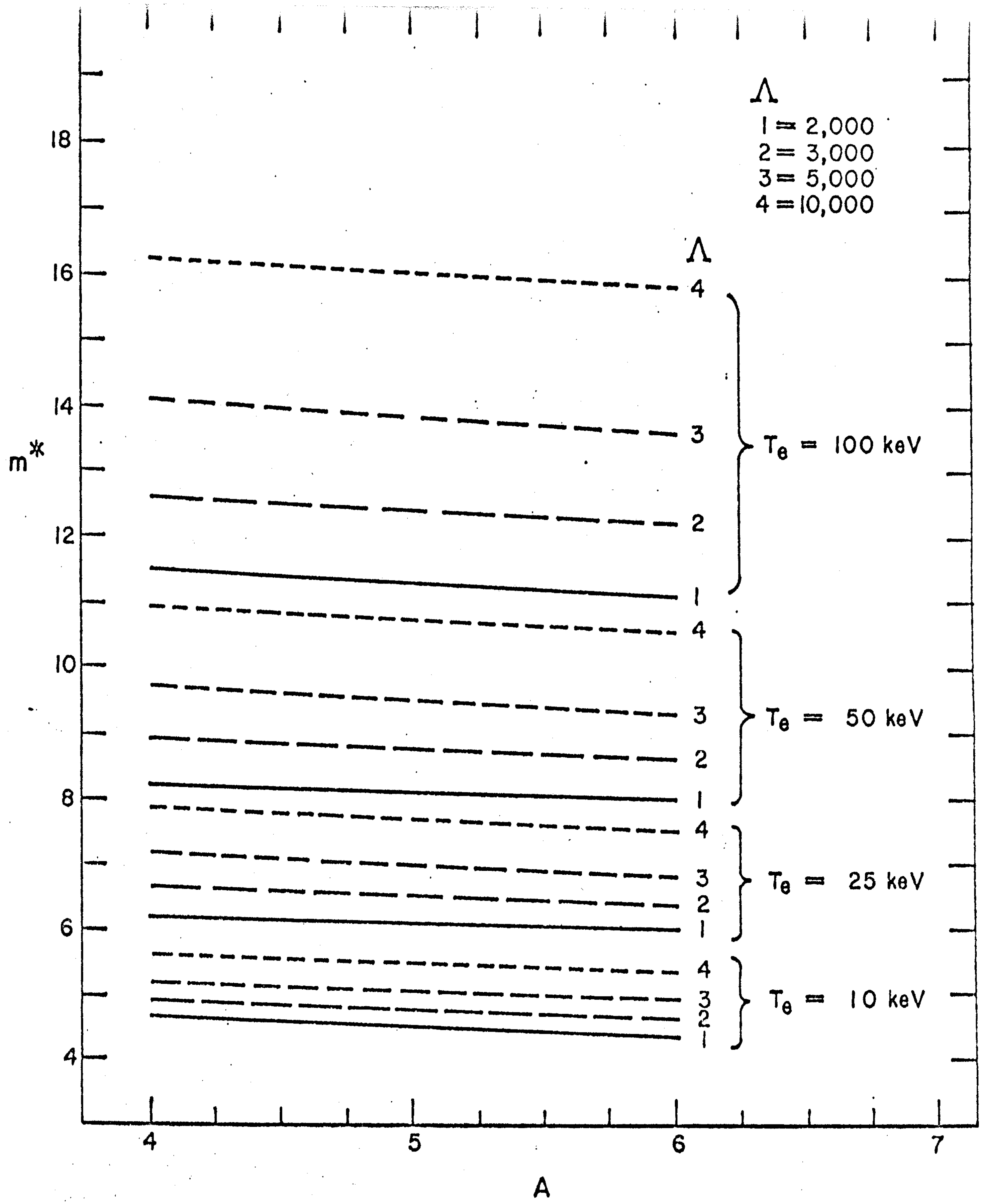

r
φ
O
ρ
R_o
℄


(a) Te = 25 KeV
UNIFORM FIELD
$\eta(\omega,\frac{\pi}{2})/\Lambda I_B(\omega)$
ω/ω_b


(b) Te = 25 KeV
TOROIDAL FIELD
$\eta(10,\frac{\pi}{2})/\Lambda I_B(\omega)$
ω/ω_b

