An Efficient and High Resolution Solver for the Two-Dimensional Numerical Simulation of the Richtmyer-Meshkov Instability

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Presented at the 8th International Workshop on the Physics of Compressible Turbulent Mixing (8th IWPCTM), California Institute of Technology, Pasadena, California, 9–14 December 2001 (submitted to *Lasers and Particle Beams*).

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1. Introduction

The numerical simulation of multi-component or multi-fluid flows has received a great deal of attention due to its ubiquitous nature in hydrodynamic flows of engineering concerns. Several different approaches have been presented in the literature for the simulation of multi-component flows associated with discontinuities and shock waves. Among them are methods, which use an extended conservative system of governing equations in which additional conservation equations are introduced to the original Euler equations to describe the conservation of parameters such as the mass fractions and the ratio of specific heats in the mixture. Unfortunately, it was found that the solution to this extended system of governing equations in conservative form, with the conventional "gamma" and "thermodynamic" models often failed to maintain pressure equilibrium and resulted in oscillations and other computational inaccuracies near material interfaces [1,4,8]. Karni [4] introduced a non-conservative scheme to capture the contact discontinuities using an additional non-conservative governing equation for the evolution of the pressure. This method reduced conservation errors and reasonable results are obtained; however, it was unable to handle strong shocks. Abgrall [1] presented a quasi-conservative approach for the simulation of multi-component flows. In his approach, the density of each component had its own governing equation for preserving positive mass fractions and included a non-conservative formulation for the ratio of specific heats of the mixture. More recently, Wang *et al.* [8] proposed a thermodynamically consistent and fully conservative approach for the treatment of contact discontinuities based on the concept of Total Enthalpy Conservation of the Mixture (ThCM). This model offers a new description of multi-component or multi-fluid flows while maintaining the favorable features of fully conservative conservation laws. This model is able to handle both strong and weak shocks, and produces relatively oscillation-free results for all variables in all test cases run to date [8]. In this paper, the ThCM model is utilized in a two-dimensional numerical simulation of the Richtmyer-Meshkov instability. To enhance the resolution of solutions in space, a high-resolution Godunov-type scheme is developed based upon a fast, exact Riemann solver [6] and the PSM (Piece-wise Spline Method) for the data reconstruction of primitive variables at cell interfaces (fourth order accuracy has been achieved even for non-uniform mesh sizes). The combination of the ThCM model with the high-resolution Godunov scheme provides an efficient and high-resolution solver that is used to support the Wisconsin shock tube investigation of hydrodynamic issues related to the Richtmyer-Meshkov (RM) instabilities (RT) [1]. The test problem #1 of the 8th IWPCTM and two Wisconsin shock tube experiments with gas pairs of CO_2-air and Ar-N_2 in the strongly shocked regime [5] are simulated using the present solver.

2. A Thermodynamically Consistent and Fully Conservative Approach

The ThCM model was developed for multi-component flow simulations under the assumptions that the individual components have the same flow variables (velocities and pressure) as the mixture. Starting from the concept of total enthalpy conservation for the mixture, a new formulation is defined for the determination of the ratio of the specific heats of the mixture, and a governing equation in conservative form for pressure is subsequently obtained. With continuity equations for the individual components, a governing equation in conservative form for the ratio of specific heats of the mixture was derived. These two equations, combined with mass and momentum balance equations, form the system for the description of multi-component flows. Details about the ThCM model can be found in [8]. In the two dimensional case for solution of the Richtmyer-Meshkov instability, the hyperbolic conservation laws with the ThCM model are given by

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = 0 \tag{1}$$

where $\mathbf{U}$ is the vector of the conservative variables, and $\mathbf{F}$ and $\mathbf{G}$ are the vectors of the conservative fluxes in the x and y directions, respectively, given by

$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho\chi \\ \rho u \\ \rho v \\ H - p \end{pmatrix}; \qquad \mathbf{F} = \begin{pmatrix} \rho u \\ \rho u \chi \\ \rho u^2 + p \\ \rho u v \\ uH \end{pmatrix}; \qquad \mathbf{G} = \begin{pmatrix} \rho v \\ \rho v \chi \\ \rho u v \\ \rho v^2 + p \\ vH \end{pmatrix}. \tag{2}$$

