Matrix Representation of Multivariate Polynomials - A Systematic Approach for Computer Algebra

S.I. Abdel-Khalik

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Socrates among the Athenians

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LOUIS PHILLIPS

ABSTRACT

A systematic method for representing the general class of functions $f = \sum_{j=1}^{M} C_j \prod_{i=1}^{N} X_i^{\alpha_{ji}}$ using matrix-like quantities is introduced. Such representation makes algebraic manipulations of these functions readily amenable to computer applications.

I. INTRODUCTION

In many branches of science, one often encounters problems involving algebraic manipulations of multivariate polynomials. The fields of reliability analysis and kinetic theory and rheology of complex macromolecules abound with such problems. A closed-form solution is usually desired; however, the algebraic manipulations are often quite tedious, time-consuming or impossible to do by hand. Algebraic manipulation packages are presently available on many computers. However, the required computing times become prohibitive as the number of variables in the polynomials increases to more than a few.

Here, we introduce a new method for representing the general class of functions $f(X_1, X_2, \ldots, X_N)$ defined by

$$f = \sum_{j=1}^{M} C_j \prod_{i=1}^{N} X_i^{\alpha_{ji}} \tag{1}$$

In equation 1, the coefficients C_j and exponents α_{ji} can, in general, be complex numbers.

The information contained in the expanded expression of the function f above can be stored in a matrix form. It is clear, however, that the theories of matrix algebra will not apply for these quantities. Hence, in order to avoid confusion, these quantities are hereafter referred to as "polytrices". A function f of the above type can be represented by an "equivalent polytrix". The basic definition of the "equivalent polytrix," along with algorithms for the addition, subtraction, and multipliction of functions of the class defined in equation 1 are presented in this article. It should be kept in mind that the main objective here is to introduce the idea of equivalent polytrices rather than to work out and present specific algorithms for different algebraic operations.

II. THE EQUIVALENT POLYTRIX

A function $f(X_1, X_2, \ldots, X_N)$ of the class defined in equation 1 can be represented by a two-dimensional matrix-like quantity called the "equivalent polytrix" $\underset{\sim}{F}$. The polytrix $\underset{\sim}{F}$ will have M rows and N+1 columns. Here, M is the number of terms in the function f, and N is the number of variables in the set $X_1, X_2, \ldots, X_N$. The elements F_{ij} of the polytrix $\underset{\sim}{F}$ are given by

$$F_{i1} = C_i \equiv \text{numerical coefficient of the ith term in the function f; } i = 1, 2, \ldots, M.$$

$$F_{ij+1} = \alpha_{ij} \equiv \text{exponent of the variable } X_j \text{ in the ith term of the function f; } j = 1, 2, \ldots, N.$$

As an example, the equivalent polytrix of the function $q(X_1, X_2, X_3)$ in equation 2 is given in equation 3:

$$q(X_1,X_2,X_3) = 3X_1^2X_3^{-2} + 5iX_2^3 - 6X_2^{1+i}X_3^{0.5}, \qquad (2)$$

$$\underset{\sim}{Q} = \begin{bmatrix} 3 & 2 & 0 & -2 \\ 5i & 0 & 3 & 0 \\ -6 & 0 & (1+i) & 0.5 \end{bmatrix}. \qquad (3)$$

The first column in the polytrix is referred to as the column of coefficients, while the remaining columns are called the power columns. For multivariate polynomials, the elements of the equivalent polytrices are real integers.

III. COMPATIBLE POLYTRICES

Two polytrices, $\underset{\sim}{R}$ and $\underset{\sim}{Q}$, are said to be compatible if they have the same number of columns and the elements in their power columns represent powers of the same set of variables. Obviously, the equivalent polytrices of the functions $q(X_1, X_2, \ldots, X_\nu)$ and $r(X_{\nu+1}, \ldots, X_N)$ can be made compatible by writing them as functions of the complete set $(X_1, X_2, \ldots, X_\nu, \ldots, X_N)$.

As an example, the equivalent polytrices of the functions $q(X_1, X_2, X_3)$ in equation 2 and $r(X_4, X_5, X_6)$ in equation 4 are given in equations 5 and 6, respectively. These are written in their compatible forms.

$$r(X_4,X_5,X_6) = X_4^2 - X_5 + 4X_6^3. \tag{4}$$

$$\underset{\sim}{Q} = \begin{bmatrix} 3 & 2 & 0 & -2 & 0 & 0 & 0 \\ 5i & 0 & 3 & 0 & 0 & 0 & 0 \\ -6 & 0 & (1+i) & 0.5 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

$$\underset{\sim}{R} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}. \tag{6}$$

This implies that one can add one or more zero columns to the right or left of the power columns in the equivalent polytrix. The resultant "expanded" polytrix will represent the same original function, provided, of course, that we associate each column with the proper variable. The process of adding zero columns to a polytrix is equivalent to multiplying each term in the original function by unity (or by a variable X_μ raised to the power zero).