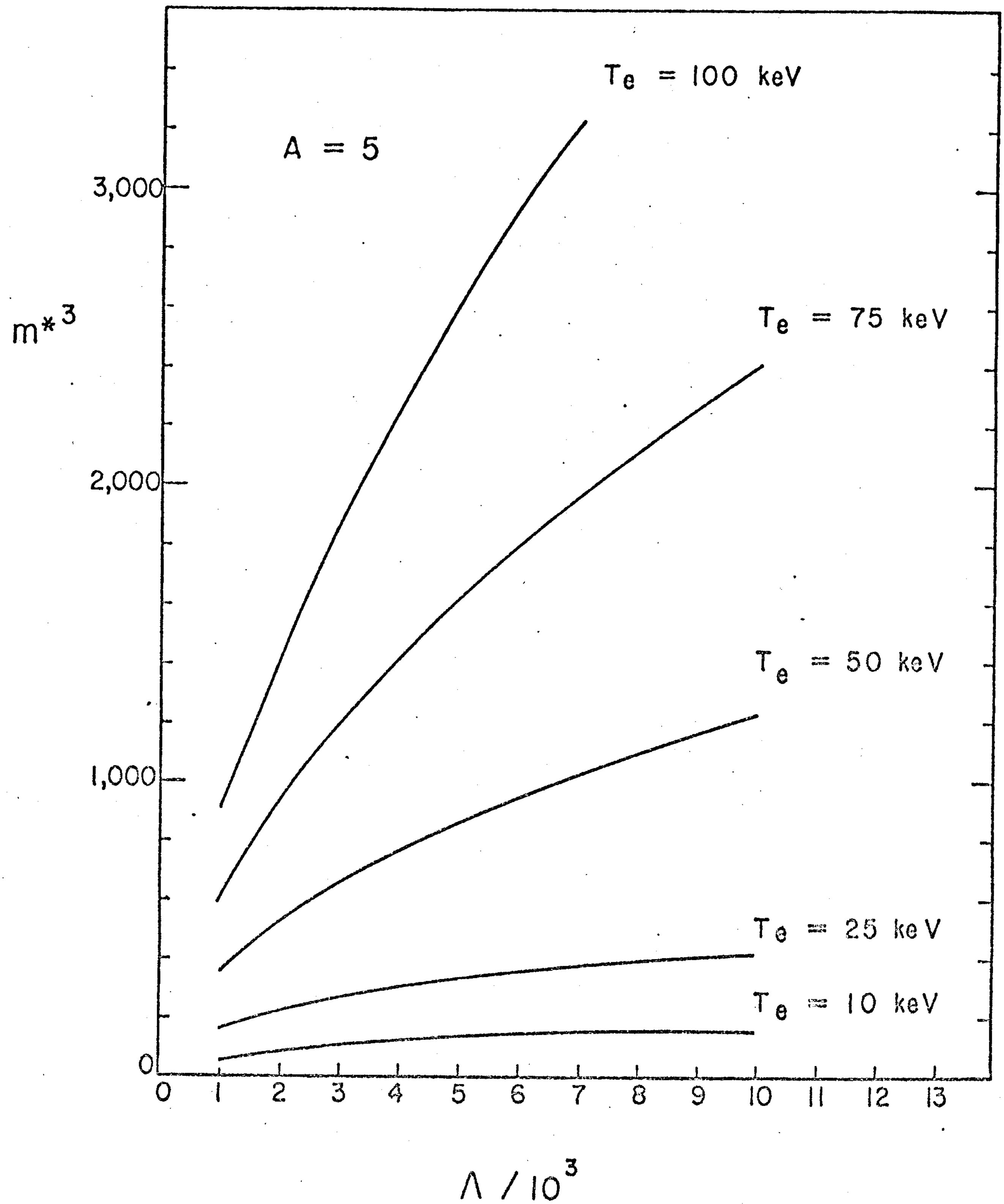

T_e = 15 keV
T_e = 25 keV
I_ω / I_B
ω / ω_b
ω / ω_b


T=50 KeV
m=5
UNFORM FIELD
30
20
10
90
80
70
60
50
40
30
20
θ (DEGREES)
η(θ)/ΛI_B (ARB. SCALE)