Here ρ, u, v, p stand for the density, velocities and pressure, respectively. H is the total enthalpy per unit volume

$$H = \chi p + \frac{1}{2}\rho\left(u^2 + v^2\right). \tag{3}$$

The equation of state for the calorically ideal gas is used in the above equations, however, the total enthalpy appearing in system (2) is not solved as a conservative variable. The solved conservative variable is $\left(H - p\right)$:

$$H - p = \left(\chi - 1\right)p + \frac{1}{2}\rho\left(u^2 + v^2\right), \tag{4}$$

where χ is introduced in the ThCM model to simplify the expression of the governing equations and is defined as

$$\chi = \frac{\gamma}{\gamma - 1} = \alpha_1\chi_1 + \alpha_2\chi_2 . \tag{5}$$

For a two component flow, γ is the ratio of specific heats of the mixture, α_1 and α_2 are the volume fractions for the individual component and χ_1 and χ_2 are gamma ratios for the individual component

$$\chi_1 = \frac{\gamma_1}{\gamma_1 - 1}; \qquad \chi_2 = \frac{\gamma_2}{\gamma_2 - 1} . \tag{6}$$

From equation (4) and the equality $\alpha_1 = 1 - \alpha_2$, one can calculate the volume fraction of each individual component, after the parameter χ for the mixture is obtained. The ratio of specific heats of the mixture can then be calculated using equation (5).

3. The Piece-wise Spline Method (PSM)

A splitting scheme is employed for equation (1) in the two spatial dimensions in which the *x*-sweep solves equation (7(A)) and the *y*-sweep solves equation (7(B)):

$$(A)\quad for \quad x-sweep: \qquad\qquad (B)\quad for \quad y-sweep:$$

$$\begin{pmatrix}\rho\\ \rho u\\ \rho\chi\\ H-p\\ \rho v\end{pmatrix}_t+\begin{pmatrix}\rho u\\ \rho u^2+p\\ \rho u\chi\\ uH\\ \rho uv\end{pmatrix}_x=0; \qquad \begin{pmatrix}\rho\\ \rho v\\ \rho\chi\\ H-p\\ \rho u\end{pmatrix}_t+\begin{pmatrix}\rho v\\ \rho v^2+p\\ \rho v\chi\\ vH\\ \rho uv\end{pmatrix}_y=0. \qquad (7)$$

The tangential velocity components, $v(u)$ in the x-sweep (y-sweep), are passively advected with the normal velocity components, $u(v)$. A two-step process, shown in equation (8a) and (8b), accomplishes the integration from time level n to $n+1$ using the Godunov scheme [3],

$$\begin{aligned}\mathbf{U}_{i,j}^{n+1/2}&=\mathbf{U}_{i,j}^{n}+\frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i-1/2,j}^{n}-\mathbf{F}_{i+1/2,j}^{n}\right); \quad \forall j \quad (a)\\ \mathbf{U}_{i,j}^{n+1}&=\mathbf{U}_{i,j}^{n+1/2}+\frac{\Delta t}{\Delta y}\left(\mathbf{G}_{i,j-1/2}^{n+1/2}-\mathbf{G}_{i,j+1/2}^{n+1/2}\right); \quad \forall i \quad (b)\end{aligned} \qquad (8)$$

with inter-cell numerical fluxes given by the local Riemann solutions at the cell interface positions: $\mathbf{F}_{i+1/2,j}=\mathbf{F}(\mathbf{U}_{i+1/2,j}(0))$. The Riemann solver used for the local Riemann solutions is the Pike fast, exact Riemann solver [6]. The Godunov scheme (8) has first order accuracy in time and the accuracy of the scheme in space depends on the accuracy of the initial conditions for the local Riemann solutions. The original Godunov scheme has first order accuracy and the accuracy can be improved using so-called "data reconstruction" of the initial conditions. The PSM method and the corresponding slope limiters developed by Ren *et al*. [7] were improved and used here for the data reconstruction of initial conditions for the local Riemann solutions at cell interfaces. The formulations for the data reconstruction using the PSM method for the x-direction are:

$$\begin{aligned}\mathbf{U}_i^L&=\mathbf{U}_i^n-\frac{\Delta x_i}{2}\mathbf{m}_i+\frac{(\Delta x_i)^2}{8}\mathbf{M}_i-\frac{(\Delta x_i)^3}{48}\mathbf{M}_i^{(3)} \quad (a)\\ \mathbf{U}_i^R&=\mathbf{U}_i^n+\frac{\Delta x_i}{2}\mathbf{m}_i+\frac{(\Delta x_i)^2}{8}\mathbf{M}_i+\frac{(\Delta x_i)^3}{48}\mathbf{M}_i^{(3)} \quad (b)\end{aligned} \qquad (9)$$