In problems involving a large number of functions, the expanded polytrices may contain many null power columns. This may be a limiting factor with regard to computer storage capacity. However, this problem can be easily solved by storing the non-zero power columns and assigning an "identifier" to each column corresponding to a given variable in the complete set $(X_1, X_2, \ldots, X_N)$.

IV. EXAMPLES

POLYTRIX ADDITION, SUBTRACTION, AND MULTIPLICATION

Consider the functions q, r, a, s, and p related by

$$a = q + r,$$
$$s = q - r,$$
$$p = q \cdot r. \qquad (7)$$

If q and r are members of the class defined in equation 1, then a, s, and p must also be members of the same class. This means that the functions q, r, a, s, and p can all be represented by their equivalent polytrices $\underset{\sim}{Q}$, $\underset{\sim}{R}$, $\underset{\sim}{A}$, $\underset{\sim}{S}$, and $\underset{\sim}{P}$. In the following we show how the elements in $\underset{\sim}{A}$, $\underset{\sim}{S}$, and $\underset{\sim}{P}$ are related to those in $\underset{\sim}{Q}$ and $\underset{\sim}{R}$.

Let M_q, M_r, M_a, M_s, and M_p be the number of terms in q, r, a, s, and p, respectively. Therefore,

$$M_a = M_q + M_r,$$
$$M_s = M_q + M_r,$$

$$M_p = M_q \cdot M_r. \tag{8}$$

Similar terms may arise in any of the functions a, s, and p. These can be combined so that the final number of terms in the functions a, s, and p may be less than M_a, M_s, and M_p, respectively. The process of combining similar terms in a polytrix is explained later.

The elements in the polytrices $\underset{\sim}{A}$, $\underset{\sim}{S}$, and $\underset{\sim}{P}$ are given by

$$\left.\begin{array}{ll} A_{ij} = Q_{ij}; & (i = 1,2,\ldots,M_q) \\ A_{kj} = R_{\ell j}; & (\ell = 1,2,\ldots,M_r), \text{ and } (k = M_q + \ell) \end{array}\right\}, \quad (j = 1,2,\ldots,N+1) \tag{9}$$

$$\left.\begin{array}{ll} S_{ij} = Q_{ij}; & (i = 1,2,\ldots,M_q), \text{ and } (j = 1,2,\ldots,N+1) \\ S_{kj} = R_{\ell j}; & (\ell = 1,2,\ldots,M_r), \text{ and } (j = 2,3,\ldots,N+1) \\ S_{k1} = -R_{\ell 1}; & (k = M_q + \ell) \end{array}\right\}, \tag{10}$$

$$\left.\begin{array}{l} P_{m1} = Q_{i1}R_{\ell 1} \\ P_{mj} = Q_{ij} + R_{\ell j} \end{array}\right\} \left.\begin{array}{l} (m = (i-1)M_r + \ell), \\ (i = 1,2,\ldots,M_q), \\ (\ell = 1,2,\ldots,M_r), \\ (m = 1,2,\ldots,M_qM_r), \\ (j = 2,3,\ldots,N+1). \end{array}\right\} \tag{11}$$

A polytrix $\underset{\sim}{F}$ can be reduced to its simplest form by collecting similar terms in the corresponding function f as follows:

$$\left.\begin{array}{ll} \text{If } F_{nj} = F_{mj}, \text{ with} & (j = 2,3,\ldots,N+1), \\ & (1 \le n \le M_f), \\ & (1 \le m \le M_f), \\ \text{Set } F_{n1} = F_{n1} + F_{m1} \text{ and } F_{m1} = 0. & \end{array}\right\}, \ (m \neq n). \tag{12}$$

One can then delete the mth row of the polytrix $\underset{\sim}{F}$.

CONCLUSIONS

A method for representing the general class of functions $f = \sum_j C_j \prod_i X_i^{\alpha_{ji}}$ using matrix-like quantities has been introduced. These have been called "polytrices" to avoid confusion since ordinary matrix algebra does not apply for them. Representing the functions f above in terms of their equivalent polytrices makes their algebraic manipulations readily amenable to computer application. It is hoped that as more "laws" of polytrix algebra are discovered, algebraic manipulations of the functions defined in equation 1 will be greatly simplified.

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