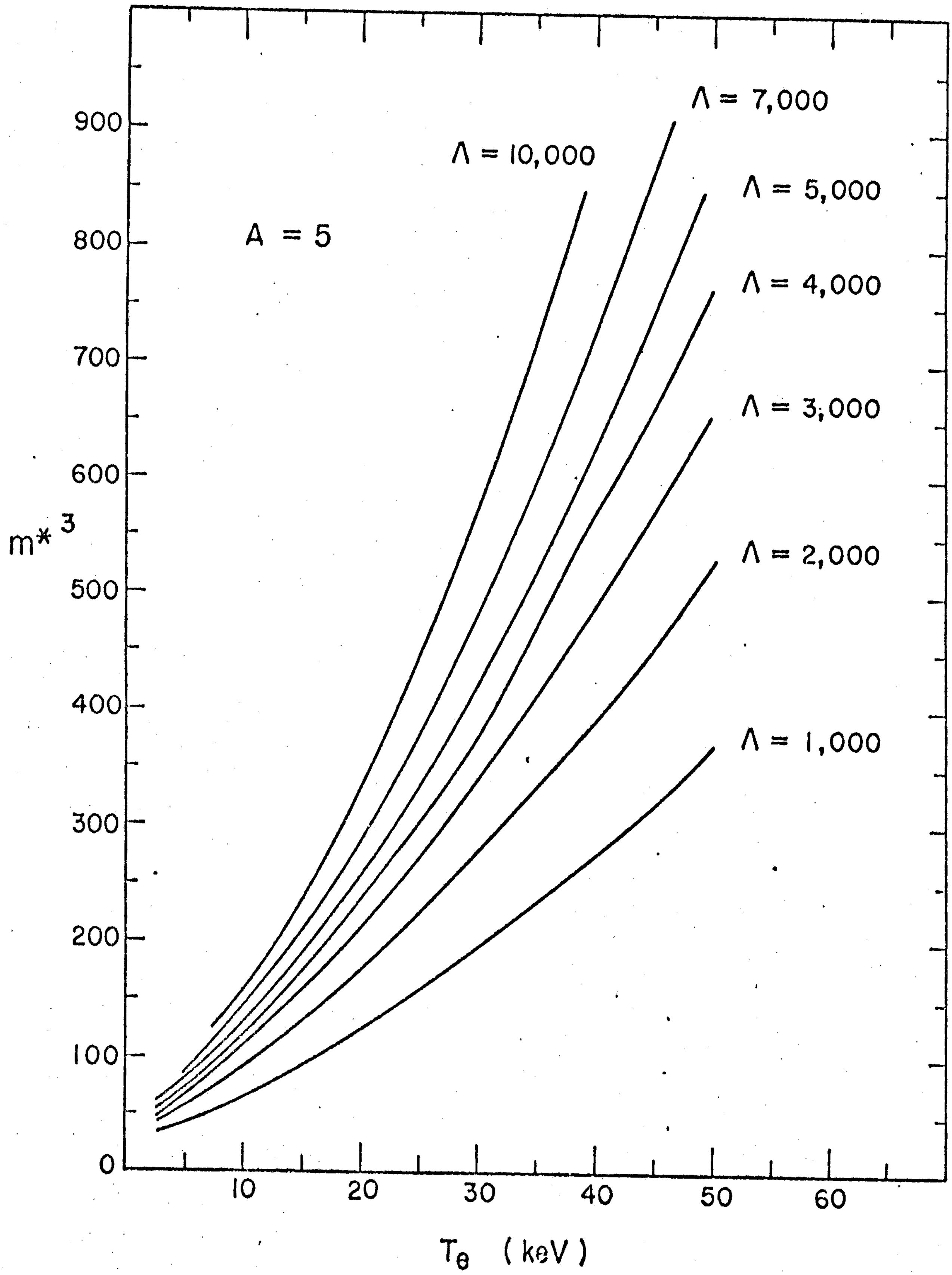
Λ
1 = 2,000
2 = 3,000
3 = 5,000
4 = 10,000
Λ
4
3
2
1
T_e = 100 keV
4
3
2
1
T_e = 50 keV
4
3
2
1
T_e = 25 keV
4
3
2
1
T_e = 10 keV
m*
18
16
14
12
10
8
6
4
4
5
6
7
A