where $\mathbf{U}$ are the vectors of the conservative primitive variables, $\mathbf{m},\mathbf{M},\mathbf{M}^{(3)}$ are the vectors of the spline approximations of the first, second and third derivatives of the corresponding vector of primitive variables, respectively, with respect to the x-direction. Generally the data reconstruction using formulation (9a) and (9b) causes spurious oscillations in the vicinity of the shock wave, therefore the basic idea of van Leer [9] was used to create slope limiters to suppress these possible oscillations. The limiter (similar to the so-called "minmod" slope limiter) for the PSM method is given below:

$$\begin{aligned}\mathbf{U}_i^L&=\mathbf{U}_i^n-\frac{1}{2}\min\mathrm{mod}\left(\mathbf{m}_{i-1},\mathbf{m}_i,\mathbf{m}_{i+1}\right)\Delta x_i+\frac{1}{8}\min\mathrm{mod}\left(\mathbf{M}_{i-1},\mathbf{M}_i,\mathbf{M}_{i+1}\right)(\Delta x_i)^2-\frac{1}{48}\min\mathrm{mod}\left(\mathbf{M}_{i-1}^{(3)},\mathbf{M}_i^{(3)},\mathbf{M}_{i+1}^{(3)}\right)(\Delta x_i)^3\\ \mathbf{U}_i^R&=\mathbf{U}_i^n+\frac{1}{2}\min\mathrm{mod}\left(\mathbf{m}_{i-1},\mathbf{m}_i,\mathbf{m}_{i+1}\right)\Delta x_i+\frac{1}{8}\min\mathrm{mod}\left(\mathbf{M}_{i-1},\mathbf{M}_i,\mathbf{M}_{i+1}\right)(\Delta x_i)^2+\frac{1}{48}\min\mathrm{mod}\left(\mathbf{M}_{i-1}^{(3)},\mathbf{M}_i^{(3)},\mathbf{M}_{i+1}^{(3)}\right)(\Delta x_i)^3\end{aligned} \quad (10)$$

and

$$\min\mathrm{mod}\left(g_{i-1},g_i,g_{i+1}\right)=\begin{cases}\min\left(|g_{i-1}|,|g_i|,|g_{i+1}|\right)\times sign(g_i) & if \ \min\left(g_{i-1},g_i,g_{i+1}\right)\times\max\left(g_{i-1},g_i,g_{i+1}\right)>0\\ 0 & else\end{cases}$$

The present PSM method has fourth order accuracy. Many test runs for the simulation of single- and multi-component flows have showed that the method is efficient, of high resolution and non-oscillating.

4. Results

4.1 IWPCTM Test Problem 1

The present solver is used for the numerical simulation of test problem #1 of the 8th IWPCTM and two Wisconsin shock tube experiments. Test problem #1 is a shock tube experiment conducted by Meshkov with an air-helium interface with a 2-D perturbation impacted by an incident shock The geometry, initial conditions, mesh size, and schematic of the experiment are given in [10]. Figure 1 shows the distribution of the total mass (per unit length) of each material in the mixing zone as a function of time. The symbols on the graph and values in Table 1 give the total masses at the specific times requested by the 8th IWPCTM. The solution indicates that the total mass of each material in the mixing zone rapidly increases with time initially and then increases linearly with a larger slope for air than for helium. Figure 2 shows an image of the structure of the shocked interface and the 5% and 95% volume fraction isolines in the mixing zone along with the velocity vectors for 1.24 ms aftershock interaction.

Table 1
Total mass (g/m) of each material in mixing zone vs. time (ms) for test problem 1 of the 8th IWPCTM

Time (ms)	He (g/m)	Air (g/m)	Air+He (g/m
0	0	0	0
0.344	0.0536	0.1185	0.1721
0.644	0.0685	0.1589	0.2274
0.94	0.0817	0.1928	0.2845
1.24	0.0995	0.2275	0.327
1.545	0.1173	0.2779	0.3952
1.838	0.135	0.314	0.449

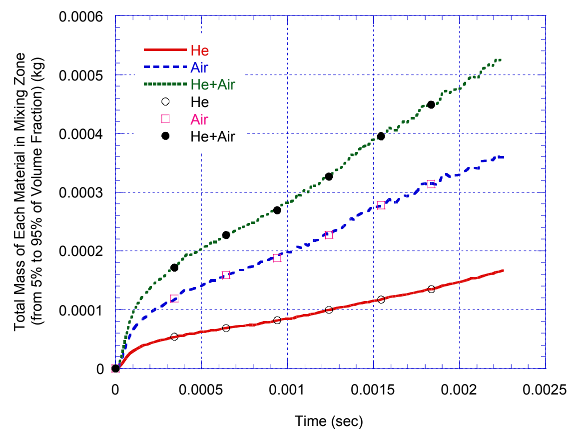

Figure 1. Total mass of each material vs. time for test problem 1 of the 8th IWPCTM.