Figure 5



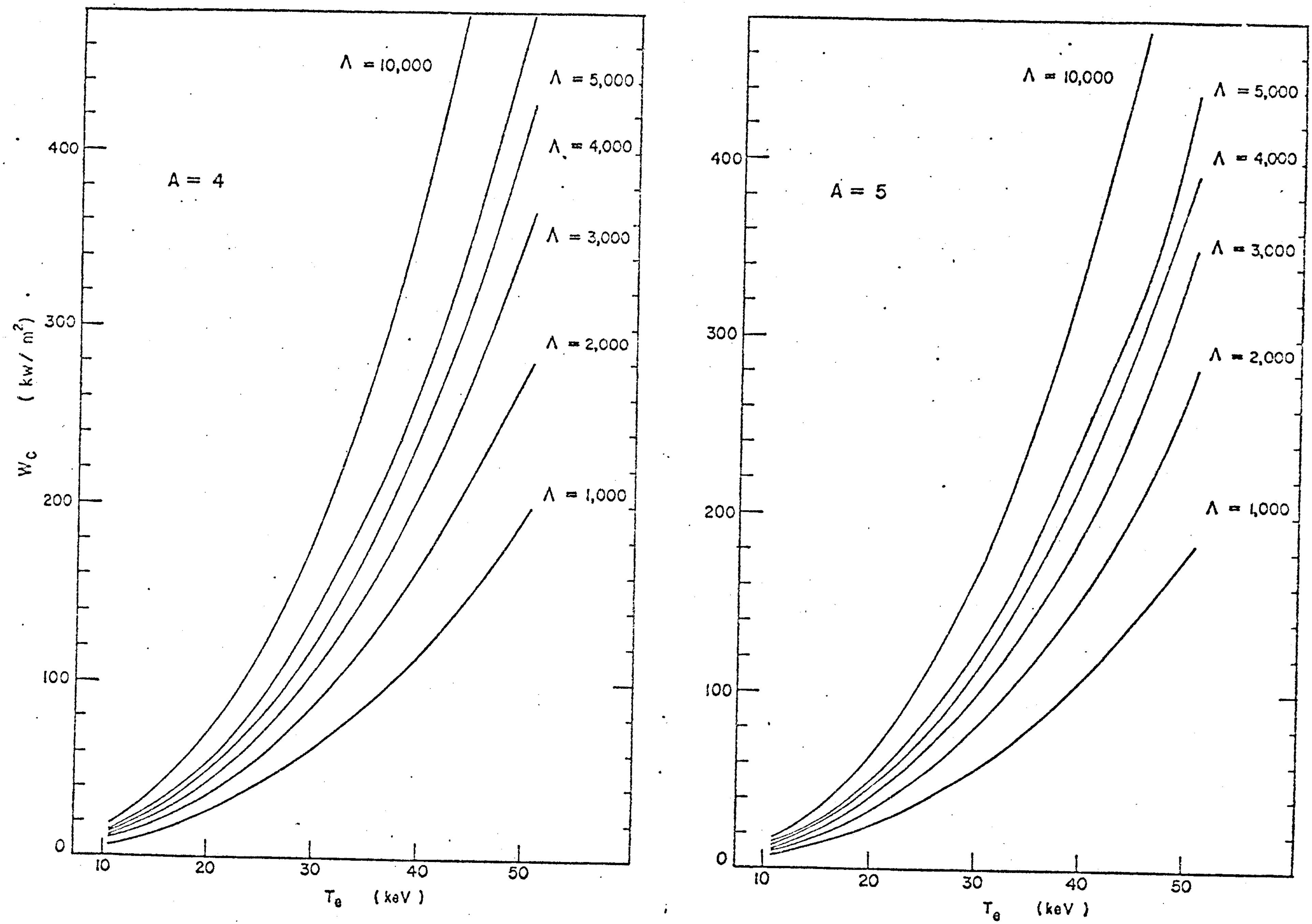
A = 5
T_e = 100 keV
T_e = 75 keV
T_e = 50 keV
T_e = 25 keV
T_e = 10 keV
3,000
2,000
1,000
0
m^{*3}
0
1
2
3
4
5
6
7
8
9
10
11
12
13
$\Lambda / 10^3$