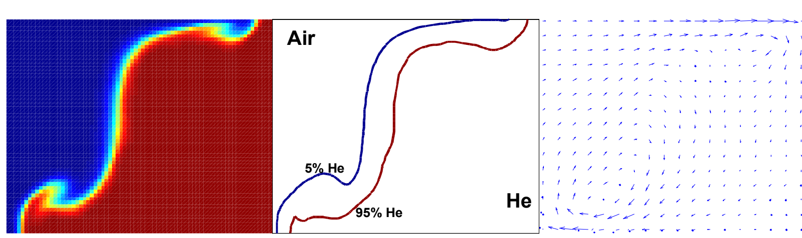

Figure 2. Volume fraction, volume fraction isolines and velocity vectors for a time of 1.24 ms after shock interaction.

4.2 Simulation of Wisconsin's RM experiments

Two Wisconsin shock tube experiments were also simulated, both with relatively strong shocked interfaces. The two experiments considered here consist of gas pairs CO_2-air (Exp. 333, M=3.08) and Ar-N_2 (Exp. 319, M=2.80), with measured initial conditions and specific geometry and experimental conditions given in [5]. The simulation of experiment 333 used a 200x2546 grid, with a resolution of 0.381 mm and took about 24 hours of CPU time on a Pentium 4 at 1.46 GHz. Figure 3 shows a comparison of the simulation of experiment 333 with the experimentally measured instability at a time of 0.66 ms after shock interaction. The initial condition shown in Figure 3a was fit to a sine series and input as the initial condition in simulation. The overall agreement between the experiment and the simulation seem good. In this particular experimental image the fine structure that is evident in the numerical solution is not seen; however, the overall "spike" and "bubble" heights seem similar. The amplitude ($\eta=(\eta_{bubble} + \eta spike)/2$) as a function of time predicted by the simulation, along with the experimentally measured amplitude at 0.66 ms and 0.70 ms are shown in Figures 4a and 4b for experiments 333 and 319 respectively. As can be seen the simulation obtains consistent results with the experiments.

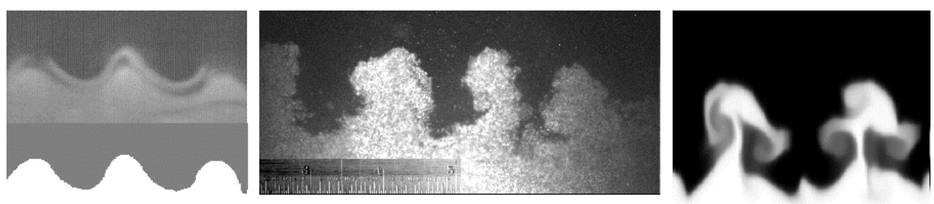

Figure 3. Experimental initial conditions, experimental image of instability 0.66 ms after shock interaction and image of simulation prediction of the instability at 0.67 ms.

5. Conclusion

An efficient and high-resolution solver was developed based upon the ThCM model and a PSM method-based Godunov scheme for the two-dimensional numerical simulation of the Richtmyer-Meshkov instability. The test problem #1 of the 8th IWPCTM and two Wisconsin shock tube experiments (gas pair CO_2-air and Ar-N_2 in a strongly shocked regime) are simulated. The primary results show that the present solver developed based upon the ThCM model and a PSM-based high-resolution Godunov scheme is a promising tool for two-dimensional numerical simulation of the Richtmyer-Meshkov instability. Future studies will consider the effect of the viscosity and the accuracy enhancement in time as well as the extension of the present solver to the three-dimensional simulations. This solver will then be compared to other codes and against experimental images with higher resolution.

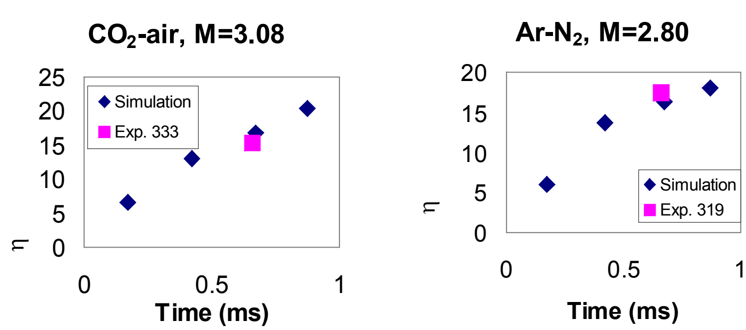

Figure 4. Comparison of simulation and experimental results.

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