Figure 6



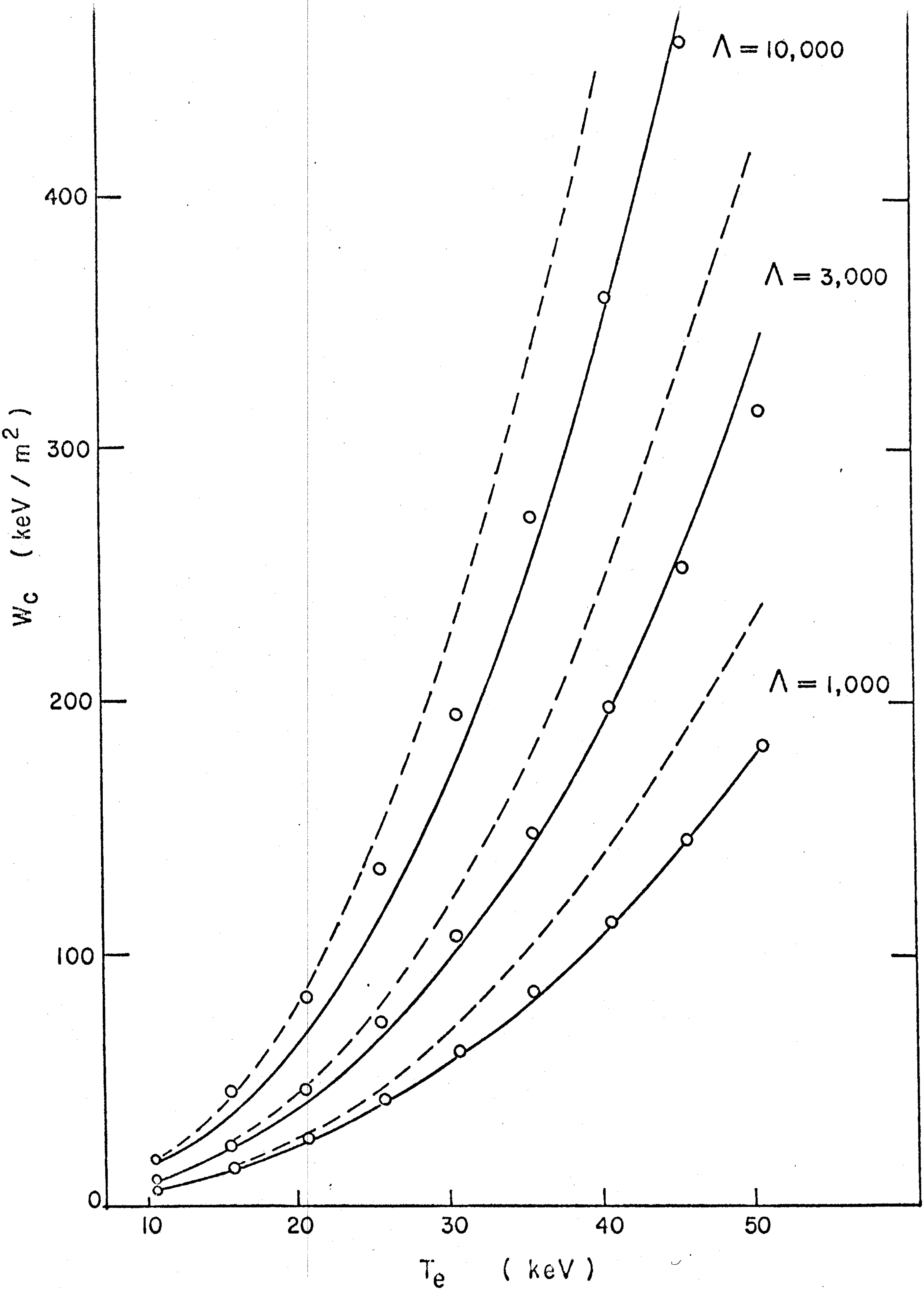
Λ = 7,000
Λ = 10,000
Λ = 5,000
A = 5
Λ = 4,000
Λ = 3,000
Λ = 2,000
Λ = 1,000
m* 3
900
800
700
600
500
400
300
200
100
0
10
20
30
40
50
60
Tθ (keV)

Figure 7



Λ = 10,000
Λ = 5,000
Λ = 4,000
Λ = 3,000
Λ = 2,000
Λ = 1,000
A = 4
400
300
200
100
0
W_c (kw/ m^2)
10
20
30
40
50
T_e (keV)
Λ = 10,000
Λ = 5,000
Λ = 4,000
Λ = 3,000
Λ = 2,000
Λ = 1,000
A = 5
400
300
200
100
0
10
20
30
40
50
T_e (keV)

Figure 8



Λ = 10,000
Λ = 3,000
Λ = 1,000
400
300
200
100
0
10
20
30
40
50
W_c (keV / m^2)
T_e (keV)